PHYSICAL REVIEW A VOLUME 35, NUMBER 2 JANUARY 15, 1987

Theory of the quantum-beat laser

Marlan O. Scully and M. S. Zubairy*
Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, West Germany
and Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
(Received 11 August 1986)

A theory of the quantum-beat laser is developed using a general Fokker-Planck approach. An explicit expression for the diffusion coefficient for the relative phase angle of the two modes is derived. It is shown that the diffusion coefficient can vanish under certain conditions.

I. INTRODUCTION

In a recent paper[1] it was shown that the linewidth and the attendant uncertainty of the difference frequency between two laser modes may be eliminated by preparing the laser medium in a coherent superexposition of two upper states as in quantum-beat experiments.[2,3] Such a "quantum-beat laser" has potential for application in various areas of precision measurement, e.g., gravitational-wave detection[4,5] and tests of metric theories of gravitation[6].

The arguments of Ref. 1 were of a general nature but expressions for the various laser parameter, e.g., gain and cross-coupling coefficients were not given. In this paper we develop a more general Fokker-Planck approach to this problem and derive an explicit expression for the diffusion constant for the relative phase angle of the two modes. The conditions under which this diffusion constant vanishes are given.

Before getting into the detailed calculations, let us first advance the notion that the linewidth and attendant uncertainty in the difference frequency between two laser modes may be eliminated by preparing the laser medium in a coherent superposition of two upper states as in quantum-beat experiments. The heterodyne beat note between the two laser modes arising from these states can, under the appropriate conditions, be freed of spontaneous-emission noise. That this might be the case can be shown semiclassically as follows. Consider the atom of Fig. 1. The atomic state vector is given by

$$|\psi\rangle=\alpha e^{-i\phi_a}|a\rangle+\beta e^{-i\phi_b}|b\rangle+\gamma e^{-i\phi_c}|c\rangle\ ,$$

which implies that for the spontaneously emitted (semiclassical) fields E_1 and E_2,

$$E_1=\mathscr{E}_1 e^{-i(\phi_a-\phi_c)-i\nu_1 t}\ ,$$

and

$$E_2=\mathscr{E}_2 e^{-i(\phi_b-\phi_c)-i\nu_2 t}\ ,$$

where $\mathscr{E}_1$ is proportional to $\alpha\gamma^*$ and ϵ_2 to $\beta\gamma^*$. Now the phase factors ϕ_a and ϕ_b are fixed by preparing the atoms in a coherent superposition of $|a\rangle$ and $|b\rangle$. However, the phase of the $|c\rangle$ level ϕ_c is a random variable, and therefore the fields $\langle E_1\rangle$ and $\langle E_2\rangle$ average to zero. We note, however, that the random phase ϕ_c cancels in the heterodyne cross term

$$\langle E_1^* E_2\rangle=\mathscr{E}_1^*\mathscr{E}_2 e^{i(\phi_a-\phi_b)+i(\nu_1-\nu_2)t}\ .$$

Similar conclusions are obtained when the field is quantized. The fully quantized state of the atom-field complex is

$$|\psi\rangle=\alpha|a\rangle|0\rangle+\beta|b\rangle|0\rangle+\gamma_1|c\rangle|1_1\rangle+\gamma_2|c\rangle|1_2\rangle\ ,$$

where $|1_i\rangle$ is the state $\hat{a}_i^\dagger|0\rangle$, $i=1,2$ and $\hat{a}_i^\dagger$ ($\hat{a}_i$) are the creation (destruction) operators for photons having frequency ν_i. Hence the expectation value of the electric field operator $\hat{E}_1$,

$$\hat{E}_1=\mathscr{E}_1(\hat{a}_1 e^{i\mathbf{k}_1\cdot\mathbf{r}-i\nu_1 t}+\text{adj.})\ ,$$

is easily seen to vanish, that is

$$\langle\hat{E}_1\rangle=\mathscr{E}_1\alpha^*\gamma\langle a|c\rangle e^{i\mathbf{k}_1\cdot\mathbf{r}-i\nu_1 t}\ \text{c.c.}=0\ .$$

This happens because the states $|a\rangle$ and $|c\rangle$ are orthogonal. Similar arguments show that $\langle E_2\rangle$ likewise vanishes. However, the expectation value of the operator product $\hat{E}_1^\dagger\hat{E}_2$ does not vanish,

$$\langle\hat{E}_1^\dagger\hat{E}_2\rangle=\mathscr{E}_1\mathscr{E}_2\gamma_1^*\gamma_2\langle c|c\rangle e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}+i(\nu_1-\nu_2)t}\ .$$

That is, the spontaneously emitted photons at ν_1 and ν_2 are correlated. This is the fully quantized analogy of the preceding semiclassical discussion.

