Stability-instability transitions in Hamiltonian systems of n dimensions

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(Received 5 May 1986; revised manuscript received 7 July 1986)

We show analytically that for a class of simple periodic motions in a general Hamiltonian system of n dimensions, if C is a parameter of the system and C_ρ one of its generally many critical values at which the motion undergoes a stability-instability transition, the behavior of the largest Lyapunov exponent μ as C approaches C_ρ from the unstable region is given by $\mu=\text{const}\times|C-C_\rho|^\beta$, where $\beta=\frac{1}{2}$, independent of the transition point, type of transitions, or the dimensionality of the system. We present numerical results for a three-dimensional Hamiltonian system which exhibits three types of stability-instability transitions, and for a two-dimensional Hamiltonian system which exhibits two types of transitions.

I. INTRODUCTION

In this paper, we shall show analytically that for a class of simple periodic motions in a general type of Hamiltonian system, if C is a parameter of the system and C_ρ one of its generally many critical values at which the motion undergoes a stability-instability transition, the behavior of the largest Lyapunov exponent μ as C approaches C_ρ from the unstable region is given by

$$\mu=\text{const}\times|C-C_\rho|^\beta, \tag{1}$$

where $\beta=\frac{1}{2}$, independent of the transition point, type of transitions, or the dimensionality of the system.

Our result is reminiscent of a similar result established numerically for a dissipative dynamical system in which in what are called the intermittent transition to chaos[1,2] of types I and III, the Lyapunov exponent μ behaves in a manner similar to that given by Eq. (1), where β was numerically determined to be equal to 0.5. It is also of interest to note that the exponent β is 0.449 806 9... for the period-doubling route to chaos.[3] The significance of our result is the universality of the value of β for the Hamiltonian systems of any dimensions, and that its validity has been established analytically.

Since a great deal of work on stability-instability and order-chaos transitions has been done on various Hamiltonian systems,[4] we should mention the following features of our analysis: (i) that our systems involve continuous times, not discrete mappings which in many respects are easier to deal with, and (ii) that our results deal with the stability-instability transitions of a class of periodic motions, as functions of the coupling parameters of the system. Thus some of our analysis can be viewed as a sequel of Sec. 3.3 of Ref. 4. A point of interest in this connection is a result[5-9] which shows that for a number of two-dimensional Hamiltonian systems, a given motion may change from stable to unstable to stable a finite or infinite number of times as the value of the coupling parameter is varied continuously from $-\infty$ to $+\infty$, and that the $\beta=\frac{1}{2}$ exponent has been established for these two-dimensional systems[6,7] and for the case when the stability of the cycle of interest is described by a Mathieu equation.[9]

We have numerically verified our analytic result (1) for a number of Hamiltonian systems. As examples, we shall present in this paper a three-dimensional and a two-dimensional Hamiltonian system for which we shall give, among other results, tables listing some accurately determined critical points C_ρ together with the types of transition at these critical points.

II. STABILITY-INSTABILITY TRANSITIONS IN HAMILTONIAN SYSTEMS OF n DIMENSIONS

Consider a general Hamiltonian system whose Hamiltonian is given by

$$H=\frac{1}{2}\sum_{j=1}^{n} m_j\dot{x}_j^2+V(x_1,x_2,\ldots,x_n), \tag{2}$$

where the potential energy V depends on the position coordinates only. The shape of the potential function can be quite arbitrary, and the motions considered may include those which start to be bounded, and become unbounded at later times. The equations of motion are given from Eq. (2) by

$$m_j\ddot{x}_j+\partial V/\partial x_j=0,\quad j=1,2,\ldots,n \tag{3}$$

which are generally nonlinear. A typical jth equation of motion, for example, may be of the form

$$\ddot{x}_j+a_1^{(j)}x_{j_1}^{r_1}+a_2^{(j)}x_{j_2}^{r_2}+\cdots+C_1^{(j)}x_{k_1}^{p_1}x_{k_2}^{p_2}\cdots+C_2^{(j)}x_{l_1}^{q_1}x_{l_2}^{q_2}\cdots+\cdots=0, \tag{4}$$

where the parameters of the system are the $a^{(j)}$'s and the coupling parameters $C^{(j)}$'s.

We assume that, under a set of initial conditions, our system has a simple periodic solution with a determinable real period T. An example in which this situation often occurs is when the initial condition is given by

$$x_j(0)=a,\ \ \dot{x}_j(0)=b,\ \ x_k(0)=\dot{x}_k(0)=0 \ \text{ for } k\neq j \ , \tag{5}$$

for which the equations of motion (5) are assumed to give a solution of the form

$$x_j(t)=\phi(t),\ \ x_k(t)=0 \ \text{ for } k\neq j \ , \tag{6}$$

where $\phi(t)$ is a periodic function of time with a period T, i.e., $\phi(t+T)=\phi(t)$. We shall consider the stability of this solution when the initial condition given by Eq. (5) is slightly changed. Since the linearized equations of motion for small perturbations Δx_r from x_r obtained from Eq. (4) and the substitutions of Eq. (6) in them will contain some or all of the parameters $a^{(j)}$ and $C^{(j)}$ of the system, the stability of our simple solution (6) will generally depend on these parameters, and the dependence is generally not simple. Let $\mathbf{w}=(\Delta x_1,\Delta\dot{x}_1,\Delta x_2,\Delta\dot{x}_2,\dots,\Delta x_n,\Delta\dot{x}_n)$ be a $2n$-dimensional column vector whose components represent small perturbations Δx_r and $\Delta\dot{x}_r$ from x_r and $\dot{x}_r$, $r=1,2,\dots,n$. The linearized equations of motion can be written as

$$\dot{\mathbf{w}}=\underline{M}(t)\mathbf{w} \ , \tag{7}$$

where the matrix $\underline{M}(t)$ can be written in an explicitly time-dependent form through $\phi(t)$ because of the solution (6). $\underline{M}(t)$ is thus periodic with a period τ, which is equal to T, or $T/2$ if only even powers of $\phi(t)$ appear in $\underline{M}(t)$. The steps from Eq. (4) to Eq. (7) in which $\underline{M}(t)$ is expressible in an explicitly time-dependent form are straightforward, if an appropriate choice of the initial condition which would yield a solution such as (6) could be made. The fact that such choices can often be made, and that the study of the stability of these simple solutions as a function of the parameters of the system can lead analytically to many important conclusions, does not seem to have been appreciated previously.

The dimension of the matrix $\underline{M}(t)$ is generally $2n$. In practice, however, it often happens that certain symmetries or simplifying features can be used to reduce $\underline{M}(t)$ to a smaller size. Literatures on the stability analysis of the type of equations given by Eq. (7), pioneered by Lyapunov, are very extensive,[10] especially on the problem concerning the analytic criterions for stability. We are, on the other hand, more interested in classifying the type of stability-instability transitions, their dependence on the parameters of the system, and the ready numerical determination of the transition points.

