Entropy production and plasma relaxation

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(Received 23 May 1986)

Plasma relaxation, as well as the relaxation of more general diffusive systems, is discussed from a dynamical point of view. It is shown that a state which minimizes the rate of entropy production, as suitably defined, attracts neighboring states and thus causes relaxation. For plasmas, the entropy principle is compared with Taylor's hypothesis of energy minimization and is shown to explain observed experimental results. An iteration scheme based on the entropy principle is constructed, and enables numerical calculation of the relaxed state. The scheme arises more naturally than the "local potential" method of Glansdorff and Prigogine which is improved upon.

I. INTRODUCTION

The tendency of a plasma spontaneously to approach some preferred steady state is usually referred to as plasma relaxation. This phenomenon was observed in laboratory systems[1] as well as in astrophysical plasmas,[2] and is now attracting considerable interest. The subject was recently reviewed by Taylor[3] who also provided the most popular explanation for this behavior. Taylor suggested[4] that through a rapid process of magnetic reconnection a slightly resistive plasma tends to relax to a state of minimum magnetic energy subject to the conservation of magnetic helicity, this quantity being defined as the volume integral of $\mathbf{A}\cdot\mathbf{B}$, where $\mathbf{B}$ is the magnetic field and $\mathbf{A}$ is its vector potential, $\mathbf{B}=\nabla\times\mathbf{A}$. (An important property of the helicity is that it is a constant of the motion for nonresistive plasmas.) The minimum energy state is force-free with $\nabla\times\mathbf{B}=\lambda_0\mathbf{B}$, where λ_0 is a constant. Direct observations in some laboratory plasmas indeed confirm[3] the prevalence of magnetic configurations close to this state, which is now known as the "Taylor state." Likewise, some numerical simulations[5,6] of fluid plasmas in laboratory-relevant configurations show a similar tendency.

While the Taylor hypothesis turns out to be remarkably successful in predicting the relaxed state, the understanding it provides is certainly incomplete. The basic deficiency of the theory is that it lacks a dynamical justification. Even if one agrees that the magnetic helicity changes more slowly than the magnetic energy,[7] it is possible to envision, in addition to the helicity, other constants of the motion of an ideal plasma which should also remain almost constant in time in the slightly resistive case.[8-10] Another puzzling question is[10,11] why the magnetic energy is the quantity to be minimized, when it is known that isolated systems in thermodynamic equilibrium are states of maximum entropy. Any consideration such as one of those just mentioned modifies the Taylor state in a way that allows an even closer agreement with experiment.[8-11]

Significant progress was made recently in understanding the dynamical behavior of weakly turbulent plasmas, that is, laminar plasmas affected by small fluctuations which vary on a faster time scale and a shorter length scale than the mean (average) fields. The effect of the fluctuations was found to give rise to an additional anomalous diffusion of the mean magnetic field.[12-16] Essentially the same effect can be seen either from the point of view of the kinetic description of electrons in a turbulent magnetic field environment,[12] from an empirical point of view following the consequences of Taylor's arguments,[13] and by following the dynamics of various plasma modes.[14-16] A detailed description of this effect, and the relaxation it causes, will be given in Sec. II. The main purpose of the present paper is to explore the relaxation of mean fields in a plasma. Indeed, it will be shown that such a relaxation expresses the tendency of the plasma to approach a state of minimum rate of entropy production, rather than a state of minimum energy,[4] which can nevertheless be shown to be similar to a Taylor state if the turbulent dissipation is strong enough. The relation between the two principles is discussed in Sec. III.

When investigating the principle of relaxation to a state of minimum rate of entropy production, it is worthwhile to note that such a principle is not valid for all systems. Indeed, Prigogine's original work on the thermodynamics of irreversible processes[17] found that such a principle holds for finite dimensional systems in the linear regime near the relaxed state and when the Onsager coefficients relating diffusion forces to fluxes form a symmetric matrix and can be taken as constant. In the nonlinear regime, and when the aforementioned conditions are violated, it was proposed[18] to use a so-called "local potential," the minimum of which should govern plasma relaxation. Many authors[19] later found that the local potential idea was at most a means for generating a useful iteration scheme for the solution of a steady state, much like Galerkin's method, but with no dynamical meaning. In particular, one could not assert that a system will be driven dynamically to relax to the steady state obtained by the local potential method. In this work we develop an alternative description of plasma relaxation, one with a dynamical content. Indeed, it will be shown in Sec. II that a state minimizing an appropriate entropy production functional is a steady state of the system and, moreover, it attracts neighboring states and brings about their relaxation.

The relaxation of more general systems is considered in Sec. IV, where, again, a steady state is shown to be reached dynamically as a result of an entropy principle. This result too is only "local," in the sense that the minimum-entropy-production state is only shown to attract states sufficiently close to it. We also construct an iteration scheme based on our entropy-production principle, and show that it has good convergence properties to the desired state. The scheme's relation to the local potential method is then discussed. Finally, in Sec. V we work out a special example and apply our iteration procedure to the nonlinear heat equation.

II. RELAXATION OF MEAN FIELDS IN A PLASMA

The magnetic field $\mathbf{B}$ evolves according to Faraday's law $\partial\mathbf{B}/\partial t+\nabla\times\mathbf{E}=\mathbf{0}$, where the electric field $\mathbf{E}$ is determined by Ohm's law for a resistive plasma

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J}\ , \tag{1}$$

with $\mathbf{J}$ being the current density, $\mathbf{J}=\nabla\times\mathbf{B}$, η is the electrical resistivity, and $\mathbf{v}$ is the plasma velocity. In this work we consider η to be a prescribed function of space. The plasma is assumed to be described by a mean state on which are superimposed small fluctuations of zero mean but varying on a faster time scale and a shorter length scale. Denoting mean quantities by subscript 0 and fluctuating quantities by subscript 1, we have $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$, $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1$, and the mean Faraday's law reads

$$\frac{\partial\mathbf{B}_0}{\partial t}+\nabla\times(\eta\mathbf{J}_0-\mathbf{v}_0\times\mathbf{B}_0-\mathscr{E})=\mathbf{0}\ , \tag{2}$$

where

$$\mathscr{E}=(\mathbf{v}_1\times\mathbf{B}_1)_0\ . \tag{3}$$

$\mathscr{E}$ is the mean of the quantity in parentheses, and the mean may be viewed as either an average over an ensemble or as a space-time average over the scales of the fluctuations. Equation (2) is also used in the magnetic dynamo problem[20] where, however, progress has been limited because of the fact that an expression for $\mathscr{E}$ is not known. In the case of confined plasmas, fortunately, further progress was made possible by the derivation of some properties of $\mathscr{E}$ and, following that, by the determination of the functional form of $\mathscr{E}(\mathbf{B}_0)$ in some cases.[15,16] Two important integral properties of $\mathscr{E}$ which hold when the fluctuation level is low enough and the resistivity η is small enough, are

$$\int\mathscr{E}\cdot\mathbf{J}_0d\tau=-\int\eta(\mathbf{J}_1^2)_0d\tau,\quad \int\mathscr{E}\cdot\mathbf{B}_0d\tau=0, \tag{4}$$

where both equalities are correct to order η, and the integrals are taken over the plasma volume. These properties suggest[13,15] that in a low pressure plasma, with $\mathbf{J}_0$ nearly parallel to $\mathbf{B}_0$, $\mathscr{E}$ satisfies

$$\mathscr{E}\cdot\mathbf{B}_0=\nabla\cdot(K^2\nabla\lambda),\quad \lambda\equiv\frac{\mathbf{J}_0\cdot\mathbf{B}_0}{\mathbf{B}_0^2} \tag{5}$$

where K^2 is some positive function. Indeed, a detailed expression for K^2 was derived from the dynamical equations of certain plasma modes.[14–16] A typical functional form of $\mathscr{E}$ was derived for a cylindrical plasma of radius a with all mean quantities depending on the radius r only.[15,16] A simplified version of this form, after excluding the radial component which does not enter Eq. (2), is

$$\mathscr{E}=\alpha\frac{1}{rB_0^2}(\eta a^2rB_0^2\lambda')'\mathbf{B}_0+\beta\frac{\eta p_0'}{B_0^2}\hat{\mathbf{r}}\times\mathbf{B}_0\ . \tag{6}$$