Motivated by such arguments in Ref. 1, we were led to

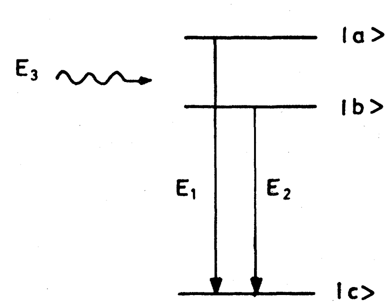

FIG. 1. Energy-level diagram for the quantum-beat laser.

investigate the relative spontaneous-emission laser linewidth when the (three-level) lasing atoms are prepared in a coherent superposition of the upper two levels. The main result of this paper is the Fokker-Planck equation for the relative phase $\theta=\theta_1-\theta_2$ and the sum phase $\mu=\theta_1+\theta_2$. This is given by Eq. (24) of the present paper. In the steady-state limit, and taking all gain coefficients and field amplitudes to be equal, the equation of motion for the phase distribution $P(\theta,\mu)$ is given by

$$\dot{P}(\theta,\mu)=\alpha\sin\psi\frac{\partial P}{\partial\theta}(\theta,\mu)+\frac{\alpha}{2\rho^2}(1-\cos\psi)\frac{\partial^2 P}{\partial\theta^2}(\theta,\mu)+\frac{\alpha}{8\rho^2}(1+\cos\psi)\frac{\partial^2 P}{\partial\mu^2}(\theta,\mu)\ ,$$

where $\psi=\theta+(\nu_1-\nu_2-\nu_3)t$, the frequency ν_3 is the "microwave" frequency as in Fig. 1, and α is the linear gain coefficient.

Hence we see that the diffusion coefficient for the relative phase angle, which is proportional to $(1-\cos\psi)$, vanishes when the angle ψ itself vanishes. Since the frequency difference ψ locks to "zero," we see that the phase diffusion in the relative phase angle can indeed vanish. However, the phase diffusion in the sum phase angle, μ, is not zero, but scales as $(1+\cos\psi)$. The correct physical picture then is that the phasor vectors corresponding to the two electric fields are strongly coupled, but are in fact fluctuating in unison so that the sum phase angle is diffusing at twice the usual diffusion rate while the difference phase angle is noise quenched.

In Sec. II we derive the density matrix equation of motion in general operation form and translate this into a P-representation Fokker-Planck equation in Sec. III.

II. EQUATION OF MOTION FOR THE DENSITY MATRIX

We consider a system of three-level atoms, as shown in Fig. 1, which are being pumped in the state $|a\rangle$ at a rate r_a. The laser cavity is arranged such that it would resonantly contain both modes v_1 and v_2. The transitions between levels $|a\rangle$-$|c\rangle$ are assumed dipole allowed. The dipole-forbidden $|a\rangle$-$|b\rangle$ transition is induced by some external means (such as, for instance, by applying a strong magnetic field for a magnetic dipole-allowed transition). The corresponding Rabi frequency is denoted by $\Omega e^{-i\phi}$ where Ω and ϕ are the real amplitude and phase. We shall treat the $|a\rangle$-$|b\rangle$ transition semiclassically and to all orders in the Rabi frequency. The $|a\rangle$-$|c\rangle$ transitions will be treated fully quantum mechanically but only to the second order in the corresponding coupling constants.

The Hamiltonian for the system is

$$H=H_0+V\ , \tag{1a}$$

$$H_0=\sum_{i=a,b,c}\hbar\omega_i|i\rangle\langle i|+\hbar\nu_1 a_1^\dagger a_1+\hbar\nu_2 a_2^\dagger a_2\ , \tag{1b}$$

$$\begin{aligned}V=&\hbar g_1(a_1|a\rangle\langle c|+a_1^\dagger|c\rangle\langle a|)\\&+\hbar g_2(a_2|b\rangle\langle c|+a_2^\dagger|c\rangle\langle b|)\\&-\frac{\hbar\Omega}{2}(e^{-i\phi-i\nu_3 t}|a\rangle\langle b|+e^{i\phi+i\nu_3 t}|b\rangle\langle a|)\ ,\end{aligned} \tag{1c}$$

where a_1, $a_1^\dagger$, a_2, $a_2^\dagger$, are the destruction and creation operators for the fields in modes of frequencies ν_1 and ν_2, respectively; g_1 and g_2 are the coupling constants associated with the $|a\rangle$-$|c\rangle$ and $|b\rangle$-$|c\rangle$ transitions, respectively; and ν_3 is the frequency of the field which induces the transition between levels $|a\rangle$ and $|b\rangle$ and it is assumed to be at resonance with the $|a\rangle$-$|b\rangle$ transition, i.e., $\nu_3=\omega_a-\omega_b$.

In order to derive the equation of motion for the reduced density matrix for the field ρ_F we first obtain an equation for the off-diagonal matrix element,

$$\begin{aligned}&\langle n_1,n_2|\rho_F|n_1',n_2'\rangle\\&\quad=\langle a,n_1,n_2|\rho|a,n_1',n_2'\rangle+\langle b,n_1,n_2|\rho|b,n_1',n_2'\rangle\\&\qquad+\langle c,n_1,n_2|\rho|c,n_1',n_2'\rangle\ ,\end{aligned} \tag{2}$$

where ρ is the atom-field density matrix and a trace over atomic states is taken. Let us define the following atom-field states:

$$|1\rangle=|a,n_1-1,n_2\rangle\ , \tag{3a}$$

$$|2\rangle=|b,n_1,n_2-1\rangle\ , \tag{3b}$$

$$|3\rangle=|c,n_1,n_2\rangle\ . \tag{3c}$$

The Schrödinger equation for the matrix element $\langle n_1,n_2|\rho_F|n_1',n_2'\rangle$ is therefore

$$\begin{aligned}&\langle n_1 n_2|\dot{\rho}_F|n_1',n_2'\rangle\\&\quad=-\frac{i}{\hbar}(V_{13}\rho_{31'}-\rho_{13'}V_{3'1'})_{\substack{n_1\to n_1+1\\ n_1'\to n_1'+1}}\\&\qquad-\frac{i}{\hbar}(V_{23}\rho_{32'}-\rho_{23'}V_{3'2'})_{\substack{n_2\to n_2+1\\ n_2'\to n_2'+1}}\\&\qquad-\frac{i}{\hbar}(V_{31}\rho_{13'}+V_{32}\rho_{23'}-\rho_{31'}V_{1'3'}-\rho_{32'}V_{2'3'})\ .\end{aligned} \tag{4}$$

We must now evaluate $\rho_{31'}$, $\rho_{13'}$, $\rho_{32'}$, and $\rho_{23'}$.