Let the column vectors $\mathbf{W}_k(t)$, $k=1,2,\dots,2n$, representing the $2n$ fundamental solutions of Eq. (7) be placed into a matrix form $\underline{W}(t)$ such that $\underline{W}(0)$ is a unit matrix, i.e., the components $w_{jk}(t)$, $j=1,2,\dots,2n$ of the kth fundamental solution $\mathbf{W}_k(t)$ have initial values given by $w_{jk}(0)=\delta_{jk}$. The special initial values are what distinguish a fundamental solution from just a solution of Eq. (7). Since $\underline{M}(t)$ is periodic with period τ, there exists a nonsingular constant matrix $\underline{P}$ such that $\underline{W}(t+\tau)=\underline{W}(t)\underline{P}$. Setting $t=0$ and noting that $\underline{W}(0)$ is a unit matrix, it follows that the elements p_{jk} of $\underline{P}$ are given by

$$p_{jk}=w_{jk}(\tau) \ . \tag{8}$$

The matrix $\underline{P}$, which from Eq. (8) can be readily numerically determined, plays an important role in the stability analysis, for it can be shown, using a similar argument which led to the Floquet theorem, that the characteristic values s_j, $j=1,2,\dots,2n$ of the matrix $\underline{P}$ determine the stability or instability of the system whose linearized equations of motion are given by Eq. (7). More specifically, denoting

$$s_j=\exp(\mu_j\tau) \ , \tag{9a}$$

or

$$\mu_j=\tau^{-1}\ln s_j \ , \tag{9b}$$

the general solution of the differential equation (7) is given by

$$\mathbf{w}(t)=\sum_{k=1}^{2n} c_k e^{\mu_k t}\boldsymbol{\psi}_k(t) \ , \tag{10}$$

where the c_k's are arbitrary constants and the $\boldsymbol{\psi}_k(t)$ are functions which are periodic in time with the period τ. (See Appendix A.)

An important feature of the matrix $\underline{M}(t)$ for our Hamiltonian system (2) is that its (k,j) element is always zero whenever $k+j$ is an even number. If $\underline{E}$ is a $2n\times 2n$ diagonal matrix whose diagonal elements are $(1,-1,1,-1,\dots,1,-1)$, it is easy to verify that $\underline{E}^{-1}=\underline{E}$, and that

$$\underline{E}^{-1}\underline{M}(t)\underline{E}=-\underline{M}(t) \ . \tag{11}$$

We assume that $\underline{M}(t)$ is an even function with respect to changing t to $-t$, i.e., $\underline{M}(-t)=\underline{M}(t)$. If $\phi(t)$ is an odd function of time, and odd powers of $\phi(t)$ appear in $M(t)$, then we choose another initial time t_0 such that the solution corresponding to (6) is $x_j(t')=\phi(t')$, $x_k(t')=0$ for $k\neq j$, where $t'=t-t_0$, and where $\phi(-t')=\phi(t')$. Then Eq. (11) can be written as

$$\underline{E}^{-1}\underline{M}(t)\underline{E}=-\underline{M}(-t) \ . \tag{12}$$

An important consequence of Eq. (12), as it can be shown by applying it to Eqs. (7) and (8), is that the inverse of the matrix $\underline{P}$ is given by

$$\underline{P}^{-1}=\underline{E}^{-1}\underline{P}\,\underline{E} \ , \tag{13}$$

from which it follows that the characteristic equation of the matrix $\underline{P}$ is reciprocal, i.e., is of the form

$$s^{2n}+a_1s^{2n-1}+a_2s^{2n-2}+\cdots+a_{2n-2}s^2+a_{2n-1}s+1=0 \ , \tag{14}$$

where $a_1=a_{2n-1}$, $a_2=a_{2n-2}$, etc. That is to say, for every characteristic root of Eq. (14), there is also the characteristic root s^{-1}. (See Appendix B.) That Eqs. (13) and (14) are a consequence of Eq. (12) is in fact a particular realization of a theorem of Lyapunov.[11]

From Eq. (9b), the stable region is characterized by the roots s_j distributed over the unit circle in the complex plane, and the unstable region is characterized by one or more of these roots having an absolute value greater than

1. As in the case of the intermittancy route to chaos,[1] the transition from a stable to unstable region can be classified according to three types:[12,13] I, a real characteristic root crosses the unit circle at $+1$; II, two conjugate characteristic roots cross the unit circle simultaneously; and III, a real characteristic root crosses the unit circle at -1.

It is always possible to write the characteristic equation (14) in the form

$$(s^2-\alpha_1 s+1)(s^2-\alpha_2 s+1)\cdots(s^2-\alpha_n s+1)=0\ , \tag{15}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_n$ can be expressed as roots of an nth-degree algebraic equation whose coefficients can be determined recursively from the coefficients $a_1,a_2,\ldots,a_{2n-1}$ of Eq. (14). (See Appendix C.) In the stable region, all the α's are real and have absolute values ≤ 2. A transition of type I to an unstable region as a result of changing the value of a parameter of the system past its critical value is characterized by one of the α's, α_j say, crossing the value $+2$ to a value greater than $+2$, while the remaining α's remain real and ≤ 2 in absolute values. Thus a transition of type I from a stable to unstable region is characterized by a complex-conjugate pair of roots on the unit circle approaching each other and closing in on the positive real axis, and becoming degenerate at the value $+1$ at the stability-instability transition point, and becoming separate again but appearing on two sides of $+1$ on the real axis, their values remaining reciprocal of each other. Similarly, a transition of type III is characterized by one of the α's in Eq. (15) crossing the value -2 to a value less than -2, which implies a complex-conjugate pair of roots closing in on the negative real axis and becoming degenerate at the value -1 at the transition point, and then becoming separate and appearing on two sides of -1 on the real axis. In either case, the roots of the equation $s^2-\alpha_j s+1=0$ given by

$$s_j=\tfrac{1}{2}[\alpha_j\pm(\alpha_j^2-4)^{1/2}] \tag{16}$$

change from complex to real, but all the α's in Eq. (15) remain real and thus remain analytic functions of the parameters of the system, since all the real coefficients of Eq. (14) are analytic functions of the parameters of the system. Thus, as the parameter C approaches the transition point C_ρ from the unstable region, α_j can be written as $\alpha_j=\pm 2\pm\epsilon$, where the positive and negative signs refer to transitions of types I and III, respectively, and where $\epsilon=\text{const}\times|C-C_\rho|\geq 0$. Thus we find, from Eq. (16), that the largest roots in absolute values are given, respectively, in type-I and -III transitions, by $s_j=\pm 1\pm\epsilon^{1/2}$, or, as $C\to C_\rho$ from the unstable region, the behavior of the largest Lyapunov exponent is given, from Eq. (9b), by Eq. (1), with $\beta=\frac{1}{2}$ independent of the transition points or the dimensionality of the system. We shall not consider the case in which, by an accident or a symmetry, the coefficient of $|C-C_\rho|^{1/2}$ happens to be equal to zero.

