Transition to a convective roll pattern as obtained from the stochastic center-manifold theory

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We prove that the transition to a convective pattern in a Rayleigh-Bénard cell with a step in the heat input can be obtained by restricting the system to the locally attractive and locally invariant center manifold in phase space. The problem of providing the adequate scaling factor for the random source in the order-parameter equation is solved and the theoretical findings reproduce satisfactorily the experimentally measured time dependence for the convective heat flow.

I. INTRODUCTION

The contraction in phase space in dissipative systems beyond a symmetry-breaking instability can be studied by means of the center-manifold (CM) theory, provided that there exists a separation of the relaxation-time scales at the onset of the convective state or dissipative structure.[1–4]

The basic tenets of this theory can be stated as follows.

(i) Let $\mathbf{X}_f=(X_{f,j})$ denote the vector of fast-relaxing degrees of freedom with associated damping constants $\lambda^{(j)}\ll 0$ and $\mathbf{X}_s=(X_{s,i})$, the vector of order parameters. Then, after a relaxation time $T_{\rm CM}$, given by

$$T_{\rm CM}=O(\sup_j\{\,|\lambda^{(j)}|^{-1}\})\ ,\qquad (1)$$

the probability density functional $P(\mathbf{X}_f,\mathbf{X}_s,t)$ is constrained to a narrow strip about the CM in such a way that

$$\langle\langle\, \mathbf{X}_f \,\rangle\rangle=\widetilde{\mathbf{X}}_f(\mathbf{X}_s)\ ,\qquad (2)$$

where $\langle\langle\ \rangle\rangle$ represents an average over an ensemble of realizations of the random source field[4] and $\mathbf{X}_f=\widetilde{\mathbf{X}}_f(\mathbf{X}_s)$ is the CM equation representing the adiabatic following[5] or statistical subordination of fast variables.

(ii) In order to allow for a continuous flow of probability about the CM and obtain the reduced Fokker-Planck (FP) equation which characterizes each particular unfolding, certain scaling relations determining the strength of the statistical fluctuations must be fulfilled. In order to clarify this point, we should first consider the fact that P is factorizable in two factors: a time-dependent factor $Q_s(X_s,t)=Q_s$, and a time-independent factor $Q_f(X_f,X_s)=Q_f$ peaked at the CM. That is,

$$P=Q_sQ_f\ .\qquad (3)$$

The factor Q_s satisfies the reduced FP equation which describes the *transition* to a convective state[5] or dissipative structure. This equation must be derived from the general FP equation satisfied by P, making use of relation (3), by integration with respect to the fast variables along the CM. In order for this reduction to be possible, the Gaussian width of Q_f about the CM should be properly scaled with the unfolding parameter and with the effective diffusion coefficient.

Each particular unfolding beyond a symmetry-breaking instability has a defined scaling associated with it. This scaling will prove to be of paramount importance in reproducing the experimentally found transition to a convective state.[5–7]

The context we shall deal with in this work consists of a Rayleigh-Bénard cell swept through its threshold by means of a controlled heat input.[5–7] The temperature of the bottom plate is time dependent and therefore the Rayleigh number R is also time dependent. We shall restrict ourselves to the case of a step in the heat input. The theory determines *the effect of the fast hydrodynamic modes which have been projected out* when obtaining the order-parameter equation. This effect is responsible for the *occurrence of an inhomogeneous term in the order-parameter equation.* The contribution of these intrinsic fluctuations allows us *to properly account for the transition to the convective state* which could not be obtained from the homogeneous order-parameter equation.

Thus, we shall not find it necessary to *model* the effect of the fast variables with a Langevin source of a phenomenological origin; instead, the homogeneous order parameter equation with the source added will be shown to be equivalent to the CM-reduced FP equation.[8]

We shall demonstrate that the empirical factor necessary to fit experimental data on the time dependence of the convective heat flow[9] can be obtained from the CM treatment.