The wave vector $|\psi\rangle$ can be expanded in terms of the eigenstates of the atom-field system,

$$|\psi\rangle=\sum_{i=a,b,c}\ \sum_{m_1,m_2}A_{im_1m_2}|i,m_1,m_2\rangle\ , \tag{5}$$

where $A_{im_1m_2}$ is the probability amplitude for finding the atom in state $|i\rangle$ and the fields of modes 1 and 2 in the states $|m_1\rangle$ and $|m_2\rangle$, respectively. We first treat the $|a\rangle$-$|b\rangle$ transition semiclassically to all orders in Ω. The equations of motion for the amplitudes $A_{an_1n_2}$ and $A_{bn_1n_2}$ are

$$\dot{A}_{an_1n_2}=-i\left[\omega'_a-\frac{i\gamma}{2}\right]A_{an_1n_2}+\frac{i\Omega}{2}e^{-i\phi-i\nu_3t}A_{bn_1n_2}\,, \tag{6}$$

$$\dot{A}_{bn_1n_2}=-i\left[\omega'_b-\frac{i\gamma}{2}\right]A_{bn_1n_2}+\frac{i\Omega}{2}e^{i\phi+i\nu_3t}A_{an_1n_2}\,, \tag{7}$$

where $\omega'_i=\omega_i+n_1\nu_i+n_2\nu_2$ $(i=a,b,c)$ and γ is the decay constant for the levels a, b, and c (for simplicity we have taken them equal). If the atoms are injected at random initial times t_0 in level $|a\rangle$, the solution of Eqs. (6) and (7) is

$$A_{an_1n_2}(t)=e^{-i(\omega'_a-i\gamma/2)(t-t_0)}\times\cos\left[\frac{\Omega}{2}(t-t_0)\right]A^F_{n_1n_2}, \tag{8a}$$

$$A_{bn_1n_2}(t)=ie^{-i\phi-\omega'_bt+i\omega'_at_0-(\gamma/2)(t-t_0)}\times\sin\left[\frac{\Omega}{2}(t-t_0)\right]A^F_{n_1n_2}\,, \tag{8b}$$

where $A^F_{n_1n_2}$ are the probability amplitudes for the field only. The equation of motion for the amplitude $A_{cn_1n_2}$ is

$$\dot{A}_{cn_1n_2}=-i\left[\omega'_c-\frac{i\gamma}{2}\right]A_{cn_1n_2}-ig_1\sqrt{n_1}A_{a\,n_1-1\,n_2}-ig_2\sqrt{n_2}A_{b\,n_1\,n_2-1}\,. \tag{9}$$

Here we treat the $|a\rangle$-$|c\rangle$ and $|b\rangle$-$|c\rangle$ transitions to the lowest order only. It follows, on integrating Eq. (9), that

$$A_{cn_1n_2}(t)=-i\int_{t_0}^{t}d\tau\,e^{-i(\omega'_c-i\gamma/2)(t-\tau)}\times[g_1\sqrt{n_1}A_{a\,n_1-1\,n_2}(\tau)+g_2\sqrt{n_2}A_{b\,n_1\,n_2-1}(\tau)]\,, \tag{10}$$

where $A_{a\,n_1-1\,n_2}$ and $A_{b\,n_1\,n_2-1}$ can be obtained by appropriately shifting n_1 and n_2 in Eqs. (8a) and (8b).

We can now determine $\rho_{13'}$ by summing the contribution $A_{a\,n_1-1\,n_2}(t)A^*_{cn'_1n'_2}(t)$ of all atoms which are injected at random times at a rate r_a, i.e.,

$$\rho_{13'}=r_a\int_{-\infty}^{t}dt_0A_{a\,n_1-1\,n_2}(t)A^*_{cn'_1n'_2}(t)\,. \tag{11}$$

On substituting from Eqs. (8a) and (10) we obtain, after some straightforward algebra,

$$\rho_{13'}=T_{11}\langle n_1-1,n_2\,|\rho_F|\,n'_1-1,n'_2\rangle+T_{12}\langle n_1-1,n_2\,|\rho_F|\,n'_1,n'_2-1\rangle\,, \tag{12}$$

where

$$T_{11}=ig_1\sqrt{n_1}r_a\left[\left[\frac{1}{\gamma-i\Omega}+\frac{1}{\gamma}\right]\frac{1}{\gamma+i(\Delta_1-\Omega/2)}+\left[\frac{1}{\gamma+i\Omega}+\frac{1}{\gamma}\right]\frac{1}{\gamma+i\left[\Delta_1+\frac{\Omega}{2}\right]}\right], \tag{13a}$$

$$T_{12}=-ig_2\sqrt{n_2}r_a\left[\left[\frac{1}{\gamma-i\Omega}-\frac{1}{\gamma}\right]\frac{1}{\gamma+i\left[\Delta_2-\frac{\Omega}{2}\right]}-\left[\frac{1}{\gamma+i\Omega}-\frac{1}{\gamma}\right]\times\frac{1}{\gamma+i\left[\Delta_2+\frac{\Omega}{2}\right]}\right]e^{i\Phi(t)}\,, \tag{13b}$$