A transition of type II from a stable to unstable region is characterized by two of the α's, α_j, and α_{j+1}, say, in Eq. (15) changing from real to a complex-conjugate pair, while the remaining α's remain real and <2 in absolute values. Since α_j and α_{j+1} change from real to complex values across the transition point, neither of them is an analytic function of the parameters of the system generally. Thus we cannot, for example, assume that $\alpha_j=\alpha_{j_0}+\epsilon$ in the neighborhood of the transition point. On the other hand, we have

$$(s^2-\alpha_j s+1)(s^2-\alpha_{j+1}s+1)=s^4+As^3+Bs^2+As+1\ , \tag{17}$$

where

$$A=-(\alpha_j+\alpha_{j+1}),\quad B=\alpha_j\alpha_{j+1}+2\ . \tag{18}$$

From Eqs. (17) and (18), we get

$$\alpha_j=-\tfrac{1}{2}[A+(A^2-4B+8)^{1/2}]\ ,\qquad \alpha_{j+1}=-\tfrac{1}{2}[A-(A^2-4B+8)^{1/2}]\ , \tag{19}$$

and

$$s_j=\tfrac{1}{2}[\alpha\pm i(4-\alpha^2)^{1/2}]\ , \tag{20}$$

where α denotes α_j or α_{j+1}. We note that since α_j and α_{j+1} are complex conjugate, A and B are always real, and hence they are always analytic functions of the parameters of the system. Thus in the neighborhood of the transition point, A and B can be written as $A=A_0+\epsilon$, $B=B_0+\epsilon$, and at the transition point, $A_0^2-4B_0+8=0$. As the parameter C approaches the transition point C_ρ from the unstable region, $\alpha_j=-\frac{1}{2}(A+i\epsilon^{1/2})$, $\alpha_{j+1}=-\frac{1}{2}(A-i\epsilon^{1/2})$, from Eq. (19). Substituting these into Eq. (20), we find $s_j=\frac{1}{4}[-A_0\pm i(16-A_0^2)^{1/2}]\pm c\epsilon^{1/2}$, where c is some complex constant. Thus the absolute square of the largest root is given by $|s_j|^2=1+\epsilon^{1/2}$, where ϵ is equal to some positive constant times $|C-C_\rho|$. Using Eq. (9b), we find exactly the same behavior given by Eq. (1) for the Lyapunov exponent as that for the type-I and -III transitions, with the same critical exponent $\beta=\frac{1}{2}$ which is independent of the transition point, type of transitions, or the dimensionality of the system. Noting that the numerical analysis of Manneville and Pomeau,[1] and Daido and Haken,[2] for a dissipative dynamical system gave a different value for β for the type-II transition, it may be useful to compare our analysis concerning the analytic or nonanalytic behavior of the relevant parameters with their basic assumptions on these parameters, and it may be worth while to redo their numerical analysis.

That there is a close analogy between the behavior of the Lyapunov exponent, Eq. (1), with the behavior of the long-range order (e.g., magnetization in a ferromagnet) in critical phenomena in statistical mechanics has been pointed out by Daido and Haken.[2] We would like to point out that the distribution of the characteristic roots s_j and its behavior as the stability-instability transition point of type I or III is approached also have their analogs in thermodynamic phase transitions in the Lee-Yang theorem[14] on the distribution of roots of the grand partition function. However, the behavior of the characteristic roots corresponding to the stability-instability transition of type II does not appear to have any analog in equilibrium thermodynamics.

III. A THREE-DIMENSIONAL HAMILTONIAN SYSTEM

As a first example, we consider a system whose Hamiltonian is given by

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2+\omega_1^2x^2+\omega_2^2y^2+\omega_3^2z^2)+Cxyz\ . \quad (21)$$

The equations of motion are

$$\begin{aligned}\ddot{x}+\omega_1^2x+Cyz&=0\ ,\\ \ddot{y}+\omega_2^2y+Cxz&=0\ ,\\ \ddot{z}+\omega_3^2z+Cxy&=0\ .\end{aligned} \quad (22)$$

It is easy to see that for an initial condition given by

$$x(0)=A,\ \ \dot{x}(0)=y(0)=\dot{y}(0)=z(0)=\dot{z}(0)=0\ , \quad (23)$$

the system gives a simple periodic motion

$$x(t)=A\cos(\omega_1 t),\ \ y(t)=z(t)=0 \ \text{ for all } t\ . \quad (24)$$

Using Eq. (24), the linearized equations of motion for small perturbations $\Delta x,\Delta\dot{x},\ldots,\Delta\dot{z}$ from $x,\dot{x},\ldots,\dot{z}$ give the following equation for Δx,

$$d^2(\Delta x)/dt^2+\omega_1^2(\Delta x)=0 \quad (25a)$$

and the following coupled equations for $\Delta y=w_1$, $\Delta\dot{y}=w_2$, $\Delta z=w_3$, and $\Delta\dot{z}=w_4$:

$$d\mathbf{w}/dt=\underline{M}\mathbf{w}\ , \quad (25b)$$

where

$$\mathbf{w}=\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}$$

and (25c)

$$\underline{M}=\begin{bmatrix}0 & 1 & 0 & 0\\ -\omega_2^2 & 0 & -CA\cos(\omega_1 t) & 0\\ 0 & 0 & 0 & 1\\ -CA\cos(\omega_1 t) & 0 & -\omega_3^2 & 0\end{bmatrix}.$$

The solution for Δx and $\Delta\dot{x}$ for some given (small) initial values of $\Delta x(0)$ and $\Delta\dot{x}(0)$ is clearly always regular from Eq. (25a). On the other hand, for some small deviations $\Delta y(0)$, $\Delta\dot{y}(0)$, $\Delta z(0)$, and $\Delta\dot{z}(0)$ from the set of initial values given by Eq. (23), the solution for $\Delta y,\Delta\dot{y},\Delta z,\Delta\dot{z}$ from Eq. (25b) may all be superpositions of oscillatory functions of time in which case the motion is said to be stable, or one or more of the deviations may grow exponentially with time in which case the motion is said to be unstable. Equation (25b) is a specific example of Eq. (7), and the stability of the solution of such equation is generally a complicated function of the parameters of the system, which in this case are $\omega_1,\omega_2,\omega_3,C$ and the initial x displacement A. The matrix $\underline{M}(t)$ given by Eq. (25c) is periodic with a period $\tau=2\pi/\omega_1$. The stability or instability of the solution of Eq. (25b) is determined by the characteristic values s_j, $j=1,2,3,4$ of the matrix $\underline{P}$ given by Eq. (8).