II. CENTER-MANIFOLD EQUATION

In general, the stochastic vector field describing the state of the system, V, near the threshold admits the decomposition

$$\mathbf{V}=\mathbf{X}_s+\mathbf{X}_f\ .\qquad (4)$$

After a relaxation time of $O(T_{\rm CM})$, this field restricts itself to the CM and therefore Eq. (4) leads to

$$\langle\langle\, \mathbf{V} \,\rangle\rangle=\mathbf{X}_s+\widetilde{\mathbf{X}}_f(\mathbf{X}_s)\ .\qquad (5)$$

This equation simply gives the adiabatic following or statistical subordination of fast variables.[8] The aim of this section is to determine the vector field $\widetilde{\mathbf{X}}_f(\mathbf{X}_s)$ for the convection problem. Following standard notation, we write:

$$\mathbf{X}_f=\sum_{i\geq 2}\sum_{|\mathbf{q}|=q_0} V_{\mathbf{q}}^{(i)}\mathbf{e}_{\mathbf{q}}^{(i)} , \tag{6}$$

$$\mathbf{X}_s=\sum_{|\mathbf{q}|=q_0} V_{\mathbf{q}}^{(1)}\mathbf{e}_{\mathbf{q}}^{(1)} , \tag{7}$$

where $\mathbf{V}=(\theta,\mathbf{u},w)$, $\mathbf{u}=(u,v)$, (u,v,w) is the velocity field, and θ is the deviation of the temperature from the linear conducting profile between the boundaries $z=0,z=1$. The field $\mathbf{V}$ obeys the Boussinesq equations.[5] The distance, time, and temperature are scaled, respectively, by d, d^2/κ, and $\kappa\nu/\alpha g d^3$, where d is the cell height, κ and ν the thermal and viscous diffusivities, and α, the thermal expansion coefficient.

The eigenvectors $\mathbf{e}_{\mathbf{q}}^{(i)}$'s depend on the vertical coordinate z and are proportional to $\exp(i\mathbf{q}\cdot\mathbf{r})$ where $\mathbf{r}$ is the horizontal vector, $\mathbf{q}$ a horizontal wave vector, and $\mathbf{q}_0$ the critical wave vector for convective onset. The linear self-adjoint Boussinesq operator with eigenvectors $\mathbf{e}_{\mathbf{q}}^{(i)}$, $i\geq 1$ is $\underline{D}^0$ defined as[5]:

$$\underline{D}^0=\begin{bmatrix} \nabla^2+\dfrac{\partial^2}{\partial z^2} & 0 & R_c \\ 0 & \sigma\left[\nabla^2+\dfrac{\partial^2}{\partial z^2}\right] & 0 \\ \sigma & 0 & \sigma\left[\nabla^2+\dfrac{\partial^2}{\partial z^2}\right] \end{bmatrix} . \tag{8}$$

The gradient ∇ refers to the horizontal vector $\mathbf{r}$, R_c is the critical Rayleigh number and σ is the Prandtl number. Free boundary conditions are assumed.[9–12] The Fourier coordinates are defined in the canonical way

$$V_{\mathbf{q}}^{(i)}=\langle \mathbf{e}_{\mathbf{q}}^{(i)},\mathbf{V}\rangle . \tag{9}$$

The inner product $\langle \mathbf{V}_1,\mathbf{V}_2\rangle$ is defined by

$$\langle \mathbf{V}_1,\mathbf{V}_2\rangle=[\sigma\theta_1^*\theta_2+R_c(\mathbf{u}_1^*\cdot\mathbf{u}_2+w_1^*w_2)]_m . \tag{10}$$

The symbol $[\ ]_m$ indicates that we are averaging over a layer, that is, along the vertical direction given by the coordinate z.