(13b)

with $\Delta_1=\omega_a-\omega_c-\nu_1$, $\Delta_2=\omega_b-\omega_c-\nu_2$, and $\phi(t)=(\nu_1-\nu_2-\nu_3)t-\phi$. In a similar manner we obtain

$$\rho_{23'}=T_{22}\langle n_1,n_2-1\,|\rho_F|\,n'_1,n'_2-1\rangle+T_{21}\langle n_1,n_2-1\,|\rho_F|\,n'_1-1,n'_2\rangle\,, \tag{14}$$

where

$$T_{22}=ig_2\sqrt{n_2}r_a\left[\left[\frac{1}{\gamma-i\Omega}-\frac{1}{\gamma}\right]\frac{1}{\gamma+i\left[\Delta_2-\frac{\Omega}{2}\right]}+\left[\frac{1}{\gamma+i\Omega}-\frac{1}{\gamma}\right]\frac{1}{\gamma+i\left[\Delta_2+\frac{\Omega}{2}\right]}\right], \tag{15a}$$

$$T_{21}=ig_1\sqrt{n_1}r_a\left[\left[\frac{1}{\gamma-i\Omega}+\frac{1}{\gamma}\right]\frac{1}{\gamma+i\left[\Delta_1-\frac{\Omega}{2}\right]}-\left[\frac{1}{\gamma+i\Omega}+\frac{1}{\gamma}\right]\times\frac{1}{\gamma+i\left[\Delta_1+\frac{\Omega}{2}\right]}\right]e^{-i\Phi(t)}\,. \tag{15b}$$

Also

$$\rho_{31'}=T^*_{11}\langle n_1-1,n_2\,|\,\rho_F\,|\,n'_1-1,n'_2\rangle + T^*_{12}\langle n_1,n_2-1\,|\,\rho_F\,|\,n'_1-1,n'_2\rangle\ , \tag{16}$$

$$\rho_{32'}=T^*_{22}\langle n_1,n_2-1\,|\,\rho_F\,|\,n'_1,n'_2-1\rangle + T^*_{21}\langle n_1-1,n_2\,|\,\rho_F\,|\,n'_1,n'_2-1\rangle\ . \tag{17}$$

It then follows, on substituting for $\rho_{13'}$, $\rho_{31'}$, $\rho_{32'}$, and $\rho_{23'}$ from Eqs. (12), (14), (15), and (16) in Eq. (4), that

$$\dot{\rho}_F=\sum_{i,j=1}^{2}\mathcal{L}_{ij}(a_i,a_i^\dagger)\rho_F\ , \tag{18}$$

where

$$\mathcal{L}_{ii}\rho_F=-\tfrac{1}{2}[\alpha^*_{ii}a_ia_i^\dagger\rho_F+\alpha_{ii}\rho_F a_i a_i^\dagger-(\alpha_{ii}+\alpha^*_{ii})a_i^\dagger\rho_F a_i]\ , \tag{19a}$$

$$\mathcal{L}_{12}\rho_F=-\tfrac{1}{2}[\alpha^*_{21}a_1^\dagger a_2\rho_F+\alpha_{12}\rho_F a_1^\dagger a_2-(\alpha_{12}+\alpha^*_{21})a_1^\dagger\rho_F a_2]e^{i\Phi}\ , \tag{19b}$$

$$\mathcal{L}_{21}\rho_F=-\tfrac{1}{2}[\alpha^*_{12}a_1 a_2^\dagger\rho_F+\alpha_{21}\rho_F a_1 a_2^\dagger-(\alpha_{21}+\alpha^*_{12})a_2^\dagger\rho_F a_1]e^{-i\Phi}\ , \tag{19c}$$

with

$$\alpha_{11}=\frac{g_1^2 r_a}{2}\left[\left[\frac{1}{\gamma}+\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_1-\dfrac{\Omega}{2}\right]}+\left[\frac{1}{\gamma}+\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_1+\dfrac{\Omega}{2}\right]}\right], \tag{20a}$$

$$\alpha_{12}=\frac{g_1 g_2 r_a}{2}\left[\left[\frac{1}{\gamma}-\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_2-\dfrac{\Omega}{2}\right]}-\left[\frac{1}{\gamma}-\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_2+\dfrac{\Omega}{2}\right]}\right], \tag{20b}$$

$$\alpha_{21}=\frac{g_1 g_2 r_a}{2}\left[\left[\frac{1}{\gamma}+\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_1-\dfrac{\Omega}{2}\right]}-\left[\frac{1}{\gamma}+\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_1+\dfrac{\Omega}{2}\right]}\right], \tag{20c}$$

$$\alpha_{22}=\frac{g_2^2 r_a}{2}\left[\left[\frac{1}{\gamma}-\frac{1}{\gamma-i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_2-\dfrac{\Omega}{2}\right]}+\left[\frac{1}{\gamma}-\frac{1}{\gamma+i\Omega}\right]\frac{1}{\gamma+i\left[\Delta_2+\dfrac{\Omega}{2}\right]}\right]. \tag{20d}$$

In Eq. (18) we ignored the cavity-loss terms because they do not contribute to the diffusion constant of the relative phase angle.