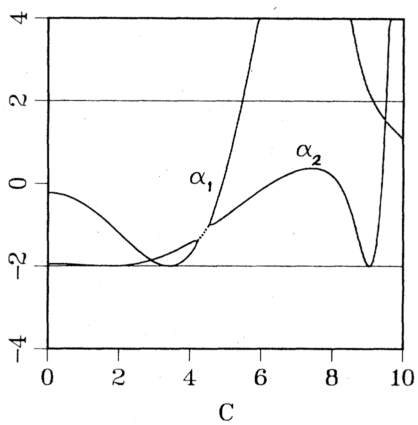

FIG. 1. The values of α_1 and α_2 of Eq. (26) in Sec. III plotted as functions of C, for $\omega_1=1$, $\omega_2=2\sqrt{3}$, $\omega_3=\sqrt{3}$, and $A=1$. The dotted line indicates that the values of α's become complex. See Table I(a) for a summary of the regions of stability and instability.

The characteristic equation of the matrix $\underline{P}$ is reciprocal, as Eqs. (11)–(15) show, and is conveniently expressed in the form

$$(s^2-\alpha_1 s+1)(s^2-\alpha_2 s+1)=0\ . \quad (26)$$

The stability of the given motion (24) as a function of ω_1, ω_2, ω_3, and C for an initial condition which differs slightly from Eq. (23) can be seen by plotting α_1 and α_2 as functions of these parameters. Figure 1 shows α_1 and α_2 as functions of the coupling parameter C, while the other parameters are kept fixed. The dotted line indicates that the values of α become complex conjugate. Regions in which both values of α are real and <2 in absolute values are the stable regions. Regions in which one or both of the α's become >2 or <-2, or those in which the α's become complex, are unstable regions. The stability-instability transition points, the types of transitions, and the regions of stability and instability are summarized in Table I. The distributions of the roots s_j when a parameter of the system is close to and at one of its stability-instability transition points of types I, II, and III are shown in Fig. 2. The behavior of the largest Lyapunov exponent in the neighborhood of each of these transition points as they are approached from the unstable region has been numerically verified to be given by Eq. (1), as we analytically predicted for Hamiltonian systems of any dimension.

In the case $\omega_2=\omega_3$, the values of α_1 and α_2 in the characteristic equation (26) become degenerate. An immediate consequence is that transitions of type II no longer occur. It is easy to verify that Eq. (25b) reduces to the following two independent equations:

$$d^2u/dt^2+[\omega_2^2+CA\cos(\omega_1 t)]u=0\ , \quad (27a)$$

$$d^2v/dt^2+[\omega_2^2-CA\cos(\omega_1 t)]v=0\ , \quad (27b)$$

where $u=\Delta y+\Delta z$, and $v=\Delta y-\Delta z$. Equations (27) can be written as Mathieu equations with a change of variable $\bar{t}=\frac{1}{2}\omega_1 t$:

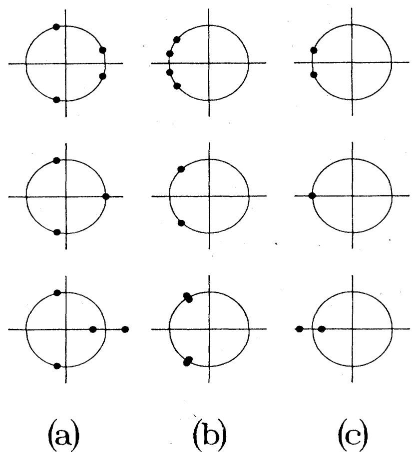

FIG. 2. The distributions of the roots s_j, $j=1,2,3,4$ of Eq. (26) in Sec. III when the value of a parameter of the system is close to one of its stability-instability transition points of types I, II, and III are shown in (a), (b), and (c) respectively. In (a), $A=1$, $\omega_1=1$, $\omega_2=2\sqrt{3}$, $\omega_3=\sqrt{3}$, and from top to bottom, $C=5.435\,55$, 5.465 55, and 5.505 55. In (b), $A=1$, $\omega_1=1$, $\omega_2=2\sqrt{3}$, $\omega_3=\sqrt{3}$, and from top to bottom, $C=3.699\,61$, 4.199 61, and 4.466 61. In (c), $A=1$, $\omega_2=\omega_3=\sqrt{3}$, and from top to bottom, $C=2.686\,33$, 2.746 33, and 2.786 33. Check with Table I for the values of the parameters indicated.

$$d^2u/d\bar{t}^2+[a+2q\cos(2\bar{t})]u=0\ , \tag{28a}$$

$$d^2v/d\bar{t}^2+[a-2q\cos(2\bar{t})]v=0\ , \tag{28b}$$

where

$$a=\frac{4\omega_2^2}{\omega_1^2},\quad q=\frac{2CA}{\omega_1^2}\ . \tag{29}$$

The occurrence of a Mathieu equation with the consequence regarding the existence of an infinite number of stability-instability domains has also appeared in the work of Churchill, Pecelli, Sacolick, and Rod,[8] Doveil and Escande,[9] Heller, Stechel, and Davis,[15] and of the present authors.[6] The stability chart for the Mathieu equation can be found in many texts. The characteristic curves $a_0,a_1,b_1,b_2,\ldots$ divide the (a,q) plane into regions of stability and instability. If the point (a,q) given by the values of ω_1, ω_2 $(=\omega_3)$, C, and A from Eq. (29) falls in the stable regions, then the solutions for u and v are oscillatory with time and the behavior of the motion is stable. On the other hand, if the point falls in the unstable region, then the solutions for u and v grow exponentially with time, signifying an unstable behavior. The stability-instability transition points are given by the intersections of the horizontal straight line $a=4\omega_2^2/\omega_1^2$ with the characteristic curves. For example, if we set $a=12$ (e.g., for $\omega_1=1$, $\omega_2=\omega_3=\sqrt{3}$), the region for which $-5.492\,66\le q\le 5.492\,66$ is stable, and the regions for which $|q|>5.492\,66$ are unstable, except for an infinite number of very narrow strips of stable regions. Thus the dependence on (i) the initial energy $\frac{1}{2}\omega_1^2A^2$, (ii) the coupling parameter C, and (iii) the natural frequencies of oscillation ω_1 and ω_2 $(=\omega_3)$, is represented conveniently by two dimensionless parameters a and q. If a_ρ and q_ρ denote the respective critical values at which the motion undergoes stability-instability transitions, the behavior of the largest Lyapunov exponent μ as any one of these critical values is approached from the unstable region is given by

$$\mu=\text{const}\times|a-a_\rho|^{\beta}\ , \tag{30}$$

or

TABLE I. The first few stability-instability transition points of periodic motion (24) of the Hamiltonian system (21), numbered arbitrarily by $\rho=1,2,3,\ldots$. The types of transitions at these points and the regions of stability (S) and instability (I) are also indicated.