The Nusselt number N is determined from the convective heat flow which is given by $(N-1)R/R_c$. Thus, we have

$$\frac{(N-1)R}{R_c}=c^2||\mathbf{X}_s||^2 . \tag{11}$$

The constant c will be given later. The scaling relations will be given in terms of the small parameter ϵ defined as

$$\epsilon=(R-R_c)/R_c . \tag{12}$$

The CM has the local dimension of the order-parameter space and it contains the locally attractive and locally invariant behavior of the system. It is tangent at the steady state to the eigenspace of $\underline{D}^0$ generated by the modes which are excited at the onset of convection. To a first approximation[1–4,8] this equation can be obtained by adiabatic elimination

$$\dot V_{\mathbf{q}}^{(j)}\approx 0,\quad j\geqq 2,\quad t\geqq T_{\mathrm{CM}} . \tag{13}$$

Or, equivalently,

$$0\approx\langle \mathbf{e}_{\mathbf{q}}^{(j)},\underline{D}^0\mathbf{V}+\delta\underline{D}\,\mathbf{V}+\mathbf{N}(\mathbf{V},\mathbf{V})-\partial P\rangle , \tag{14}$$

Where $\delta\underline{D}$ is the matrix with zero in every entry except in the place of R_c in $\underline{D}^0$. In that entry there is the element $\delta R=R-R_c$. The term $\mathbf{N}(\mathbf{V},\mathbf{V})$ corresponds to the nonlinear part of the Boussinesq operator[5] and it is given by

$$\mathbf{N}(\mathbf{V}_1,\mathbf{V}_2)=-[\mathbf{V}_1\cdot\boldsymbol{\partial}]\mathbf{V}_2 , \tag{15}$$

where

$$\boldsymbol{\partial}=(0,\nabla,\partial/\partial z) . \tag{16}$$

Neglecting terms of $O(\epsilon)$ and noticing that the integration by parts gives $\langle \mathbf{e}_{\mathbf{q}}^{(j)},\boldsymbol{\partial}P\rangle=0$, we get

$$V_{\mathbf{q}}^{(j)}=|\lambda_{\mathbf{q}}^{(j)}|^{-1}\langle \mathbf{e}_{\mathbf{q}}^{(j)},\mathbf{N}(\mathbf{V},\mathbf{V})\rangle,\quad j\geqq 2 . \tag{17}$$

The equation can be simplified further by retaining only the slow modes in the nonlinear part,[5]

$$V_{\mathbf{q}}^{(j)}=|\lambda_{\mathbf{q}}^{(j)}|^{-1}\sum_{\mathbf{q}',\mathbf{q}''}\langle \mathbf{e}_{\mathbf{q}}^{(j)},\mathbf{N}(\mathbf{e}_{\mathbf{q}'}^{(1)},\mathbf{e}_{\mathbf{q}''}^{(1)})\rangle V_{\mathbf{q}'}^{(1)}V_{\mathbf{q}''}^{(1)} . \tag{18}$$

III. SCALING FACTOR FOR THE RANDOM SOURCE DETERMINING THE TRANSITION TO THE CONVECTIVE STATE

This section is devoted to showing how the probability density is distributed *along* and *about* the CM at the onset of convection.

The distribution *along* the CM, Q_s, is determined by the reduced FP equation which is equivalent to the order-parameter equation with a Langevin term added. The crux of the argument is that, in order to determine this distribution, we must properly display the relative size of each term in the FP equation integrated with respect to the fast variables. This implies that the distribution *about* the CM determines the distribution *along* the CM. Alternatively, the relative size of the terms is displayed by proper scaling of the Gaussian width of Q_f.

The starting point of the CM theory is the general FP equation for P,

$$\partial_t P=-\sum_{|\mathbf{q}|=q_0}\sum_{i\geqq 1}\partial_{V_{\mathbf{q}}^{(i)}}(\dot V_{\mathbf{q}}^{(i)}P) +\sum_{\mathbf{q},\mathbf{q}'}\sum_{i,j\geqq 1} d_{\mathbf{q}}^{(i)}d_{\mathbf{q}'}^{(j)}\partial^2_{V_{\mathbf{q}}^{(i)}V_{\mathbf{q}'}^{(j)}}P . \tag{19}$$