III. FOKKER-PLANCK EQUATION AND VANISHING OF DIFFUSION CONSTANT FOR RELATIVE PHASE

We now derive the Fokker-Planck equation for the coherent-state representation for the field $P(\mathcal{E}_1,\mathcal{E}_2)$ which is defined by

$$\rho_F=\int P(\mathcal{E}_1,\mathcal{E}_2)\,|\,\mathcal{E}_1,\mathcal{E}_2\rangle\langle\mathcal{E}_1,\mathcal{E}_2\,|\,d^2\mathcal{E}_1 d^2\mathcal{E}_2\ . \tag{21}$$

Here $|\,\mathcal{E}_1,\mathcal{E}_2\rangle$ is the coherent state which is an eigenstate of a_1 and a_2 with eigenvalues $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. By using the relations[7]

$$a_i\,|\,\mathcal{E}_1,\mathcal{E}_2\rangle=\mathcal{E}_i\,|\,\mathcal{E}_1,\mathcal{E}_2\rangle\ , \tag{22a}$$

$$a_i^\dagger\,|\,\mathcal{E}_1,\mathcal{E}_2\rangle=\left[\frac{\partial}{\partial\mathcal{E}_i}+\frac{\mathcal{E}_i^*}{2}\right]|\,\mathcal{E}_1,\mathcal{E}_2\rangle\ , \tag{22b}$$

we can reduce the density operator equation (18) to a c-number equation for P. The resulting Fokker-Planck equation in terms of the variables ρ_1, ρ_2, θ and μ, where

$$\rho_i=|\,\mathcal{E}_i\,|\quad(i=1,2)\ , \tag{23a}$$

$$\theta=\theta_1-\theta_2\ , \tag{23b}$$

$$\mu=\tfrac{1}{2}(\theta_1+\theta_2)\ , \tag{23c}$$

$$\theta_i=i\ln\left[\frac{\mathcal{E}_i}{\rho_i}\right]\quad(i=1,2)\ , \tag{23d}$$

is

$$\begin{aligned}\dot{P}=&\,d_0P+d(\rho_1)\frac{\partial P}{\partial\rho_1}+d(\rho_2)\frac{\partial P}{\partial\rho_2}+d(\mu)\frac{\partial P}{\partial\mu}\\&+d(\theta)\frac{\partial P}{\partial\theta}+D(\theta)\frac{\partial^2 P}{\partial\theta^2}+D(\mu)\frac{\partial^2 P}{\partial\mu^2}+D(\rho_1)\frac{\partial^2 P}{\partial\rho_1^2}\\&+D(\rho_2)\frac{\partial^2 P}{\partial\rho_2^2}+D(\theta,\mu)\frac{\partial^2 P}{\partial\theta\partial\mu}+D(\rho_1,\rho_2)\frac{\partial^2 P}{\partial\rho_1\partial\rho_2}\\&+D(\rho_1,\mu)\frac{\partial^2 P}{\partial\rho_1\partial\mu}+D(\rho_2,\mu)\frac{\partial^2 P}{\partial\rho_2\partial\mu}\\&+D(\rho_1,\theta)\frac{\partial^2 P}{\partial\rho_1\partial\theta}+D(\rho_2,\theta)\frac{\partial^2 P}{\partial\rho_2\partial\theta_2}\ ,\end{aligned} \tag{24}$$

with

$$d_0 = -\tfrac{1}{2}(\alpha_{11}+\alpha_{22})+\text{c.c.} , \tag{25a}$$

$$d(\rho_1) = -\frac{1}{4}\left[\alpha_{11}\left[\rho_1-\frac{1}{2\rho_1}\right]+\alpha_{12}\rho_2 e^{-i\psi}\right]+\text{c.c.} , \tag{25b}$$

$$d(\rho_2) = -\frac{1}{4}\left[\alpha_{22}\left[\rho_2-\frac{1}{2\rho_2}\right]+\alpha_{21}\rho_1 e^{i\psi}\right]+\text{c.c.} , \tag{25c}$$

$$d(\mu) = \frac{i}{8}\left[\alpha_{11}+\alpha_{22}+\alpha_{12}\frac{\rho_2}{\rho_1}e^{-i\psi} + \alpha_{21}\frac{\rho_1}{\rho_2}e^{i\psi}\right]+\text{c.c.} , \tag{25d}$$

$$d(\theta) = \frac{-i}{4}\left[\alpha_{11}-\alpha_{22}+\alpha_{12}\frac{\rho_2}{\rho_1}e^{-i\psi}-\alpha_{21}\frac{\rho_1}{\rho_2}e^{i\psi}\right] +\text{c.c.} , \tag{25e}$$

$$D(\theta) = \frac{1}{8}\left[\frac{\alpha_{11}}{\rho_1^2}+\frac{\alpha_{22}}{\rho_2^2}-\frac{\alpha_{12}}{\rho_1\rho_2}e^{-i\psi}-\frac{\alpha_{21}}{\rho_1\rho_2}e^{i\psi}\right]+\text{c.c.} , \tag{25f}$$