	(a) $A=1$, $\omega_1=1$, $\omega_2=2\sqrt{3}$, $\omega_3=\sqrt{3}$		
ρ	Transition point C_ρ	Type of transition	Region
1	4.199 61	II	S: $0\le C\le C_1$
2	4.576 14	II	I: $C_1<C<C_2$
3	5.465 55	I	S: $C_2\le C\le C_3$
4	9.144 96	I	I: $C_3<C<C_4$
5	9.524 31	I	S: $C_4\le C\le C_5$
	(b) $A=1$, $\omega_1=1$, $\omega_3=\sqrt{3}$, $C=1$		
ρ	Transition point $\omega_2^{(\rho)}$	Type of transition	Region
1	0.491 80	I	I: $0<\omega_2<\omega_2^{(1)}$
2	0.861 49	I	S: $\omega_2^{(1)}\le\omega_2\le\omega_2^{(2)}$
3	0.953 25	I	I: $\omega_2^{(2)}<\omega_2<\omega_2^{(3)}$
4	1.288 97	II	S: $\omega_2^{(3)}\le\omega_2\le\omega_2^{(4)}$
5	1.332 28	II	I: $\omega_2^{(4)}<\omega_2<\omega_2^{(5)}$
	(c) $A=1$, $\omega_1=1$, $\omega_2=\omega_3=\sqrt{3}$		
ρ	Transition point C_ρ	Type of transition	Region
1	2.746 33	III	S: $-C_1\le C\le C_1$
			I: $\lvert C\rvert>C_1$

$$\mu = \text{const} \times |q - q_p|^{\beta} , \quad (31)$$

where $\beta = \frac{1}{2}$, as our numerical result verified.

IV. A TWO-DIMENSIONAL HAMILTONIAN SYSTEM: HÉNON-HEILES SYSTEM WITH VARIABLE COUPLING

The Hénon-Heiles system[16] has occupied a position of historical importance in the studies of order-chaos transitions in coupled Hamiltonian systems. It is well known that when the energy of the Hénon-Heiles system is increased the system undergoes an order-chaos transition at some point. We shall show that there is an infinite number of stability-instability transitions in the Hénon-Heiles system as a function of coupling and energy.[17]

The Hamiltonian of the generalized Hénon-Heiles system is given by Eq. (2) with $m_1 = m_2 = 1$ and the nonlinear potential

$$V(x,y) = \tfrac{1}{2}(x^2 + y^2) + Cx^2 y - \frac{y^3}{3} , \quad (32)$$

where C is the coupling parameter. The case $C=1$ reduces to the usual Hénon-Heiles system. The case $C=0$ clearly decouples the system into two independent oscillators. By making a 45° coordinate transformation $x' = (x+y)/\sqrt{2}$, $y' = (x-y)/\sqrt{2}$ one can show that the case $C = -1$ also decouples the system. Thus the system (32) is integrable and hence is always regular when $C=0$ and $C=-1$, no matter what the initial energy of the system is.

Assuming initially $x(0) = \dot{x}(0) = 0$ we then have $x(t) = 0$ for all t and

$$\ddot{y} + y - y^2 = 0 . \quad (33)$$

The small perturbations Δx and Δy from these initial values of x and y will now evolve according to

$$d^2(\Delta x)/dt^2 + (1 + 2Cy)(\Delta x) = 0 \quad (34a)$$

and

$$d^2(\Delta y)/dt^2 + (1 - 2y)(\Delta y) = 0 . \quad (34b)$$

The evolution of Δx and Δy may be oscillatory with time in which case the behavior of the system is stable. If, on the other hand, one or both of these perturbations grow exponentially with time, then the behavior of the system is very sensitive to any small perturbations of the initial condition (for the given value of C) which is characteristic of a locally chaotic but not necessarily globally chaotic behavior.

Assuming $\dot{y}(0) = 0$, there are three possible values of $y(0)$ for a given value of initial energy E and they are given, from Eq. (2) with the nonlinear potential given by Eq. (32), by the roots of the cubic equation $[y(0)]^2/2 - [y(0)]^3/3 = E$. The discriminant of this cubic equation is $3E(6E-1)/8$. Thus for $0 < E < \frac{1}{6}$, the equation gives three real roots for $y(0)$ which, when arranged in descending order of magnitude, will be called a,b,c, i.e., $a > b > c$. For $E > \frac{1}{6}$, the equation gives one real root and two complex roots. For $E=0$, and $E = \frac{1}{6}$, two of the three real roots are equal, and the equal roots can be verified to be $b=c$ and $a=b$, respectively.

For the case $E > \frac{1}{6}$, the real roots a and b are replaced by a complex-conjugate pair, and the root c becomes less than -0.5. The initial condition $y(0) = c$ with $c < -0.5$, will result in an unbounded solution for $y(t)$. The two cases $[1 < y(0) \le 1.5,\ 0 < E \le \frac{1}{6}$; and $y(0) < -0.5,\ E > \frac{1}{6}]$ for which the motion is unbounded from the start will not be discussed in this paper.

Let us consider the initially bounded solution for y, i.e., the case $0 < E < \frac{1}{6}$ and $-0.5 < y(0) < 1$, and analyze the behavior of Δx and Δy. The solution of Eq. (33) is given by

$$y(\bar{t}) = c + (a-c)k^2 \mathrm{sn}^2(\bar{t}, k^2) , \quad (35)$$

where $\bar{t} = \Omega t$, $\Omega = \sqrt{(a-c)/6}$, $k^2 = (b-c)/(a-c)$, and $K(k)$ is the complete elliptic integral of the first kind. That is, $y(\bar{t})$ is a periodic function of $\bar{t}$ with period $2K(k)$. Substituting Eq. (35) into Eqs. (34), we have

$$d^2(\Delta x)/d\bar{t}^2 = p_1(\bar{t})\Delta x \quad (36a)$$

and

$$d^2(\Delta y)/d\bar{t}^2 = p_2(\bar{t})\Delta y , \quad (36b)$$

where

$$p_1(\bar{t}) = -12Ck^2 \mathrm{sn}^2 \bar{t} - G , \quad (37a)$$

$$p_2(\bar{t}) = 12k^2 \mathrm{sn}^2 \bar{t} - H , \quad (37b)$$

$$G = -4C(1+k^2) + 4(1+C)(1-k^2+k^4)^{1/2} , \quad (38a)$$

and

$$H = 4(1+k^2) . \quad (38b)$$

Equations (36) are of the form of the Lamé equation[18]

$$d^2u/dt^2 = p(t)u ,$$

where

$$p(t) = \nu(\nu+1)k^2 \mathrm{sn}^2 t - h ,$$

with a real period $2K(k)$. The occurrence of a Lamé equation in the stability analysis has also appeared in the work of Churchill, Pecelli, and Rod,[19] Pecelli and Thomas,[20] and of the present authors.[7]

Equation (36) can be written in the form of Eq. (7) with $\mathbf{w} = \mathrm{col}(\Delta x, \Delta\dot{x})$ or $\mathbf{w} = \mathrm{col}(\Delta y, \Delta\dot{y})$ and $\underline{M}(t)$ a 2×2 matrix. If $f(\bar{t})$ and $g(\bar{t})$ are two fundamental solutions of Eq. (36a) or (36b) such that $f(0)=1$, $\dot{f}(0)=0$, $g(0)=0$, $\dot{g}(0)=1$, then the α in the characteristic equation (15), $s^2 - \alpha s + 1 = 0$, is given by (see Appendix B)

$$\alpha = \mathrm{tr}\underline{P} = f(\tau) + \dot{g}(\tau) = w_{11}(\tau) + w_{22}(\tau) ,$$

where $w_{jk}(t)$ are components of the fundamental solutions used in Eq. (8) and $\tau = 2K(k)$ in this case.