$\mathscr{E}$ deviates from (6) near $r=a$ so that it vanishes at the boundary, as follows from (3), when we require that $\mathbf{v}$ and $\mathbf{B}$ be tangential to the boundary. In Eq. (6) $B_0=|\mathbf{B}_0|$, primes denote d/dr, α and β are some positive constants, and p is the plasma pressure. p_0 is determined from $\mathbf{B}_0$ via the equilibrium relation

$$\nabla p_0=\mathbf{J}_0\times\mathbf{B}_0 \tag{7}$$

with the additional condition $p_0=0$ on the plasma boundary which is a magnetic flux surface. We note that Eq. (7) is correct only up to order 1 and that it implies that, even though plasma profiles diffuse in time, the diffusion takes place through a sequence of equilibrium states. Indeed, $\mathbf{v}_0$ in Eq. (2) is determined from the requirement that Eq. (7) be satisfied for all time, as in the diffusion theory of Grad and Hogan.[21]

Before proceeding with the question of relaxation we point out that if $\mathscr{E}$ is linear in η, and vanishes as $\eta\to0$, then $\mathscr{E}$ must be a homogeneous function of degree one in $\mathbf{B}_0$. That is, replacing $\mathbf{B}_0$ by $c\mathbf{B}_0$ with some constant c causes a change from $\mathscr{E}_0$ to $c\mathscr{E}_0$. This is of course true in the special case of Eq. (6) and results from the fact that the full fluid equations are invariant under the transformation $\mathbf{B}\to c\mathbf{B}$, $\mathbf{v}\to c\mathbf{v}$, $p\to c^2p$, $t\to t/c$, $\mathbf{x}\to\mathbf{x}$, and $\eta\to c\eta$. From its definition $\mathscr{E}\to c^2\mathscr{E}$, and our assertion is verified when the assumed dependence of $\mathscr{E}$ on η is taken into account. In the following we assume $\mathscr{E}$ to have this property.

We now start our investigation of the relationship between relaxation and entropy production, by investigating a simple example. Consider a solid toroidal conductor, subject to an applied electric field. For simplicity we consider the torus to be a straight periodic cylinder along the z axis, and the external electric field to be applied in the z direction only. The applied field enters the problem via the boundary conditions

$$\oint_T\mathbf{E}\cdot d\mathbf{l}=V,\quad \oint_P\mathbf{E}\cdot d\mathbf{l}=0\ , \tag{8}$$

where T is any simple periodic boundary curve in the axial direction of the conductor, and V is the applied axial voltage per unit length. P is any closed boundary curve the short way around the cylinder. A fuller discussion of the boundary conditions for this problem is given in Appendix A, where it is seen that we can impose the additional requirements

$$\mathbf{B}\cdot\hat{\mathbf{n}}=0,\quad \mathbf{J}\cdot\hat{\mathbf{n}}=0 \tag{9}$$

on the boundary, with $\hat{\mathbf{n}}$ being a normal vector. We also specify the total axial magnetic flux Φ. The magnetic field evolves according to

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times\mathbf{E}=\mathbf{0},\quad \mathbf{E}=\eta\mathbf{J} \tag{10}$$

subject to the boundary conditions (8), (9), and a given Φ. Multiplying Eq. (10) by $\partial\mathbf{B}/\partial t$ and integrating, we have

$$\frac{\partial}{\partial t}\int\left(\tfrac{1}{2}\eta\mathbf{J}^2-\mathbf{E}_a\cdot\mathbf{J}\right)d\tau=-\int\left[\frac{\partial\mathbf{B}}{\partial t}\right]^2 d\tau\ , \tag{11}$$

where $\mathbf{E}_a\equiv V\hat{\mathbf{z}}$ is the applied electric field, and this term enters through the boundary conditions as in Appendix A. Equation (11) implies that the quantity $\int d\tau(\eta\mathbf{J}^2/2-\mathbf{E}_a\cdot\mathbf{J})$ decreases monotonically during the evolution of the system and ceases to decrease only when $\partial\mathbf{B}/\partial t=\mathbf{0}$, that is, when steady state is reached. We refer to this integral as the entropy production integral. It should be noted that a similar result was derived by Prigogine for some simple nondriven systems.[17] It is suggestive that the entropy production integral continues to decrease until it reaches its minimum with respect to all possible "admissible states." Indeed, solving the variational problem

$$\text{Minimize}\ \int\left(\tfrac{1}{2}\eta\mathbf{J}^2-\mathbf{E}_a\cdot\mathbf{J}\right)d\tau \tag{12}$$

subject to $\nabla\cdot\mathbf{J}=0$, $\mathbf{J}\cdot\hat{\mathbf{n}}=0$ on the boundary yields the steady-state solution $\eta\mathbf{J}=\mathbf{E}_a$. $\mathbf{B}$ is found by solving $\nabla\times\mathbf{B}=\mathbf{J}$, $\nabla\cdot\mathbf{B}=0$, with $\mathbf{B}\cdot\hat{\mathbf{n}}=0$ on the boundary and Φ specified. [Note that in the more general toroidal geometry the integral $\mathbf{E}_a\cdot\mathbf{J}$ is replaced by the toroidal current times the toroidal voltage, and the Euler equation for the minimization is $\nabla\times(\eta\mathbf{J})=\mathbf{0}$.] Moreover, the linearity of problem (10) and its equivalence with the usual heat equation implies that *every* initial state approaches the unique steady state exponentially fast in time. [The rate of exponential decay depends on the eigenvalues of problem (10) with homogeneous boundary conditions.] Finally, we mention that one may use the relation known to hold in a steady state

$$\int\eta\mathbf{J}^2 d\tau=\int\mathbf{E}_a\cdot\mathbf{J}\,d\tau\ , \tag{13}$$

and to impose it as an additional constraint on the variational problem (12). An additional constraint which is satisfied by the solution of the original problem does not, of course, change that solution. But using (13) we see that the entropy production integral has the negative value $\int -\eta\mathbf{J}^2/2\,d\tau$, so that (12) may be written as

$$\text{Maximize}\ \int\tfrac{1}{2}\eta\mathbf{J}^2 d\tau \tag{14}$$

subject to (13), with $\nabla\cdot\mathbf{J}=0$ and $\mathbf{J}\cdot\hat{\mathbf{n}}=0$ on the boundary. Expression (14) has the traditional meaning of entropy production, and the steady state is found by *maximizing* the rate of entropy production, subject to equality (13) which is a statement of conservation of energy in steady state. To conclude, we find in the case of the electrically driven solid conductor that the relaxation of the magnetic field is intimately related to some entropy production principle. A state $\mathbf{B}$ represents the unique relaxed state if and only if it minimizes the entropy production integral as defined in (12) or, equivalently, maximizes (14) subject to Eq. (13).