This equation is subject to the conditions[1–4,8]

$$P=Q_f(\{V_{\mathbf{q}'}^{(i)}\}_{\mathbf{q}',i})Q_s(\{V_{\mathbf{q}}^{(1)}\}_{\mathbf{q}},t) , \tag{20}$$

$$Q_f=\prod_{|\mathbf{q}|=q_0}\prod_{i\geqq 2}(g_{\mathbf{q}}^{(i)}/\pi)^{1/2}\exp[-g_{\mathbf{q}}^{(i)}(V_{\mathbf{q}}^{(i)}-\langle\langle V_{\mathbf{q}}^{(i)}\rangle\rangle)^2] . \tag{21}$$

In order to be able to scale each term in the resulting equation, the diffusion coefficients $d_{\mathbf{q}}^{(j)}$'s are factorized as $d_{\mathbf{q}}^{(j)}=k\widetilde{d}_{q}^{(j)}$, where k is a small parameter to be properly scaled, and $\widetilde{d}_{\mathbf{q}}^{(j)}=O(1)$. Making use of relations (20) and (21), the general FP equation (19) can be integrated along the CM as given by Eq. (18), to yield the following relation:

$$\partial_t Q_s=-\sum_{|\mathbf{q}|=q_0}\left[\partial_{V_{\mathbf{q}}^{(1)}}(\langle\langle \dot{V}_{\mathbf{q}}^{(1)}\rangle\rangle Q_s)+\langle\langle \dot{V}_{\mathbf{q}}^{(1)}\rangle\rangle\sum_{j\geq 2}\sum_{\mathbf{q}'}\frac{\partial_{V_{\mathbf{q}}^{(1)}}g_{\mathbf{q}'}^{(j)}}{2g_{\mathbf{q}'}^{(j)}}Q_s\right]-\sum_{|\mathbf{q}|=q_0}\sum_{j\geq 2}(\partial_{V_{\mathbf{q}}^{(j)}}\langle\langle \dot{V}_{\mathbf{q}}^{(j)}\rangle\rangle)Q_s$$
$$-\sum_{|\mathbf{q}|=q_0}\sum_{j\geq 2}2k^2(\widetilde{d}_{\mathbf{q}}^{(j)})^2 g_{\mathbf{q}}^{(j)}Q_s+\sum_{\mathbf{q},\mathbf{q}'}\sum_{j\geq 2}4k^2\widetilde{d}_{\mathbf{q}}^{(1)}\widetilde{d}_{\mathbf{q}'}^{(j)}g_{\mathbf{q}'}^{(j)}\partial_{V_{\mathbf{q}}^{(1)}}(\langle\langle V_{\mathbf{q}'}^{(j)}\rangle\rangle)Q_s+k^2\sum_{\mathbf{q},\mathbf{q}'}\widetilde{d}_{\mathbf{q}}^{(1)}\widetilde{d}_{\mathbf{q}'}^{(1)}\partial^2_{V_{\mathbf{q}}^{(1)}V_{\mathbf{q}'}^{(1)}}Q_s$$
$$+k^2\sum_{\mathbf{q},\mathbf{q}',\mathbf{q}''}\widetilde{d}_{\mathbf{q}}^{(1)}\widetilde{d}_{\mathbf{q}'}^{(1)}\sum_{j\geq 2}\left\{\frac{\partial_{V_{\mathbf{q}}^{(1)}}g_{\mathbf{q}''}^{(j)}}{g_{\mathbf{q}''}^{(j)}}\partial_{V_{\mathbf{q}}^{(1)}}Q_s+\left[\frac{\partial^2_{V_{\mathbf{q}}^{(1)}V_{\mathbf{q}'}^{(1)}}g_{\mathbf{q}''}^{(j)}}{2g_{\mathbf{q}''}^{(j)}}-\left[\frac{\partial_{V_{\mathbf{q}}^{(1)}}g_{\mathbf{q}''}^{(j)}}{g_{\mathbf{q}''}^{(j)}}\right]^2\frac{1}{4}-2g_{\mathbf{q}''}^{(j)}(\partial_{V_{\mathbf{q}}^{(1)}}\langle\langle V_{\mathbf{q}''}^{(j)}\rangle\rangle)^2\right]Q_s\right\}. \quad (22)$$