Equation (36b) is independent of C, and its α value is equal to 2. This is because $H = 4(1+k^2)$ is a characteristic value of the Lamé equation of order $\nu=3$ for any value of $0 < k^2 < 1$. The solution for Δy is the Lamé polynomial[21] $Es_3^2(\bar{t}, k^2)$ having a period $2K(k)$ and thus it is

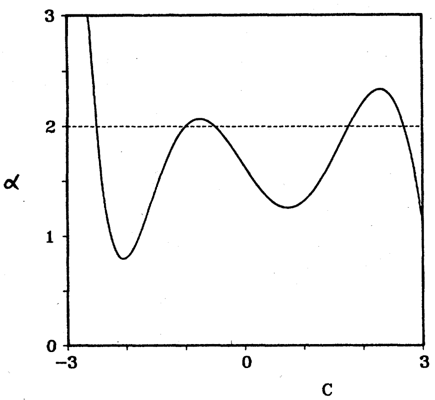

FIG. 3. The parameter α plotted as a function of the coupling for the case in Sec. IV with the initial condition $x(0)=\dot{x}(0)=\dot{y}(0)=0$, $y(0)=0.5$, and hence $E=\frac{1}{12}=0.083\,333$ and $k^2=0.5$, $\Delta x(0),\Delta y(0)$ small, $\Delta\dot{x}(0)=\Delta\dot{y}(0)=0$. The intersection points of this curve with the straight dashed lines $\alpha=2$ and $\alpha=-2$ give the stability-instability transition points.

always stable.

The α values for Eq. (36a) plotted as functions of C for the case $k^2=0.5$ $(E=\frac{1}{12})$ are shown in Fig. 3. The stable regions are those characterized by $|\alpha|\le 2$ and the unstable regions are those characterized by $|\alpha|>2$. A number of stability-instability transition points for C are presented in Table II. The intervals of α values $(2,-2)$ or $(-2,2)$ generally give the stable regions and the intervals $(2,2)$ or $(-2,-2)$ generally give the unstable regions. There are three exceptions to this rule in Table II: the regions $-2.5\le C\le -1$ and $-0.5\le C\le 1.76$ for $E=0.083\,333$ are stable, as can be seen from Fig. 3, and the region $-0.500\le C\le 0.700$ for $E=0.163\,59$ is stable. It is seen that as a function of the coupling parameter C, the generalized Hénon-Heiles system for the initial condition $x(0)=\dot{x}(0)=\dot{y}(0)=0$, $-0.5<y(0)<1$ $(0<E<\frac{1}{6})$ exhibits an infinite number of stable and unstable regions. The widths of successive stable regions are seen to become narrower as $|C|$ increases. For most practical purposes, the stable regions become points on the C axis as $|C|$ increases beyond 10, and except for these points, all values of $|C|>10$ give rise to unstable motion.

For $C=0$, Eq. (36a) clearly always gives a periodic solution of the form $\Delta x=A\cos(\omega\bar{t}+\phi)$ having a period $2\pi/\omega=\pi/(1-k^2+k^4)^{1/4}$. There are three other special values of C, $C=-\frac{1}{2},-1,-\frac{5}{2}$ for which the solutions for Δx are Lamé polynomials (i.e., characteristic functions of integral order) having periods $2K(k)$. The values of G corresponding to these values of C are the characteristic values[21] a_2^2, b_3^2, and b_5^2 with the corresponding Lamé polynomials $Ec_2^2(\bar{t},k^2)$, $Es_3^2(\bar{t},k^2)$, and $Es_5^2(\bar{t},k^2)$, respectively, for the solution for Δx. The α value for these cases is equal to 2 for any value of k.

Generally, for negative values of $C=-|C|$, we can write Eq. (36a) in the form of the Lamé equation by writing $|C|=\nu(\nu+1)/2$ or $\nu=[(1+8|C|)^{1/2}-1]/2$. G given by Eq. (38a) is generally not a characteristic value of Lamé equation of integral order. However, periodic functions of periods $2K$ or $4K$ (i.e., characteristic functions of the Lamé equation) exist for nonintegral values of ν and for certain values of G. The characteristic values and functions can be determined using the transcendental equations given by Ince.[21] The results agree with those listed in Table II which have been obtained using the α value.

From Table II, we note that the Hénon-Heiles system $(C=1)$ is stable for the case $E=0.083\,333$, but is unstable for the case $E=0.163\,59$. For the Hénon-Heiles system, the α-value curve as a function of $y(0)$ in the range $-0.5<y(0)<1$ is shown in Fig. 4, where the corresponding values of E are also given. We have specified the value of $y(0)$ in addition to the value of E to emphasize the point that the range $1<y(0)\le 1.5$, for which even

TABLE II. The stability-instability transition points for the generalized Hénon-Heiles system, Eq. (32).

$E=0.083\,333$ C	α	$E=0.163\,59$ C	α
−34.785	2	−13.069	−2
−34.784	−2	−13.068	2
−16.035	−2	−2.500	2
−16.025	2	−2.499	−2
−2.500	2	−1.034	−2
−1.000	2	−1.000	2
−0.500	2	−0.500	2
1.760	2	0.700	2
2.692	2	1.475	2
3.515	−2	1.873	−2
9.328	−2	3.238	−2
9.369	2	3.337	2
17.062	2	5.055	2
17.069	−2	5.086	−2
26.643	−2	7.121	−2
26.644	2	7.137	2

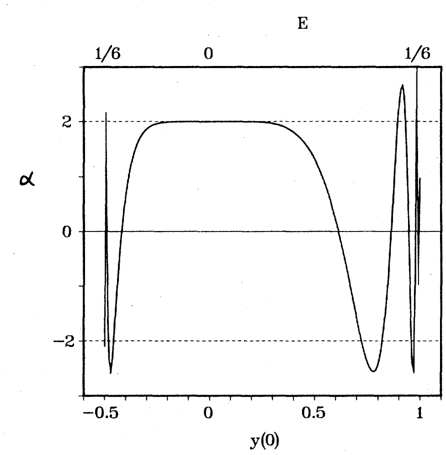

FIG. 4. The parameter α plotted as a function of $y(0)$ or E for the Hénon-Heiles system $C=1$. The initial condition is $x(0)=\dot{x}(0)=0$, $\Delta x(0),\Delta y(0)$ small, and $\Delta\dot{x}(0)=\Delta\dot{y}(0)=0$. The intersection points of this curve with the straight dashed lines $\alpha=2$ and $\alpha=-2$ give the stability-instability transition points.

though $0 \le E < \frac{1}{6}$, is excluded. It is seen that as E increases from zero, the region $0 \le E \le 0.1352$ is stable, but the system clearly displays several unstable regions before E reaches the value $\frac{1}{6}$. Two of these unstable regions are $0.1352 < E < 0.1525$ and $0.1615 < E < 0.1645$.