We now return to the case of the plasma. For simplicity we deal with an incompressible plasma. This assumption reduces the complexity of the problem by decoupling the temperature equation from the equation for $\mathbf{B}_0$, the only one to be followed, and is consistent with the consideration of η as a given function of space. (η should actually depend on the temperature.) For additional simplicity we deal with a plasma inside a cylindrical conductor as in Appendix A and, furthermore, we consider all mean profiles to depend on the radius r alone. The effect of this assumption is to cause the term $\mathbf{v}_0\times\mathbf{B}_0$ in Eq. (2) to drop out, since only the radial component of $\mathbf{v}_0$ enters (2), but div $\mathbf{v}_0=0$ implies that the radial component of $\mathbf{v}_0$ is zero. Likewise, $\mathbf{B}_0$ and $\mathbf{J}_0$ have zero radial components. (Appendix B describes the relaxation in a simple configuration where the flow does play a role.) The evolution of the mean magnetic field is determined by

$$\frac{\partial\mathbf{B}}{\partial t}+\nabla\times(\eta\mathbf{J}-\mathscr{E})=\mathbf{0}\ , \tag{15}$$

where from now on the subscript 0 is dropped. The term in parentheses, namely the electric field, satisfies Eq. (8) which here takes the form

$$\eta\mathbf{J}-\mathscr{E}=\mathbf{E}_a\quad\text{on}\quad r=a\ , \tag{16}$$

with $\mathbf{E}_a=V\hat{\mathbf{z}}$ as before, and $\mathscr{E}(\mathbf{B})$ is assumed to be a known vector-valued function, as in Eq. (6) for example. A finite plasma pressure may affect $\mathscr{E}$ but is again determined from $\mathbf{B}$ by Eq. (7). We note here that since the tangential component of $\mathscr{E}$ vanishes on the boundary, Eq. (16) actually involves only $\mathbf{J}$. This property of $\mathscr{E}$ becomes important when we integrate by parts and use $\int\mathbf{B}\cdot(\nabla\times\mathscr{E})d\tau=\int\mathbf{J}\cdot\mathscr{E}\,d\tau$, while replacing $\mathscr{E}$ with $\eta\mathbf{J}$ yields additional contributions from boundary terms.

We proceed now with the question of relaxation. Equation (15) is nonlinear in $\mathbf{B}$, and moreover, the precise functional form of $\mathscr{E}(\mathbf{B})$ is not known. We can, nevertheless, proceed formally by considering the state at the initial time to be sufficiently close to a steady state such that a linearized form of $\mathscr{E}(\mathbf{B})$ is valid. Let $\mathbf{B}^0$ be a steady state of (15), (16), and let $\delta\mathbf{B}=\mathbf{B}-\mathbf{B}^0$. We write

$$\nabla\times\mathscr{E}=\nabla\times\mathscr{E}^0+\underline{L}\,\delta\mathbf{B}+O(|\delta\mathbf{B}|^2)\ , \tag{17}$$

where superscript 0 denotes steady-state quantities and where $\underline{L}$ is a linear operator ($\nabla\times\mathscr{E}$ is assumed to have a Frechet derivative). We write $\underline{L}$ as $\underline{L}=\underline{S}+\underline{A}$, where $\underline{S}$ is a symmetric operator and $\underline{A}$ is antisymmetric, both defined with respect to the usual inner product of square integrable functions. Note that $\underline{L}$ depends on $\mathbf{B}^0$ but not on time. Multiplying (17) by $\partial\mathbf{B}/\partial t$, we have near steady state

$$\frac{\partial}{\partial t}\int d\tau\left(\tfrac{1}{2}\eta\mathbf{J}^2-\mathbf{E}_a\cdot\mathbf{J}-\mathscr{E}^0\cdot\mathbf{J}-\tfrac{1}{2}\underline{S}\delta\mathbf{B}\cdot\delta\mathbf{B}\right)$$
$$-\int d\tau\,\underline{A}\delta\mathbf{B}\cdot\frac{\partial\delta\mathbf{B}}{\partial t}=-\int d\tau\left[\frac{\partial\mathbf{B}}{\partial t}\right]^2+O(|\delta\mathbf{B}|^3)$$
$$\leq 0\,. \tag{18}$$

The second integral on the left-hand side vanishes at $\mathbf{B}=\mathbf{B}^0$ ($\delta\mathbf{B}=0$). Moreover, its time derivative also vanishes when $\mathbf{B}=\mathbf{B}^0$ because of the antisymmetry of $\underline{A}$. If we think of time derivatives as variations of a functional, the last observation implies that the second integral does not contribute to either the first or the second variations. Thus, we disregard this integral and view the first integral in (18) as an "entropy production integral," the minimization of which should yield the steady state $\mathbf{B}^0$. Moreover, the inequality in (18) suggests, as in the case of the solid conductor, that nearby states will be attracted, and relax, to $\mathbf{B}^0$. These points are to be explored now.

We first simplify the entropy production integral. Using the assumed first degree homogeneity of $\mathscr{E}$ in $\mathbf{B}$, we assert that $\nabla\times\mathscr{E}^0=\underline{L}\mathbf{B}^0$. This is seen by taking in Eq. (17) $\delta\mathbf{B}=\alpha\mathbf{B}^0$, with some small constant α, so that

$$\nabla\times\mathscr{E}(\mathbf{B})=(1+\alpha)\nabla\times\mathscr{E}(\mathbf{B}^0)=(1+\alpha)\nabla\times\mathscr{E}^0\,.$$

The result then follows immediately from Eq. (17) by comparing first-order terms in α. Using it in the entropy production integral and rewriting it while keeping only terms up to second order in $\delta\mathbf{B}$, the integral takes the form (correct to second order)

$$\mathscr{S}\equiv\int d\tau\left[\tfrac{1}{2}\eta\mathbf{J}^2-\tfrac{1}{2}\mathscr{E}\cdot\mathbf{J}-\mathbf{E}_a\cdot\mathbf{J}+\tfrac{1}{2}(\mathscr{E}\cdot\mathbf{J}^0-\mathscr{E}^0\cdot\mathbf{J})\right]\,. \tag{19}$$