This equation has been obtained making use of the fact that Q_f behaves like a δ function peaked at the CM when the strength of the statistical fluctuations is small. It suffices to show [cf. Eq. (21)] that the Gaussian width tends to zero, or the $g_q^{(j)}$'s tend to infinity, when the noise intensity tends to zero. Specifically, we have used the facts

$$\text{(a)}\ \langle\langle M(\mathbf{X}_s,\mathbf{X}_f)\rangle\rangle=M(\mathbf{X}_s,\widetilde{\mathbf{X}}_f)\ , \quad (23)$$

$$\text{(b)}\ \int MQ_f\,d\mathbf{X}_f=M(\mathbf{X}_s,\widetilde{\mathbf{X}}_f)\ . \quad (24)$$

We must adjust the Gaussian widths $g_q^{(j)}$'s so that relation (22) becomes an equation of continuity for Q_s, that is, we have a conserved flow of probability on a strip about the CM. The aim is to reduce Eq. (22) to a FP equation equivalent to the empirical Langevin equation proposed previously,[9-12]

$$\frac{\partial\psi}{\partial t}=\frac{1}{\tau_0}[\epsilon-\widetilde{\xi}_0^4(\nabla^2+q_0^2)^2-g\psi^2]\psi+f(\mathbf{r},t)\ , \quad (25)$$

where the inhomogeneous term was regarded in previous treatments[9-12] of this problem as a *phenomenological* source term. This forcing field *modeled* the effect of the fast hydrodynamic modes which adjust themselves in the adiabatic following. The parameters in Eq. (25) were evaluated, cf. Ref. 5 and 9, $\tau_0^{-1}=(3\pi^2/2)[\sigma/(\sigma+1)]$, $\xi_0^2=8/3\pi^2$, $\xi_0^2=4q_0\widetilde{\xi}_0^4$, $\sigma=0.78$, $q_0d=\pi/\sqrt{2}$, $g=0.5$. The radius of the cell under consideration is $L=4.72$. The order parameter is defined as follows:

$$\psi=c\sum_{\mathbf{q}}V_{\mathbf{q}}^{(1)}\exp(i\mathbf{q}\cdot\mathbf{r})\ . \quad (26)$$

The constant c is obtained from the following relations (cf. Ref. 9):

$$e_{\mathbf{q}}^{(1)}(\mathbf{r},z)=\frac{1}{c}\begin{bmatrix} i\mathbf{q}\overline{u}_0(z)\\ w_0(z)\\ \theta_0(z)\end{bmatrix}e^{i\mathbf{q}\cdot\mathbf{r}}\ , \quad (27)$$

where

$$u_0(z)=4i\cos(\pi z)\ ,$$
$$w_0(z)=2\sqrt{2}\sin(\pi z)\ , \quad (28)$$
$$\theta_0(z)=9\sqrt{2}\pi^2\sin(\pi z)\ .$$

Thus

$$\overline{c}=[\sigma(\theta_0^2)_m+R_c(|u_0|^2+w_0^2)_m]^{1/2}\ , \quad (29)$$

$$c=[(w_0\theta_0)_m/R_c]^{1/2}\overline{c}^{\,-1}\ . \quad (30)$$

Thus, the reduced FP equation is

$$\partial_t Q_s=-\sum_{\mathbf{q}}\partial_{V_{\mathbf{q}}^{(1)}}\{\langle\langle \dot{V}_{\mathbf{q}}^{(1)}\rangle\rangle Q_s\}+k^2\sum_{\mathbf{q},\mathbf{q}'}\widetilde{d}_{\mathbf{q}}^{(1)}\widetilde{d}_{\mathbf{q}'}^{(1)}\partial^2_{V_{\mathbf{q}}^{(1)}V_{\mathbf{q}'}^{(1)}}Q_s\ . \quad (31)$$