As we have discussed, our analysis is based on a special form of initial conditions $[x(0)=\dot{x}(0)=0]$ which decouples x and y and allows us to solve $x(t)$ and $y(t)$ separately. It is expected that if we give a small but finite value to x initially, the stability-instability transition points will not be far from the prediction with $x(0) \to 0$. This is indeed what we have found numerically.

In the two-dimensional Hamiltonian system, stability-instability transitions of types I and III only occur. As the coupling parameter C approaches one of the stability-instability transition points C_n or as E approaches one of the stability-instability transition point E_n from the unstable region, $|\alpha|$ approaches 2 from above, and that μ approaches zero from a positive value as

$$\mu = \text{const} \times |C - C_n|^{\beta} \quad \text{as } C \to C_n \tag{39}$$

and

$$\mu = \text{const} \times |E - E_n|^{\beta} \quad \text{as } E \to E_n , \tag{40}$$

where the constants in Eqs. (39) and (40) are positive and $\beta = \frac{1}{2}$.

V. SUMMARY

In summary, for a class of periodic motions in a general Hamiltonian system of any dimensions, we have shown analytically that there is a universal critical exponent $\beta = \frac{1}{2}$ associated with the behavior of the largest Lyapunov exponent as any critical parameter of the system is approached from the unstable region. We have numerically verified this analytic result for a number of Hamiltonian systems and we have presented many of the interesting features of a three-dimensional and a two-dimensional Hamiltonian system. There are reasons to believe that the universality of $\beta = \frac{1}{2}$ is probably true for a wider class of motions and systems than the ones we have discussed.

ACKNOWLEDGMENTS

This research is supported in part by the U.S. Department of Energy (Division of Chemical Sciences), under Grant No. DE-FG02-84ER13243.

APPENDIX A

In this appendix, we derive the general solution, Eq. (10), of Eq. (7). Let $\mathbf{w}(t)$ be a solution of Eq. (7) for an arbitrary initial condition. Then $\mathbf{w}(t)$ can always be expressed as a linear combination of the fundamental solutions:

$$\mathbf{w}(t) = \sum_k c_k \mathbf{W}_k(t) \tag{A1}$$

or

$$\mathbf{w}(t+\tau) = \sum_k c_k \mathbf{W}_k(t+\tau) . \tag{A2}$$

Writing out the jth component of $\mathbf{w}$ in Eqs. (A1) and (A2), we get

$$w_j(t) = \sum_k c_k w_{jk}(t) \tag{A3}$$

and

$$\begin{aligned} w_j(t+\tau) &= \sum_k c_k w_{jk}(t+\tau) \\ &= \sum_k c_k \sum_l w_{jl}(t) p_{lk} \\ &= \sum_l \left[\sum_k c_k p_{lk} \right] w_{jl}(t) , \end{aligned} \tag{A4}$$

where we have made use of Eq. (8). From Eqs. (A3) and (A4), a constant s can be found such that

$$\sum_k c_k p_{lk} = s c_l , \quad l = 1, 2, \ldots, 2n . \tag{A5}$$

Equation (A5) would give a nontrivial solution for s if

$$\begin{vmatrix} p_{11}-s & p_{12} & \cdots & p_{1,2n} \\ p_{21} & p_{22}-s & \cdots & p_{2,2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{2n,1} & p_{2n,2} & \cdots & p_{2n,2n}-s \end{vmatrix} = 0 . \tag{A6}$$

Let s_j, $j = 1, 2, \ldots, 2n$, be the characteristic values of $\underline{P}$ from Eq. (A6), and let us set

$$s_j = \exp(\mu_j \tau) , \tag{A7}$$

or

$$\mu_j = \tau^{-1} \ln s_j , \tag{A8}$$

and let

$$\boldsymbol{\psi}(t) = \exp(-\mu_j t) \mathbf{w}(t) . \tag{A9}$$

Then

$$\begin{aligned} \boldsymbol{\psi}(t+\tau) &= \exp[-\mu_j (t+\tau)] \mathbf{w}(t+\tau) \\ &= \exp[-\mu_j (t+\tau)] s \mathbf{w}(t) \\ &= \exp[-\mu_j (t+\tau)] \exp(\mu_j \tau) \mathbf{w}(t) \\ &= \exp(-\mu_j t) \mathbf{w}(t) \\ &= \boldsymbol{\psi}(t) . \end{aligned} \tag{A10}$$

Hence $\boldsymbol{\psi}(t)$ is periodic in t with a period τ. Since $\mathbf{w}(t)$ is a solution of the differential equation (7), $\exp(\mu_j t)\boldsymbol{\psi}(t)$ is a solution. The general solution of Eq. (7) is thus given by Eq. (10).

APPENDIX B

In this appendix, we shall briefly outline the steps leading to Eqs. (13) and (14) from Eq. (12). We begin with the matrix $\underline{W}(t)$ representing the fundamental solutions of Eq. (7) and which satisfies the equation

$$d\underline{W}/dt=\underline{M}(t)\underline{W} \ . \tag{B1}$$

Multiplying Eq. (B1) from the left by $\underline{E}$ and from the right by $\underline{E}^{-1}$, and denoting by $\underline{V}(t)$ the matrix $\underline{E}\,\underline{W}(t)\underline{E}^{-1}$, we get

$$d\underline{V}/d(-t)=\underline{M}(-t)\underline{V} \ , \tag{B2}$$

where we have made use of Eq. (12). Since the matrix $\underline{P}$ is the nonsingular matrix which relates $\underline{W}(t+\tau)$ and $\underline{W}(t)$ by

$$\underline{W}(t+\tau)=\underline{W}(t)\underline{P} \ , \tag{B3}$$

it then follows from Eq. (B2) that we have

$$\underline{V}(-t-\tau)=\underline{V}(-t)\underline{P} \ . \tag{B4}$$

Multiplying Eq. (B4) from the left by $\underline{E}^{-1}$ and from the right by $\underline{E}$, we get

$$\underline{W}(-t-\tau)=\underline{W}(-t)\underline{E}^{-1}\underline{P}\,\underline{E} \ . \tag{B5}$$

Setting $t=-\tau$ in Eq. (B5), we get

$$\underline{W}(0)=\underline{W}(\tau)\underline{E}^{-1}\underline{P}\,\underline{E} \ . \tag{B6}$$

Comparing Eq. (B6) with the following equation obtained from setting $t=0$ in Eq. (B3),

$$\underline{W}(\tau)=\underline{W}(0)\underline{P} \ , \tag{B7}$$

clearly shows that $\underline{E}^{-1}\underline{P}\,\underline{E}$ corresponds to the inverse of $\underline{P}$, namely, Eq. (13).