In this form no longer do we need to know the details of $\underline{S}$ and $\underline{A}$, since only $\mathscr{E}$ appears in (19). Note that the dissipation due to fluctuations, $-\int d\tau\,\mathscr{E}\cdot\mathbf{J}$, appears on an equal footing with the dissipation due to the mean field. (This is a result of the first degree homogeneity of $\mathscr{E}$ in $\mathbf{B}$.) The first variation of $\mathscr{S}$ with respect to $\mathbf{B}$ when keeping $\mathbf{J}^0$ and $\mathscr{E}^0$ fixed reads (after identifying $\mathbf{J}$ with $\mathbf{J}^0$ and $\mathscr{E}$ with $\mathscr{E}^0$) $\delta\mathscr{S}=\int d\tau\,\delta\mathbf{J}\cdot(\eta\mathbf{J}^0-\mathscr{E}^0-E_a)$. Integration by parts yields the Euler equation

$$\nabla\times(\eta\mathbf{J}^0-\mathscr{E}^0)=\mathbf{0}\,, \tag{20}$$

and the natural boundary condition, which is exactly Eq. (16). Thus, the first variation of (19) does indeed yield the steady state. The second variation equals

$$\delta^2\mathscr{S}=\int d\tau(\eta\,|\delta\mathbf{J}|^2-\underline{S}\delta\mathbf{B}\cdot\delta\mathbf{B})\,, \tag{21}$$

and its positivity is necessary in order for $\mathbf{B}^0$ to correspond to a local minimum state of $\mathscr{S}$. Returning to Eq. (15) and linearizing it so that it reads

$$\frac{\partial\delta\mathbf{B}}{\partial t}+\nabla\times(\eta\delta\mathbf{J})-\underline{L}\,\delta\mathbf{B}=\mathbf{0} \tag{22}$$

we have, after multiplying by $\delta\mathbf{B}$,

$$\frac{\partial}{\partial t}\int\tfrac{1}{2}\,|\delta\mathbf{B}|^2 d\tau=-\int d\tau(\eta\,|\delta\mathbf{J}|^2-\underline{S}\delta\mathbf{B}\cdot\delta\mathbf{B})\,. \tag{23}$$

A comparison with (21) shows that the positivity of $\delta^2\mathscr{S}$ implies that a state $\mathbf{B}$ sufficiently close to $\mathbf{B}^0$ will be *attracted* to it by the dynamics. We summarize the result as follows.

Conclusion. A state $\mathbf{B}^0$ corresponding to a strict local minimum of the entropy production integral $\mathscr{S}$ is a relaxed state of the plasma, in the sense that (a) it is a steady state of the system and (b) states sufficiently near it converge to it in time (in the energy norm).

It is important to notice that our analysis is only local, and applicable to states in the vicinity of the steady state $\mathbf{B}^0$. We also emphasize that knowledge of $\mathbf{B}^0$ has to be assumed in the minimization of $\mathscr{S}$. This does not mean that we need to know the solution before we find it. Rather, the minimization must be achieved via some iterative procedure, such as guessing $\mathbf{B}^0$, carrying out the minimization and updating the guess. Section IV deals with the question of the convergence of some natural iteration schemes, but we mention now that the same kind of procedure occurs in the Glansdorff-Prigogine[18] local potential approach, which in our case corresponds to minimizing (as one possibility) $\int d\tau(\eta\mathbf{J}^2/2-\mathscr{E}^0\cdot\mathbf{J}-\mathbf{E}_a\cdot\mathbf{J})$. Although the first variation of this functional yields the correct steady-state equation, its second variation, which is always positive, has no relation to dynamical stability. Further discussion of the Glansdorff-Prigogine method will appear in later sections. We now consider the likely steady states resulting from the tendency of the plasma to minimize the rate of entropy production.

III. A GLOBAL ENTROPY PRINCIPLE

While in Sec. II we have shown that the plasma dynamics causes relaxation to a state which yields a local minimum of the rate of entropy production, we do not have enough information to determine that state. The reason is more than the fact that our analysis holds only for small deviations from a steady state, and allows in principle the existence of multiplicity of relaxed states. The real difficulty is that we do not actually know the functional form $\mathscr{E}(\mathbf{B})$. The special form (6) is an approximation[15,16] based on a model of the turbulence, but the actual $\mathscr{E}$ could be much more complex. In this section we attempt to develop an approximation to principle (19) which requires as little information on $\mathscr{E}$ as possible.

We first rewrite the entropy production principle as

$$\text{Minimize } \tilde{\mathscr{S}}=\int d\tau\left(\tfrac{1}{2}\eta\mathbf{J}^2-\tfrac{1}{2}\mathscr{E}\cdot\mathbf{J}-\mathbf{E}_a\cdot\mathbf{J}\right)\,, \tag{24}$$

subject to the usual boundary conditions, and to $\int d\tau(\mathscr{E}\cdot\mathbf{J}^0-\mathscr{E}^0\cdot\mathbf{J})=0$. Since the integral constraint is satisfied identically by the solution of problem (19), imposing it does not change the result. To this constraint we may add additional constraints which are also satisfied by the solution. We choose to impose

$$\int d\tau(\eta\mathbf{J}^2-\mathscr{E}\cdot\mathbf{J}-\mathbf{E}_a\cdot\mathbf{J})=0\ , \tag{25}$$

$$\int d\tau(\eta\mathbf{J}\cdot\mathbf{B}-\mathbf{E}_a\cdot\mathbf{B})=0\ . \tag{26}$$

These constraints imply that the amounts of energy and helicity dissipated in steady state equal the amounts replenished by the applied electric field. Moreover, they correspond to the two properties expressed by Eq. (4) which we know to hold for $\mathscr{E}$. We now *drop* the constraint $\int d\tau(\mathscr{E}\cdot\mathbf{J}^0-\mathscr{E}^0\cdot\mathbf{J})=0$, and thus dispose of the need to know the detailed $\mathscr{E}$, but at the expense of a possible error. The error is generated by allowing the stationary state of $\tilde{\mathscr{S}}$ to have too low a minimum value, and to be, perhaps, dynamically inaccessible. What we gain is that now the only information on $\mathscr{E}$ that needs to be supplied is the single integral $\int \mathscr{E}\cdot\mathbf{J}\,d\tau$. This may be approximated by a phenomenological expression such as $\int d\tau(K^2\,|\,\nabla\lambda\,|^2+L^2\,|\,\nabla p\,|^2)$ with K^2 and L^2 being positive functions of $\mathbf{B}$, as in form (6). To summarize then, we propose to minimize (24) subject to constraints (25), (26), and a given axial magnetic flux. Corresponding to the case of the solid conductor, we may also use relation (25) to change the problem into the maximization of $\int d\tau(\eta\mathbf{J}^2-\mathscr{E}\cdot\mathbf{J})$ subject to (25) and (26). This integral has the more traditional meaning of entropy production.

It is interesting to note that the error discussed before is undetectable by the information we have. Thus, suppose that the modified variational problem has been solved. Denote the solution by $\mathbf{B}^0$ and *define* $\mathscr{E}^0(\mathbf{x})\equiv\eta\mathbf{J}^0(\mathbf{x})-\mathbf{E}_a(\mathbf{x})$. With this $\mathscr{E}^0$ the equilibrium condition (20) is satisfied and, since we do not know the functional form of $\mathscr{E}$, it is not possible to tell whether our $\mathscr{E}^0$ is indeed $\mathscr{E}(\mathbf{B}^0)$. If we now minimize $\mathscr{S}$ in (19), the solution is again $\mathbf{B}^0$. The only possible detection of the error hereby generated using information we presently have can be made by checking properties (4), but these properties were imposed as constraints and are therefore trivially satisfied.