$$D(\mu) = \frac{1}{32}\left[\frac{\alpha_{11}}{\rho_1^2}+\frac{\alpha_{22}}{\rho_2^2}+\frac{\alpha_{12}}{\rho_1\rho_2}e^{-i\psi}+\frac{\alpha_{21}}{\rho_1\rho_2}e^{i\psi}\right]+\text{c.c.} , \tag{25g}$$

$$D(\rho_1) = \frac{\alpha_{11}}{8\rho_1^2}+\text{c.c.} , \tag{25h}$$

$$D(\rho_2) = \frac{\alpha_{22}}{8\rho_2^2}+\text{c.c.} , \tag{25i}$$

$$D(\theta,\mu) = \frac{1}{8}\left[\frac{\alpha_{11}}{\rho_1^2}-\frac{\alpha_{22}}{\rho_2^2}\right]+\text{c.c.} , \tag{25j}$$

$$D(\rho_1,\rho_2) = \tfrac{1}{8}(\alpha_{12}e^{-i\psi}+\alpha_{21}e^{i\psi})+\text{c.c.} , \tag{25k}$$

$$D(\rho_1,\mu) = \frac{i}{16\rho_2}(\alpha_{21}e^{i\psi}-\alpha_{12}e^{-i\psi})+\text{c.c.} , \tag{25l}$$

$$D(\rho_2,\mu) = \frac{i}{16\rho_1}(\alpha_{12}e^{-i\psi}-\alpha_{21}e^{i\psi})+\text{c.c.} , \tag{25m}$$

$$D(\rho_1,\theta) = \frac{-i}{8\rho_2}(\alpha_{21}e^{i\psi}+\alpha_{12}e^{-i\psi})+\text{c.c.} , \tag{25n}$$

$$D(\rho_2,\theta) = \frac{-i}{8\rho_1}(\alpha_{12}e^{-i\psi}+\alpha_{21}e^{i\psi})+\text{c.c.} , \tag{25o}$$

and ($\psi=\theta+\phi$).

We are interested in finding conditions under which the diffusion constant $D(\theta)$ for the relative phase angle $\theta=\theta_1-\theta_2$ of the two modes vanish. Here we mention two such conditions.

When $\psi=0$, $\rho_1=\rho_2=\rho$ then $D(\theta)=0$ if

$$\text{Re}(\alpha_{11}+\alpha_{22}-\alpha_{12}-\alpha_{21})=0 . \tag{26}$$

This equation is satisfied (with $g_1=g_2$) when

$$\Delta_1 = \frac{\Omega}{2}+\frac{2\gamma^2}{\Omega} , \tag{27a}$$

$$\Delta_2 = -\frac{3\Omega}{2} . \tag{27b}$$

This gives a condition on the detunings, the Rabi frequency of the driving field that couples levels $|a\rangle$ and $|b\rangle$, and the decay constants of the atomic levels.

Another interesting condition under which the spontaneous emission in the two modes gets highly correlated is (with $g_1=g_2=g$, $\rho_1=\rho_2=\rho$)

$$\Omega \gg \gamma , \tag{28a}$$

$$\Delta_1=\Delta_2=\frac{\Omega}{2} , \tag{28b}$$

i.e., the field detunings from the corresponding atomic lines are equal to half the Rabi frequency of the driving field that coherently mixes the level $|a\rangle$ and $|b\rangle$ and they are much larger than the atomic decay constant. In this case (for details, see Appendix A)

$$D(\theta) = \frac{\alpha_0}{4\rho^2}(1-\cos\psi) , \tag{29}$$

where

$$\alpha_0 = \frac{g^2 r_a}{2\gamma^2} . \tag{30}$$

When $\psi=0$, the vanishing of the diffusion constant takes place.

In order to establish the stability of these conditions, a nonlinear theory of the quantum-beat laser needs to be formulated. The present approach, however, shows in a simple way that it is possible for a two-mode laser to have a vanishing diffusion constant for the relative phase angle. It is interesting to note that the conditions under which $D(\theta)=0$ does not lead to a vanishing of $D(\mu)$ where $\mu=(\theta_1+\theta_2)/2$. Physically we can understand the quenching of the spontaneous emission fluctuations in the relative phase θ by referring to Fig. 2. Here we consider the "random walk" of the tips of electric field phases of the two modes in the complex α phase. If we ignore the amplitude fluctuations, the phase fluctuations in the field associated with the spontaneous emission allows the tips of the field to diffuse out around a circle in the complex

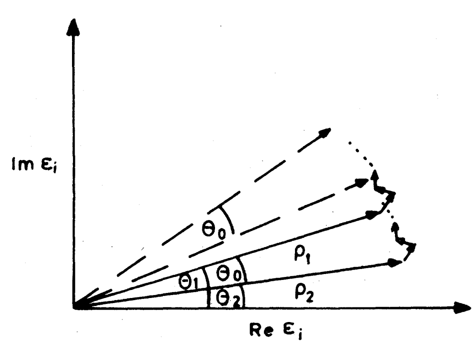

FIG. 2. Random walk of the correlated electric field phasors of magnitudes ρ_1 and ρ_2 in the complex $\mathscr{E}$ plane.

plane. When $D(\theta)=0$, the spontaneous emission in the two modes becomes highly correlated so that the relative phase angle θ is "locked" to a particular value (say θ_0). The average phase variable μ has, however, nonvanishing diffusion.