In order to obtain Eq. (31) from Eq. (22) we have to introduce the following scaling relations:

$$g_{\mathbf{q}}^{(j)}=\frac{\widetilde{g}_{\mathbf{q}}^{(j)}}{(\epsilon/\tau_0)^2},\quad k=G^{-1/2}\widetilde{k}\ . \quad (32)$$

The quantities $\widetilde{k}$ and $\widetilde{g}_{q}^{(j)}$, j bigger than 1, are of $O(1)$. We also have

$$G=\lim_{N\to\infty}\left[\frac{1}{N}\sum_{j=1}^{N}g_{\mathbf{q}_0}^{(j)}\right]. \quad (33)$$

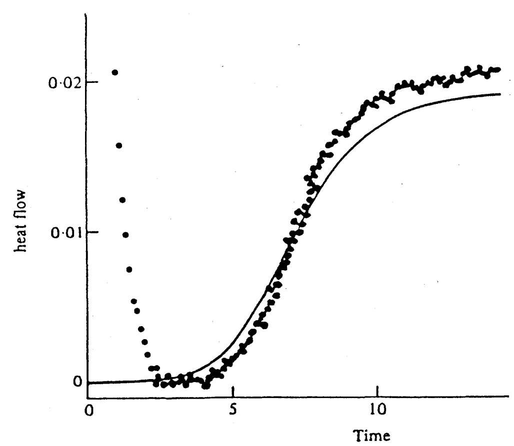

FIG. 1. Integration of Eq. (31) making use of the scaling relations (32) and (33) and the relation (34). The solid line represents the convective heat flow given directly by $c^2||\mathbf{X}_s||^2$. The experimental data was taken from Ref. 9 in the text. The bifurcation parameter for the unfolding takes the value $\epsilon=0.049+1.049\pi^2\xi_0^2/L^2$.

Then, to $O(\epsilon)$, Eq. (22) reduces to Eq. (31) if and only if

$$g_{\mathbf{q}}^{(j)} = -\lambda_{\mathbf{q}}^{(j)}/(k\tilde{d}_{q}^{(j)})^2 \ . \tag{34}$$

This relation represents the competition between the fast drift towards the CM, given by the relaxation of the fast hydrodynamic modes, and the statistical fluctuations which give the diffusive effect. The relations (32)—(34) and (21) justify relations (23) and (24). The parameters bearing a tilde can be chosen arbitrarily with the only constraint being that they are of order 1. In this work they have been chosen equal to 1.

The integration of Eq. (31), or, alternatively, of Eq. (25), can be carried out making use of relations (32)—(34). The solid line in Fig. 1 corresponds to the convective heat flow given by $c^2||\mathbf{X}_s||^2$. The experimental data was taken from Ref. 9. The step in the heat input determines ϵ, $\epsilon = 0.049(1+\pi^2\xi_0^2/L^2)+\pi^2\xi_0^2/L^2$.

We observe that the theoretical prediction with the aid of the scaling given by Eq. (32) exhibits a very good agreement with the experimental data. Particularly, vis-à-vis previous results[9–12] in which Eq. (25) was integrated. In previous treatments the intensity of the Langevin source was corrected starting from the value for equilibrium thermal fluctuations. For the analytic integration of Eq. (25) see Refs. 13 and 14. We have adjusted the Gaussian widths so that they exhibit an adequate competition between the fast drift towards the CM and the diffusive effect provided by the random source as given in Eq. (34). The fast transient in the experimental data corresponds to the time interval required to reach the lowest steady state (cf. Refs. 9—12).

IV. CONCLUSION

In this work we have been concerned with a Ginzburg-Landau regime in the order-parameter space for the onset of a convective-roll pattern.[5] In correspondence with the order-parameter equation for the two-dimensional smectic, there exists a CM-reduced FP equation which reproduces the transition to the convective state. The competition between the fast drift towards the CM and the diffusion provided by the inherent fluctuations determines scaling relations among the small characteristic parameters of the system. These parameters are (a) the Gaussian width of probability about the CM, (b) the unfolding parameter for the pitchfork bifurcation in the order-parameter space, (c) the effective-diffusion coefficient representing the strength of the intrinsic fluctuations.

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