We now discuss the relation between an entropy principle, such as the one proposed in this section, or the exact one, Eq. (19), and Taylor's minimum energy principle.[4] A typical form for the turbulent dissipation, like the expression in Eq. (6), is

$$-\int \mathscr{E}\cdot\mathbf{J}\,d\tau=\int (\alpha K^2\,|\,\nabla\lambda\,|^2+\beta L^2\,|\,\nabla p\,|^2)d\tau\ , \tag{27}$$

where α and β are positive constants and K^2 and L^2 are positive functions of $\mathbf{x}$ and $\mathbf{B}$. Equation (27) reflects the fact that resistive instabilities[22] responsible for the turbulence are driven by current or pressure gradients. If α and β become large, $\mathscr{S}$ and $\tilde{\mathscr{S}}$ can only be minimized by a state for which $|\,\nabla\lambda\,|$ and $|\,\nabla p\,|$ are small, such that the turbulent dissipation terms remain comparable to other terms. (In fact, it was shown[16] that $\alpha=1$ is sufficient to yield very flat λ in some fusion devices.) In the limit of $\alpha\rightarrow\infty$ and $\beta\rightarrow\infty$, the minimizing state must therefore be a Taylor state, with $\mathbf{J}=\lambda_0\mathbf{B}$ and $\lambda_0=$const. Comparing now constraints (25) and (26) for the Taylor state limit, we find that in that limit $\int \mathscr{E}\cdot\mathbf{J}\,d\tau\rightarrow 0$ as well. The minimum problem is thus reduced to minimizing $\int d\tau(\eta\mathbf{J}^2/2-\mathbf{E}_a\cdot\mathbf{J})$ subject to a given amount of axial magnetic flux, with $\mathbf{B}$ being a Taylor state. Equation (26) is automatically satisfied by the minimum state and from it we get

$$\lambda_0=\frac{\int \mathbf{E}_a\cdot\mathbf{B}\,d\tau}{\int \eta\mathbf{B}^2 d\tau}\ . \tag{28}$$

Substituting in the entropy production integral we find

$$\mathscr{S}=\tilde{\mathscr{S}}=-\frac{\left[\int \mathbf{E}_a\cdot\mathbf{B}\,d\tau\right]^2}{2\int \eta\mathbf{B}^2 d\tau}\ . \tag{29}$$

Notice that $\int \mathbf{E}_a\cdot\mathbf{B}\,d\tau$ equals the applied voltage times the axial magnetic flux, and has therefore a given value. The minimization of entropy production rate is then achieved by minimizing $\int \eta\mathbf{B}^2 d\tau$. Thus, if subject to a given toroidal flux, more than one Taylor state can be established in a device, the entropy principle predicts that the relaxed state will be the one with lower $\int \eta\mathbf{B}^2 d\tau$. For $\eta=$const, namely for a constant temperature which approximately holds in turbulent plasmas, this criterion corresponds to minimizing the magnetic energy. The energy criterion is considered to have been verified experimentally,[3] most dramatically in multipinch experiments[23] where, in case of bifurcated solutions, the plasma appears to choose the state of lower energy. It is as consistent to conclude that the plasma is merely minimizing the rate of entropy production.

IV. ITERATION SCHEMES

We return now to the problem of minimizing the entropy production integral $\mathscr{S}$, defined in Eq. (19), which has the peculiar feature of depending on the test function $\mathbf{B}$ as well as on the solution $\mathbf{B}^0$. In this section we consider $\mathscr{E}(\mathbf{B})$ to be known so that the solution of the exact problem may be sought. Our main result here is the construction of an iteration scheme which, starting with a sufficiently accurate initial guess, is shown to converge to a steady-state solution as long as this state is a local minimum of the entropy production integral. We will also compare our method with the Glansdorff-Prigogine (GP) scheme.[18]

The discussion may be simplified if we cast the problem in a general form rather than deal with the special form suitable for a plasma. In order to develop a better understanding we will describe in Sec. V the special case of the heat equation. Consider then the equation

$$P\frac{\partial u}{\partial t}+N(u)=0\ , \tag{30}$$

where P is a positive definite linear operator which may depend on u, and N is a nonlinear operator. We use the usual inner product for square-integrable functions and proceed formally while assuming that the domain of the operators considered is rich enough to yield a meaningful problem. An example of Eq. (30) is the nonlinear heat equation

$$\frac{\partial T}{\partial t}-\nabla\cdot[\kappa(T)\nabla T]=0\ , \tag{31}$$

where $\kappa(T)$ is a positive function, and the temperature T is required to have a prescribed value on the boundary of some spatial domain. We first derive the appropriate entropy production principle. Multiplying (30) by $\partial u/\partial t$, we have

$$\int d\tau N(u)\frac{\partial u}{\partial t}\leq 0\ . \tag{32}$$

Expanding $N(u)$ about a state u_0 we have

$$N(u)=N(u_0)+(S+A)(u-u_0)+O(\,|u-u_0|^2)\ , \tag{33}$$

where S is a symmetric and A an antisymmetric linear operator, both depending on u_0. Using this expansion in (32) and dropping the term containing A, as in Sec. II, the problem reduces to minimizing

$$\int \{N(u_0)u+\tfrac{1}{2}[S(u-u_0)](u-u_0)\}d\tau\ , \tag{34}$$

which is analogous to the minimization in Eq. (19). The minimum is sought with respect to functions u, while u_0 is held fixed. The Euler equation for minimizing (34) is

$$N(u_0)+S(u-u_0)=0\ . \tag{35}$$

Notice that the minimum of (34) exists if and only if S is positive definite. Equation (35) is, of course, the steady-state equation of Eq. (30) when $u=u_0$.

A natural iteration scheme for the variational problem (34) is to guess u_0, minimize (34) or, equivalently, solve (35) for u, rename it u_0 and continue until convergence is achieved. Let us explore the convergence of such a scheme. Let $\overline{u}$ be the actual solution of $N(\overline{u})=0$, and consider u_0 to be close to $\overline{u}$. Also, let $\delta u=u-\overline{u},\delta u_0=u_0-\overline{u}$. Expanding all terms in (35) about $\overline{u}$ and keeping only first-order terms (in the deviation from $\overline{u}$) we have,

$$N(\overline{u})+(\overline{S}+\overline{A})\delta u_0+\overline{S}(\delta u-\delta u_0)=0\ . \tag{36}$$

The bar over S and A indicates that they are evaluated at $\overline{u}$. Using $N(\overline{u})=0$, Eq. (36) can be written as $\overline{S}\delta u+\overline{A}\delta u_0=0$. This equation determines the convergence of our scheme. The scheme converges if the spectrum λ of the problem

$$-\overline{A}v=\lambda\overline{S}v \tag{37}$$

satisfies $|\lambda|<1$. Here we assume, for simplicity, completeness of the spectral expansion based on (37). Note that in order to be consistent with the minimization problem (34) we still require $\overline{S}$ to be positive definite. If $\overline{A}$ is bounded with respect to $\overline{S}$, which implies that the spectrum λ is bounded, and which is often the case (as will be seen in Sec. V for the heat equation), the iteration scheme may be improved. Instead of using u as the next guess in place of u_0, let us "back-average" and take as the next guess a linear combination $w=\alpha u+(1-\alpha)u_0$, with α some real constant. Then $\delta w\equiv w-\overline{u}=\alpha\delta u+(1-\alpha)\delta u_0$, and δw solves $\overline{S}\delta w=[-\alpha\overline{A}+(1-\alpha)\overline{S}]\delta u_0$. This iteration converges if the spectrum μ of the problem

$$[-\alpha\overline{A}+(1-\alpha)\overline{S}]v=\mu\overline{S}v\ , \tag{38}$$

satisfies $|\mu|<1$. We rewrite this problem as $-\overline{A}v=\alpha^{-1}(\mu+\alpha-1)\overline{S}v$, which has the form (37). Thus $\mu=\alpha\lambda+1-\alpha$. Since λ is purely imaginary because of the symmetry properties of the operators in (37) (as long as $\overline{S}$ is also definite), we have

$$|\mu|^2=\alpha^2|\lambda|^2+(1-\alpha)^2\ . \tag{39}$$

Minimizing the right-hand side with respect to α, we find that the minimizing α equals to $(1+|\lambda|^2)^{-1}$, and then

$$|\mu|^2=\frac{|\lambda|^2}{1+|\lambda|^2}<1\ . \tag{40}$$

As seen from (39), if the value of α taken corresponds to the largest $|\lambda|$, then the entire μ spectrum will be inside the unit circle. We summarize as follows.