IV. CONCLUDING REMARKS

We have shown that in a quantum-beat laser the spontaneous-emission noise in the two modes can be made highly correlated under certain conditions. It is worthwhile to mention that in a recent paper, Kennedy and Swain[8] show the high correlation between the two modes well above threshold in a coupled two-mode laser.[9] A careful analysis of their results, however, indicates that the diffusion constant for the relative phase angle of the two modes $D(\theta)$ [$\mu(1,-1)$ in the notation of Ref 8] in the coupled two-mode laser is equal to $C/2\bar{n}$, independent of how far above threshold the laser is operating. (Here C is the cavity-loss parameter and $\bar{n}$ is the mean number of photons in either mode). An alternative derivation of their results using a Fokker-Planck approach is given in Appendix B. Well above threshold ($\alpha \gg C$; α being the gain coefficient) there is therefore a reduction of order C/α. This is in contrast with the results of this paper where we show a cancellation of all the terms proportional to $\alpha/\bar{n}$ (since $\alpha=C$ near threshold) in the diffusion constant $D(\theta)$.

ACKNOWLEDGMENTS

The authors wish to thank J. Gea-Banacloche, G. Leuchs, L. Pedrotti, W. Sandle, and H. Walther for many useful discussions. This research was supported by the U. S. Office of Naval Research and the U. S. Air Force Office of Scientific Research.

APPENDIX A: SIMPLE ASYMPTOTIC FORM FOR α_{ij} WHEN $\Delta_1=\Delta_2=\Omega/2$ AND $\Omega \gg \gamma$

It follows from Eq. (20a) that, when $\Delta_1=\Delta_2=\Delta$ and $\Delta=\Omega/2$,

$$\alpha_{11}=\frac{g^2 r_a}{2}\left[\left(\frac{1}{\gamma}+\frac{\gamma+i\Omega}{\gamma^2+\Omega^2}\right)\frac{1}{\gamma}+\left(\frac{1}{\gamma}+\frac{\gamma-i\Omega}{\gamma^2+\Omega^2}\right)\frac{\gamma-i\Omega}{\gamma^2+\Omega^2}\right]. \tag{A1}$$

We have assumed $g_1=g_2=g$. It follows from Eq. (A1) that

$$\mathrm{Re}\alpha_{11}=\frac{g^2 r_a}{2}\left[\frac{1}{\gamma^2}+\frac{2}{\gamma^2+\Omega^2}+\frac{\gamma^2-\Omega^2}{(\gamma^2+\Omega^2)^2}\right]. \tag{A2}$$

When $\Omega \gg \gamma$, Eq. (A2) simplifies considerably and we obtain

$$\mathrm{Re}\alpha_{11}\simeq\frac{g^2 r_a}{2}\left[\frac{1}{\gamma^2}+O\left(\frac{1}{\Omega^2}\right)\right]. \tag{A3}$$

Similarly

$$\mathrm{Re}\alpha_{22}\simeq\frac{g^2 r_a}{2}\left[\frac{1}{\gamma^2}+O\left(\frac{1}{\Omega^2}\right)\right]. \tag{A4}$$

Also

$$\alpha_{12}+\alpha_{21}^{*}=\frac{g^2 r_a}{2}\left[\frac{2}{\gamma^2}-\frac{2(\gamma+i\Omega)}{\gamma(\gamma^2+\Omega^2)}-\frac{4i\Omega\gamma}{(\gamma^2+\Omega^2)^2}\right]\simeq\frac{g^2 r_a}{2}\left[\frac{2}{\gamma^2}+O\left(\frac{1}{\Omega\gamma}\right)\right]. \tag{A5}$$

It follows, on substituting for Re α_{11}, Reα_{22}, and $\alpha_{12}+\alpha_{21}$ from Eqs. (A3)–(A5) in Eq. (25f), that $D(\theta)$ is given by Eq. (29).

APPENDIX B: DERIVATION OF $D(\theta)$ IN THE COUPLED TWO-MODE LASER MODEL OF REF. 8 USING THE FOKKER-PLANCK APPROACH

Here we derive the diffusion coefficient for the relative phase angle of the two modes in the coupled two-mode laser model of Ref. 9. The equation of motion for the matrix elements $\rho(n_1,n_2;m_1,m_2)=\langle n_1,n_2|\rho_F|m_1,m_2\rangle$ of the reduced density matrix for the field is 8,

$$\begin{aligned}\dot{\rho}(n_1,n_2;m_1,m_1)=&-\left[\frac{A}{2}(n_1+1+m_1+1)+\frac{B}{16}(m_1-n_1+m_2-n_2)^2\right]\mu(n_1,n_2;m_1,m_2)\\&+A\sqrt{nm}\,\mu(n_1-1,n_2;m_1-1,m_2)-\frac{C}{2}(n_1+m_1)\rho(n_1,n_2;m_1,m_2)\\&+C[(n_1+1)(m_1+1)]^{1/2}\rho(n_1+1,n_2;m_1+1,m_2)+[1\leftrightarrow 2]\,,\end{aligned} \tag{B1}$$

where A,B,C are the gain, saturation, and cavity-loss parameters, respectively, for either mode and

$$\mu(n_1,n_2;m_1,m_2)=\left[1+\frac{B}{2A}(m_1+n_1+m_2+n_2+4)+\frac{B^2}{16A^2}(m_1-n_1+m_2-n_2)^2\right]^{-1}\rho(n_1,n_2;m_1,m_2)\,. \tag{B2}$$