If s_j is a characteristic value of $\underline{P}$ with characteristic vector $\mathbf{v}_j$, or

$$\underline{P}\mathbf{v}_j=s_j\mathbf{v}_j \ , \tag{B8}$$

then it follows from Eq. (13) that

$$\underline{P}\,\underline{E}\mathbf{v}_j=\underline{E}\,\underline{P}^{-1}\mathbf{v}_j \ , \tag{B9}$$

or

$$\underline{P}(\underline{E}\mathbf{v}_j)=s_j^{-1}(\underline{E}\mathbf{v}_j) \ , \tag{B10}$$

which shows that s_j^{-1} is a characteristic value of $\underline{P}$ with characteristic vector $\underline{E}\mathbf{v}_j$.

If $\underline{X}(t)$ is a matrix solution of Eq. (7), one can readily derive, from the formula for the derivative of a determinant, the following basic result:

$$\frac{d}{dt}\,|\,\underline{X}(t)\,|=|\,\underline{X}(t)\,|\,\mathrm{tr}\underline{M}(t) \ , \tag{B11}$$

and thus

$$|\,\underline{X}(t)\,|=|\,\underline{X}(t_0)\,|\exp\left[\int_{t_0}^{t}\mathrm{tr}\underline{M}(t')dt'\right] \ . \tag{B12}$$

Since the matrix $\underline{M}(t)$ for our Hamiltonian has the property that its (k,j) element is zero whenever $k+j$ is equal to an even number, we have $\mathrm{tr}\underline{M}(t)=0$, and hence

$$|\,\underline{X}(t)\,|=|\,\underline{X}(t_0)\,| \ . \tag{B13}$$

Replacing $\underline{X}$ by the fundamental solution $\underline{W}$, we find

$$|\,\underline{P}\,|=|\,\underline{W}(\tau)\,|=|\,\underline{W}(0)\,|=1 \tag{B14}$$

which is an important property of the matrix $\underline{P}$.

Let $\underline{P}^{(k)}$ denote the kth compound[22] of $\underline{P}$. The elements of $\underline{P}^{(k)}$ are minors of $|\,\underline{P}\,|$ of order k which come from the same group of k rows (or columns) of $\underline{P}$ placed in lexical order. The order of $\underline{P}^{(k)}$ is $\binom{2n}{k}\times\binom{2n}{k}$, where $\binom{2n}{k}=(2n)!/[k!(2n-k)!]$. Applying the Binet-Cauchy theorem[22] to Eq. (13), we have

$$\begin{aligned}(\underline{P}^{-1})^{(k)}&=(\underline{E}^{-1})^{(k)}\underline{P}^{(k)}\underline{E}^{(k)}\\&=(\underline{E}^{(k)})^{-1}\underline{P}^{(k)}\underline{E}^{(k)}\end{aligned} \tag{B15}$$

from which it follows that

$$\mathrm{tr}(\underline{P}^{-1})^{(k)}=\mathrm{tr}\underline{P}^{(k)} \ . \tag{B16}$$

Let $\mathrm{adj}^{(k)}\underline{P}$ denote the kth adjugate compound of $\underline{P}$. The matrix $\mathrm{adj}^{(k)}\underline{P}$ is obtained by replacing every element in $\underline{P}^{(k)}$ by its cofactor in $|\,\underline{P}\,|$ and transposing the resulting matrix. Clearly we have

$$\mathrm{tr}\,\mathrm{adj}^{(k)}\underline{P}=\mathrm{tr}\underline{P}^{(2n-k)} \ . \tag{B17}$$

Also, the Cauchy theorem gives

$$\underline{P}^{(k)}\mathrm{adj}^{(k)}\underline{P}=|\,\underline{P}\,|\,\underline{I} \ , \tag{B18}$$

where $\underline{I}$ denotes a unit matrix. Since $|\,\underline{P}\,|=1$, we h--

$$(\underline{P}^{(k)})^{-1}=\mathrm{adj}^{(k)}\underline{P} \ . \tag{B19}$$

From Eqs. (B16) and (B19), we get

$$\mathrm{tr}\underline{P}^{(k)}=\mathrm{tr}(\mathrm{adj}^{(k)}\underline{P}) \ . \tag{B20}$$

Now generally we have

$$\begin{aligned}|\,\underline{P}-s\underline{I}\,|&=s^{2n}-\mathrm{tr}\underline{P}^{(1)}s^{2n-1}+\mathrm{tr}\underline{P}^{(2)}s^{2n-2}\\&\quad-\cdots+\mathrm{tr}\underline{P}^{(2n-2)}s^2-\mathrm{tr}\underline{P}^{(2n-1)}s+|\,\underline{P}\,| \ .\end{aligned} \tag{B21}$$

Using Eqs. (B17) and (B20), we have

$$\mathrm{tr}\underline{P}^{(k)}=\mathrm{tr}\underline{P}^{(2n-k)} \ , \tag{B22}$$

or writing $a_k=(-1)^k\mathrm{tr}\underline{P}^{(k)}$, we have shown that the characteristic value equation of the matrix $\underline{P}$, Eq. (14), is reciprocal, i.e., $a_1=a_{2n-1}$, $a_2=a_{2n-2}$, etc.

APPENDIX C

In this appendix, we show that $\alpha_1,\alpha_2,\ldots,\alpha_n$ of Eq. (15) can be expressed as roots of an nth-degree algebraic equation whose coefficients can be determined recursively from the coefficients $a_1,a_2,\ldots,a_{2n-1}$ of Eq. (14).

Put $s+s^{-1}=\alpha$, so that $\alpha^2=s^2+2+s^{-2}$, $\alpha^3=s^3+3s+3s^{-1}+s^{-3}$, and so on. Then

$$s+s^{-1}=\alpha,\ \ s^2+s^{-2}=\alpha^2-2,\ \ s^3+s^{-3}=\alpha^3-3\alpha \ , \tag{C1}$$

and so on. By this iteration we express

$$s^r+s^{-r}\ \text{ as }\ \alpha^r-r\alpha^{r-2}+\cdots \ , \tag{C2}$$

that is, as a polynomial in α of degree r, for each value of r in succession. Divide Eq. (14) throughout by s^n, so that it can be rearranged as

$$(s^n+s^{-n})+a_1(s^{n-1}+s^{-(n-1)})+\cdots+a_n=0 . \tag{C3}$$

Substituting for each expression in parentheses its equivalent as a polynomial in α, we obtain an equation in α of degree n whose roots $\alpha_1,\alpha_2,\ldots,\alpha_n$ are related to the roots of Eq. (14) by $s+s^{-1}=\alpha_j$, or $s^2-\alpha_j s+1=0$, $j=1,2,\ldots,n$. In other words, Eq. (14) may be replaced by Eq. (15).

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