Conclusion. If $\overline{S}$ is positive definite and $\overline{A}$ is bounded with respect to $\overline{S}$, there exists a back-averaged modification of the simple iteration scheme based on the variational problem (34), which converges locally to the steady state $\overline{u}$ with $N(\overline{u})=0$.

The interesting feature of this result is its relation to the dynamics of the problem, Eq. (30). The linearized form of this equation reads

$$\overline{P}\frac{\partial\delta u}{\partial t}+(\overline{S}+\overline{A})\delta u=0\ , \tag{41}$$

and implies that $\delta u\to 0$ in time (i.e., $u\to\overline{u}$) if the real part of the spectrum of $\overline{S}+\overline{A}$ with weight $\overline{P}$ is positive. A sufficient condition for this to occur is that $\overline{S}$ be positive definite. Our result means that we are guaranteed convergence of the iteration scheme as long as we are guaranteed relaxation of the dynamical system via the positivity of $\overline{S}$. An additional advantage is that the iteration proceeds by solving the symmetric problem (35) for u thus, for example, enabling the utilization of a spectral expansion. Let us compare our approach with that of the GP scheme.[18] While in general their minimizing functional can be arbitrary as long as it produces the correct Euler equation, and thus has no relation to the dynamics of the problem, occasionally a particular choice may be very useful. Thus, it may happen (as in the heat equation) that we can write $N(u)=L_u u$, where L_u is a positive definite linear operator which depends on u. The GP scheme seeks to minimize successively $\int d\tau(L_{u_0}u)u/2$, which amounts to successive solutions of $L_{u_0}u=0$, with u_0 taken as a guess and u is the updated guess. Expanding all terms about $\overline{u}$ and keeping only first-order terms, we have (in our previous notation)

$$\overline{L}\delta u+(\overline{L}_1\delta u_0)\overline{u}=0\ , \tag{42}$$

where $(\overline{L}_1\delta u_0)$ is the first-order term in the expansion of L. Since $N(u_0)=L_{u_0}u_0$ for all u_0, we can expand the relation about $\overline{u}$ and get to first order $(\overline{S}+\overline{A})\delta u_0$

$=\bar{L}\delta u_0+(\bar{L}_1\delta u_0)\tilde{u}$. Substituting the result in (42), we have $\bar{L}\delta u+(\bar{S}+\bar{A}-\bar{L})\delta u_0=0$. Thus, the GP scheme converges if the spectrum ν of the problem

$$[\bar{L}-(\bar{S}+\bar{A})]v=\nu\bar{L}v \qquad (43)$$

satisfies $|\nu|<1$. The positivity of $\bar{S}$ only implies $\mathrm{Re}\nu<1$ and does not guarantee convergence. However, back averaging can still produce a convergent scheme. Using $w=\alpha u+(1-\alpha)u_0$ as the new guess in place of u, with a real constant α, we obtain convergence if all $|\mu|<1$, where $\mu=\alpha\nu+1-\alpha$. Minimizing $|\mu|^2$ with respect to α for a particular $\nu=\nu_0$ yields $\alpha=(1-\nu_{0r})/|1-\nu_0|^2$, where subscript r indicates the real part. For this α we have

$$|\mu|^2=1-\frac{2(1-\nu_{0r})(1-\nu_r)}{|1-\nu_0|^2}+\frac{(1-\nu_{0r})^2|1-\nu|^2}{|1-\nu_0|^4} .$$

Using $\nu_r<1$, the requirement $|\mu|^2<1$ implies

$$\frac{1-\nu_{0r}}{|1-\nu_0|^2}<2\frac{1-\nu_r}{|1-\nu|^2} . \qquad (44)$$

This inequality can easily hold for all ν if we choose ν_0 such that it minimizes $(1-\nu_{0r})/|1-\nu_0|^2$ over the spectrum ν. The minimized quantity is simply $\mathrm{Re}(1-\nu)^{-1}$. The transformation $z=(1-\nu)^{-1}$ maps the half plane $\mathrm{Re}\nu<1$ onto the half plane $\mathrm{Re}z>0$, in which is contained the image of the spectrum ν. Thus, an appropriate ν_0 exists and the back-averaged GP method converges if $\bar{S}$ is positive definite. We again remind the reader that this result has only been shown to hold when $N(u)=L_u u$ with L_u positive definite.

We conclude this section with a remark about some freedom we have in writing Eq. (30). The equation may be multiplied by QP^{-1}, with Q some positive definite operator, and then N is replaced by $M=QP^{-1}N$. The entire discussion can then proceed as before, but the linear part of M, yielding S and A, is now changed. Likewise, u itself may be transformed. Clearly, one should use this freedom to make A as "small" as possible compared to S, having the maximal $|\lambda|$ in (37) become as small as possible for fastest convergence (as long as S remains positive definite). Once the entropy functional has been determined, however, a transformation of u will not change matters. Indeed, transforming u to w gives rise to the linear transformation of $\delta u=T\delta w$, with S transforming to T^*ST (T^* is the adjoint operator to T), and similarly for A. The convergence of the scheme is not affected by such a change.

V. THE NONLINEAR HEAT EQUATION

A simple illustration of the iteration scheme discussed in Sec. IV may be obtained by considering the nonlinear heat equation (31) which is widely used as a model for diffusing systems. By transforming from T to $\tau(T)$ such that $d\tau/dT=\kappa(T)$, the equation takes the almost linear form $\kappa^{-1}\partial\tau/\partial t=\nabla\cdot(\nabla\tau)$. It is possible to derive an exact entropy principle for this equation by multiplying it by $\partial\tau/\partial t$ and showing that the integral of $|\nabla\tau|^2$ (up to a boundary term) decreases monotonically in time. We will not proceed in this way in order to generate a nontrivial problem. The GP approach[18] uses the variable $u=T^{-1}$ (the gradient of which is a diffusion "force"), so that Eq. (31) is rewritten as

$$u^{-2}\frac{\partial u}{\partial t}-\nabla\cdot[\kappa(u^{-1})u^{-2}\nabla u]=0 , \qquad (45)$$

which is a form useful for back averaging, as discussed before. We consider u to be defined in some finite domain with a prescribed value on the boundary (a Dirichlet boundary condition). Let $f(u)\equiv\kappa(u^{-1})u^{-2}$ and expand $f(u)$ about some u_0: $f(u)=f_0+f_1(u-u_0)+\cdots$, with $f_0=f(u_0)$, $f_1=df(u_0)/du_0$. Also, denote $\tilde{u}=u-u_0$. The linear part of the operator $N(u)$, as in Eq. (33), is $(S+A)\tilde{u}=-\nabla\cdot(f_1\tilde{u}\nabla u_0+f_0\nabla\tilde{u})$. Note that $\tilde{u}$ vanishes on the boundary. By examining the quadratic form of $S+A$ which eliminates the contribution of A, we find

$$S\tilde{u}=-\nabla\cdot(f_0\nabla\tilde{u})-\tfrac{1}{2}\tilde{u}\nabla\cdot(f_1\nabla u_0) , \qquad (46)$$

and S is a second-order operator, while A is only first order. Following Eq. (35), our convergence scheme requires that we solve successively

$$\nabla\cdot(f_0\nabla u)+\tfrac{1}{2}(u-u_0)\nabla\cdot(f_1\nabla u_0)=0 . \qquad (47)$$

The GP method, in contrast, requires the successive solution of the somewhat simpler equation $\nabla\cdot(f_0\nabla u)=0$.