This equation for the density matrix can be translated into an equivalent equation for the coherent-state representation $P(\mathscr{E}_1,\mathscr{E}_2)$ via the relation

$$\rho(n_1,n_2;m_1,m_2)=\int P(\mathscr{E}_1,\mathscr{E}_2)e^{-|\mathscr{E}_1|^2-|\mathscr{E}_2|^2}\frac{\mathscr{E}_1^{n_1}(\mathscr{E}_1^*)^{m_1}\mathscr{E}_2^{n_2}(\mathscr{E}_2^*)^{m_2}}{\sqrt{n_1!m_1!n_2!m_2!}}d^2\mathscr{E}_1d^2\mathscr{E}_2\ . \tag{B3}$$

The resulting equation for $P(\mathscr{E}_1,\mathscr{E}_2)$ is

$$\begin{aligned}\dot{P}=-\frac{A}{2}\Bigg[&\frac{\partial}{\partial\mathscr{E}_1}\mathscr{E}_1+\frac{\partial}{\partial\mathscr{E}_2}\mathscr{E}_2+\frac{\partial}{\partial\mathscr{E}_1^*}\mathscr{E}_1^*+\frac{\partial}{\partial\mathscr{E}_2^*}\mathscr{E}_2^*-2\frac{\partial^2}{\partial\mathscr{E}_1\partial\mathscr{E}_1^*}-2\frac{\partial^2}{\partial\mathscr{E}_2\partial\mathscr{E}_2^*}\\&+\frac{B}{4A}\left[\frac{\partial}{\partial\mathscr{E}_1}\mathscr{E}_1+\frac{\partial}{\partial\mathscr{E}_2}\mathscr{E}_2-\frac{\partial}{\partial\mathscr{E}_1^*}\mathscr{E}_1^*-\frac{\partial}{\partial\mathscr{E}_2^*}\mathscr{E}_2^*\right]^2\Bigg]M\\&+\frac{C}{2}\left[\frac{\partial}{\partial\mathscr{E}_1}\mathscr{E}_1+\frac{\partial}{\partial\mathscr{E}_2}\mathscr{E}_2+\frac{\partial}{\partial\mathscr{E}_1^*}\mathscr{E}_1^*+\frac{\partial}{\partial\mathscr{E}_2^*}\mathscr{E}_2^*\right]P\ ,\end{aligned} \tag{B4}$$

where

$$\begin{aligned}M=\Bigg[1&-\frac{B}{2A}\left[\frac{\partial}{\partial\mathscr{E}_1}\mathscr{E}_1+\frac{\partial}{\partial\mathscr{E}_2}\mathscr{E}_2+\frac{\partial}{\partial\mathscr{E}_1^*}\mathscr{E}_1^*+\frac{\partial}{\partial\mathscr{E}_2^*}\mathscr{E}_2^*\right]-2(2+|\mathscr{E}_1|^2+|\mathscr{E}_2|^2)\\&+\frac{1}{16}\frac{B^2}{A^2}\left[\frac{\partial}{\partial\mathscr{E}_1}\mathscr{E}_1+\frac{\partial}{\partial\mathscr{E}_2}\mathscr{E}_2-\frac{\partial}{\partial\mathscr{E}_1^*}\mathscr{E}_1^*-\frac{\partial}{\partial\mathscr{E}_2^*}\mathscr{E}_2^*\right]^2\Bigg]^{-1}P\ .\end{aligned} \tag{B5}$$

We now change the variables from $\mathscr{E}_i,\mathscr{E}_i^*$ $(i=1,2)$ to ρ_i $(i=1,2)$, θ, and μ in accordance with Eqs. (23a)–(23d). It can then be shown, in a straightforward manner, that the diffusion constant is

$$D(\theta)=\frac{A}{4\bar{n}\left[1+\frac{2B}{A}\bar{n}\right]}\ . \tag{B6}$$

In deriving Eq. (B6) we assumed $\rho_1^2=\rho_2^2\simeq\bar{n}$ and $\bar{n}\gg1$. Since[9]

$$\bar{n}=\frac{A(A-C)}{2BC} \tag{B7}$$

it follows that

$$D(\theta)=\frac{C}{2\bar{n}}\ . \tag{B8}$$

This result, which is independent of how far above threshold the laser is operating, is in agreement with the results of Kennedy and Swain.[8]

*Permanent address: Department of Physics, Quaid-i-Azam University, Islamabad, Pakistan.

[1]M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).
[2]W. W. Chow, M. O. Scully, and J. Stoner, Phys. Rev. A **11**, 1380 (1975).
[3]M. O. Scully and K. Druhl, Phys. Rev. A **25**, 2208 (1982).
[4]See, for example, C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[5]M. O. Scully and J. Gea-Banacloche, Phys. Rev. A **34**, 4043 (1986).
[6]M. O. Scully, M. S. Zubairy, and M. P. Haugen, Phys. Rev. A **24**, 2009 (1981).
[7]See, for example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974).
[8]T. A. B. Kennedy and S. Swain, J. Phys. B. **17**, L751 (1984).
[9]S. Singh and M. S. Zubairy, Phys. Rev. A **21**, 281 (1980).