We now consider a special case suggested by Kruskal for which the convergence of the GP method has been investigated.[24] Let u be a function of x only, $0\le x\le l$, and the prescribed boundary conditions for T are

$$T(0)=a,\quad T(l)=b . \qquad (48)$$

We also take $\kappa(T)\equiv 1$ so that $f(u)=u^{-2}$ and thus $f_0=u_0^{-2}, f_1=-2u_0^{-3}$. (Note that always, $f_1\nabla u_0=\nabla f_0$.) It was shown in Ref. 24 that the GP scheme converges for

$$e^{-2\pi/(3)^{1/2}}<\frac{b}{a}<e^{2\pi/(3)^{1/2}} . \qquad (49)$$

In our back-averaged method we need only be concerned with the positivity of $\bar{S}$. Let $\bar{T}(x)$ be the exact steady-state solution of (45), $d^2\bar{T}/dx^2=0$, so

$$\bar{T}(x)=a\left[1-\frac{x}{l}\right]+b\frac{x}{l} . \qquad (50)$$

Writing the equation $\bar{S}\tilde{u}=0$ explicitly, we have from (46)

$$\frac{d}{dx}\left[\bar{T}^2\frac{d\tilde{u}}{dx}\right]+\frac{1}{2}\frac{d^2(\bar{T}^2)}{dx^2}\tilde{u}=0 . \qquad (51)$$

By a standard oscillation theorem, $\bar{S}$ is positive definite if and only if any solution $\tilde{u}$ of (51) vanishes no more than once in the interval $0\le x\le l$. A general solution can be obtained by considering the function $\tilde{u}=\bar{T}^c$, where c is constant. Notice that $d\bar{T}/dx=(b-a)/l$, a constant, so c must satisfy the indicial equation

$$c^2+c+1=0 , \qquad (52)$$

except for the case $d\bar{T}/dx=0$ for which $\bar{S}$ is trivially positive definite. The solution of (52) is $c=(-1$

$\pm i\sqrt{3})/2$. Thus, $\tilde{u}$ oscillates as $\sin(\frac{1}{2}\sqrt{3}\ln\overline{T})$ and the nonoscillation of $\tilde{u}$ in the interval implies exactly that criterion (49) holds. In this special case, therefore, the local convergence properties of our back-averaged scheme and the GP scheme are identical. If we back average the GP scheme in this model example, it can be shown to converge for a wider range of b/a than our scheme. Nevertheless, in general, as in the plasma case for example, there does not appear to be a particularly useful GP functional, and our scheme is apparently the natural one to use. In particular, if $\overline{A}=0$ then Eq. (37) implies that the scheme converges and has a better accuracy than first order. In fact, in this case it reduces to the usual Newton method for solutions of $N(u)=0$.

VI. CONCLUSIONS

This article presents a dynamical description of the observed tendency of a plasma to relax to some preferred steady state. The present understanding of this phenomenon, which is based on Taylor's minimum energy principle,[4] involves neither a dynamical nor a strictly thermodynamical description of the relaxation. Our approach, in contradistinction, does address these two aspects. What is shown to relax is the *mean* magnetic field, the evolution of which is governed by Eq. (2). The relaxation occurs if the system is in the vicinity of a state representing the minimum of some entropy production integral, as defined in Eq. (19). The physical effect causing the relaxation is the resistivity, both the collisional resistivity η, and the turbulent eddy resistivity due to the fluctuations which affect the mean field through the generation of $\mathscr{E}$.

We caution that reaching these rigorous conclusions requires some strong assumptions about the plasma. In particular, our analysis is carried out for incompressible plasmas in a cylinder. The incompressibility assumption brings about vast simplification, as we are relieved of the need to consider the evolution of the temperature, especially since heat conduction in confined plasmas is so poorly understood. The addition of a temperature equation would appear at first glance to pose only technical difficulties. The really new element that would be introduced, however, is the effect on the evolution of mass flow. We have eliminated the flow from our problem by making the cylindrical assumption. In general configurations, or even for a compressible plasma in a cylinder, the flow has to be taken into account. We do not know what the possible effects of the flow could be, especially since there is no independent evolution equation for it, and it is determined indirectly by the requirement that Eq. (7) be satisfied at all times. A very simple case which includes flow is treated in Appendix B and suggests that our results may still hold. Likewise, employing an aspect ratio expansion which is suitable for tokamaks, it has been shown[25] that the evolution of the magnetic field is independent of the flow. Whether such conclusions hold in the general case is left as an open question.

A second set of results involves the construction of iteration schemes based on the entropy principle to calculate the relaxed state. One such scheme is shown to converge as long as the steady state represents the state of minimum rate of entropy production. We also compare our scheme with the "local potential" method of Glansdorff and Prigogine.[18] While the latter method is not based on the dynamics of the evolution problem, we find that by a suitable "back averaging" it is possible to improve the method in some cases so that it also converges if the relaxed state is a state of minimum entropy production.

ACKNOWLEDGMENTS

The authors wish to thank Professor H. Weitzner for reading the manuscript and for helpful comments. This work was supported by U.S. Department of Energy, Grant Nos. DE-FG02-86ER53223 and DE-FG02-86ER53222.

APPENDIX A: BOUNDARY CONDITIONS FOR A DRIVEN PLASMA

The common idealization whereby the solid wall surrounding a plasma is taken to be a perfect conductor, is not suitable when the plasma is driven by external fields. A perfect conductor isolates the system from outer electromagnetic fields and does not allow the driving force to penetrate and affect the plasma. This can be easily seen from Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times\mathbf{E}=\mathbf{0}\ , \tag{A1}$$

which implies that the magnetic energy evolves according to

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}\mathbf{B}^2 d\tau=-\int \mathbf{E}\cdot\mathbf{J}\,d\tau-\oint (\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{n}}\,dS\ , \tag{A2}$$

where the last surface integral is taken over the boundary. On a perfect conductor, $\mathbf{E}$ is parallel to the unit normal $\hat{\mathbf{n}}$, the last integral vanishes, and no electromagnetic energy enters or leaves the system. Seeing it even more clearly, the loop voltage $\oint \mathbf{E}\cdot dl$ measured as a line integral over any closed curve on the boundary, trivially vanishes with no possibility for an electric drive.

We get around this difficulty by imposing the following boundary conditions (in a toroidal configuration):

$$\mathbf{B}\cdot\hat{\mathbf{n}} \text{ is given and time independent}\ , \tag{A3}$$

$$\mathbf{J}\cdot\hat{\mathbf{n}}=0\ , \tag{A4}$$

$$\oint_T \mathbf{E}\cdot dl=V_T,\quad \oint_P \mathbf{E}\cdot dl=V_P\ , \tag{A5}$$

where the subscripts T and P refer to simple curves lying on the boundary and closing on themselves in the toroidal (the long way) and poloidal (the short way) directions, respectively. V_T and V_P are the applied voltages. For confined plasmas, the physically necessary form of (A3) is $\mathbf{B}\cdot\hat{\mathbf{n}}=0$. Condition (A4) may be interpreted as approximating a thin vacuum (or cold plasma) layer between the plasma and the wall, which prevents currents from flowing into the boundary. Condition (A5) requires some explanation. The reason the line integrals used there do not depend on the particular curves but only on their homology class, is that from (A1) it follows, after using (A3), that $\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{E})=0$ on the boundary. Thus, the component tangential to the boundary of any $\mathbf{E}$ satisfying (A1) is a surface gradient for which (A5) may be prescribed. Notice that (A4) implies that $\mathbf{B}$ must also be a surface gradient. It now follows that

$$\oint (\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{n}}\,dS=V_P\oint_T \mathbf{B}\cdot dl - V_T\oint_P \mathbf{B}\cdot dl\ . \tag{A6}$$

Here, the outer normal, the poloidal, and the toroidal directions form a right-handed system. We see that the electromagnetic energy supplied to the system is expressible in terms of the applied voltages times the total currents in the appropriate directions. [The line integrals of $\mathbf{B}$ yield total currents and are also curve independent by (A4).] Relation (A6) means that the boundary is only "imperfect" with respect to the applied electric field, but otherwise retains its insulating property. Physically, condition (A6) may model the case of a highly conducting boundary through which the applied electric field is able to penetrate over a time longer than the wall skin time, but faster changes still come up against a perfectly conducting wall.

An indication that the boundary conditions (A3)–(A5) give rise to a well-posed problem (up to boundary conditions on the velocity) may be seen when Eq. (A1) is applied to a solid conductor rather than a plasma. In this case $\mathbf{E}=\eta\mathbf{J}$, where η is some fixed function. Equation (A1) becomes essentially a vector heat equation for $\mathbf{B}$, and conditions (A3) and (A4), together with the restriction (which is merely an initial condition) $\nabla\cdot\mathbf{B}=0$, count as the three required boundary conditions. The necessity of the two period conditions (A5) is seen from a uniqueness proof. If $\mathbf{B}_1$ and $\mathbf{B}_2$ are two solutions of the system, and are equal at $t=0$, define $\mathbf{B}=\mathbf{B}_1-\mathbf{B}_2$ and use Eq. (A2). Noticing that the boundary voltages corresponding to $\mathbf{B}$ both vanish, and using (A6), we get

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}\mathbf{B}^2 d\tau=-\int \eta\mathbf{J}^2 d\tau\le 0\ . \tag{A7}$$

$\mathbf{B}=0$ initially implies $\mathbf{B}=0$ for all time. We note that the proof would have also worked if instead of (A5) the values of $\oint_T \mathbf{B}\cdot dl$ and $\oint_P \mathbf{B}\cdot dl$ were imposed. This corresponds to driving the system by passing a prescribed amount of current in it. Likewise, a combination of prescribed currents and voltages may also be imposed. Finally, we note that if the system is only driven toroidally, with $V_p=0$, the total toroidal magnetic flux is constant in time. Any relaxed state of the system carries the toroidal flux it had initially. As seen in Sec. II, the current in a relaxed state may be independent of initial conditions, but solving for $\mathbf{B}$ requires that the toroidal flux be known.

APPENDIX B: EXAMPLE OF PLASMA RELAXATION WITH FLOW

We consider a two-dimensional, incompressible plasma in which the flow is laminar. Both the velocity field $\mathbf{v}$ and the magnetic field have only x,y components, and z is an ignorable coordinate. Also, η is taken to be a constant. The system is driven by a voltage V applied in the z direction. The evolution equations are

$$\frac{\partial\mathbf{B}}{\partial t}+\nabla\times(\eta\mathbf{J}-\mathbf{v}\times\mathbf{B})=0\ , \tag{B1}$$

$$\nabla p=\mathbf{J}\times\mathbf{B}\ , \tag{B2}$$

$$\nabla\cdot\mathbf{v}=0\ , \tag{B3}$$

and $\mathbf{v}\cdot\hat{\mathbf{n}}=0$ on the boundary. Introducing a magnetic flux function ψ, we can represent

$$\mathbf{B}=\hat{\mathbf{z}}\times\nabla\psi,\quad \mathbf{J}=(\Delta\psi)\hat{\mathbf{z}}\ , \tag{B4}$$

and Eqs. (B1) and (B2) take the form

$$\frac{\partial\psi}{\partial t}+\mathbf{v}\cdot\nabla\psi=\eta\Delta\psi-V\ , \tag{B5}$$

$$\Delta\psi=-\frac{\partial p}{\partial\psi}\ . \tag{B6}$$

$p=p(\psi,t)$, and $\partial p/\partial\psi$ is a derivative with t held fixed. ψ is given on the boundary, a condition appropriate for a perfectly conducting wall. Multiplying Eq. (B1) by $\partial\mathbf{B}/\partial t$, we have

$$\int [\mathbf{B}_t^2+\mathbf{J}_t\cdot(\eta\mathbf{J}-\mathbf{E}_a)+\mathbf{J}_t\cdot(\mathbf{B}\times\mathbf{v})]d\tau=0\ , \tag{B7}$$

where $\mathbf{E}_a\equiv V\hat{\mathbf{z}}$ and the subscript t indicates a time derivative. Using (B2) we see that

$$\int \mathbf{J}_t\cdot(\mathbf{B}\times\mathbf{v})d\tau=-\int(\mathbf{J}\times\mathbf{B}_t)\cdot\mathbf{v}\,d\tau+\int \mathbf{v}\cdot\nabla p_t d\tau\ ,$$

but we note that the last integral vanishes because of (B3). The first integral on the right-hand side may be evaluated by using Eqs. (B5) and (B6). We have

$$\int \mathbf{J}_t\cdot(\mathbf{B}\times\mathbf{v})d\tau=\int \Delta\psi(\mathbf{v}\cdot\nabla\psi_t)d\tau$$
$$=\int \frac{\partial p}{\partial\psi}[\mathbf{v}\cdot\nabla(\eta\partial p/\partial\psi)+\mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla\psi)]d\tau\ .$$

The first term integrates to zero and the second term yields $\int -|\mathbf{v}\cdot\nabla\psi|^2\partial^2 p/\partial\psi^2 d\tau$. Thus, Eq. (B7) takes the form

$$\frac{\partial}{\partial t}\int(\tfrac{1}{2}\eta\mathbf{J}^2-\mathbf{E}_a\cdot\mathbf{J})d\tau$$
$$=-\int\left[\left[\frac{\partial\mathbf{B}}{\partial t}\right]^2-\frac{\partial^2 p}{\partial\psi^2}|\mathbf{v}\cdot\nabla\psi|^2\right]d\tau\ . \tag{B8}$$

The pressure profile of a confined plasma is typically parabolic, going from a maximum value at the center to zero on the boundary. If the steady-state pressure has this property then $\partial^2 p/\partial\psi^2<0$ and the right-hand side of (B8) is negative for neighboring states until $\mathbf{B}_t=0$ and $\mathbf{v}\cdot\nabla\psi=0$. This result suggests relaxation via the minimization of the same entropy production functional used in the solid conductor case of Sec. II.

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