PHYSICAL REVIEW A VOLUME 35, NUMBER 12 JUNE 15, 1987

Rapid Communications

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 3½ printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but, because of the accelerated schedule, publication is not delayed for receipt of corrections unless requested by the author or noted by the editor.

Quantum chaos of periodically pulsed systems: Underlying complete integrability

K. Nakamura
Department of Physics, Fukuoka Institute of Technology, Higashi-ku, Fukuoka 811-02, Japan

H. J. Mikeska
Institut für Theoretische Physik, Universität Hannover, Hannover 3000, Federal Republic of Germany
(Received 2 March 1987)

The integrability underlying the quantum mechanics of nonintegrable pulsed systems is examined along the line exploited by Nakamura and Lakshmanan [Phys. Rev. Lett. **57**, 1661 (1986)]. Coupled dynamical equations for both quasienergies and quasieigenfunctions (rather than matrix elements) with a nonintegrability parameter λ taken as "time" are shown to be reduced to a classical Sutherland's system with internal complex-vector space. Their complete integrability together with constants of motion are also exhibited.

Although quantum mechanics of classically nonintegrable systems has received more and more interest,[1] its understanding still remains phenomenological. Theoretical tools borrowed from a field of random systems are not always effective here. The Wigner-Dyson distribution in random matrix theory is sometimes inconsistent with level-spacing distribution of quantum chaos.[2] The novel concept of fractals is not always practical in quantifying "chaotic" wave functions,[3] because of the finiteness of $\hbar$.

About a quarter of a century ago, Dyson proposed a theory of Brownian motion for energy levels by taking a nonintegrability parameter as "time."[4] Recent challengers have derived coupled dynamical equations for energy eigenvalues and matrix elements.[5] Eventually, however, most of their efforts have been directed toward confirming the effectiveness of the random matrix theory in the field of quantum chaos. On the contrary, Nakamura and Lakshmanan obtained dynamical equations for energy eigenvalues and eigenfunctions (rather than matrix elements), showing their equivalence to the completely integrable Calogero-Moser's system with internal complex vector space.[6] Knowledge of quantum mechanics for an arbitrary strength of nonintegrability can thereby be provided by the solutions of Lax-form equations. Since these are findings limited only to autonomous Hamiltonian systems, the next and natural question is to ask an integrability behind nonintegrable, driven nonautonomous systems.

In this Rapid Communication, we study quantum mechanics of periodically pulsed systems which has received considerable interest recently.[7] We shall derive equations of motion for both quasienergies and quasieigenfunctions, which will then be shown to be reduced to a Sutherland's system with internal complex-vector space. Finally, its completely integrable nature will be exhibited.

Let us consider a quantum Hamiltonian[3,7]

$$H(t)=H_0+\lambda\hat{V}\sum_{j=-\infty}^{\infty}\delta(t-2\pi j) \ ,$$

which describes any quantum bound system subjected to periodically pulsed field. H_0 and $\hat{V}$ correspond to classically integrable part and nonintegrable perturbation, respectively, and both of them are time (t)-independent Hermitian operators ($H_0^\dagger=H_0$, $\hat{V}^\dagger=\hat{V}$). λ denotes the strength of nonintegrability. For the time-dependent Schrödinger equation $i\hbar d|\Psi\rangle/dt=H(t)|\Psi\rangle$, its solution just after the jth pulse is given by $|\Psi_j\rangle=U^j|\Psi_0\rangle$. Here U is a one-period unitary operator defined in terms of time-ordering operator T as follows:[3,7]

$$U\equiv U(\lambda)=T\exp\left[\int_{+0}^{2\pi+0}(-i/\hbar)H(t')dt'\right]$$
$$=\exp(-i\lambda V)U_0 \ ,$$

where $V=\hat{V}/\hbar$ and $U_0=\exp[(-i/\hbar)2\pi H_0]$. So, the eigenvalue problem

$$U(\lambda)|n(\lambda)\rangle=\exp[-i\phi_n(\lambda)]|n(\lambda)\rangle \tag{1}$$

and its quasienergies $\{\phi_n(\lambda)\}$ (which are discrete because of the nature of bound spectra) and quasieigenfunctions $\{|n(\lambda)\rangle\}$ determine the quantum dynamics. Let us consider a manifold with a definite symmetry. Then $\{\phi_n(\lambda)\}$ can be assumed to be nondegenerate, by ignoring a negligible possibility of accidental degeneracies.[8] $\{|n(\lambda)\rangle\}$ are complex orthonormal and form a complete set. Taking λ as time, equations of motion for $\phi_n(\lambda)$, $p_n(\lambda)[\equiv V_{nn}\equiv\langle n(\lambda)|V|n(\lambda)\rangle]$, $|n(\lambda)\rangle$, and $\langle n(\lambda)|$ can be obtained from the time derivative of Eq. (1) as follows:

$$d\phi_n/d\lambda = p_n \ , \tag{2a}$$

$$dp_n/d\lambda = i\sum_{m\neq n} V_{nm}V_{mn}(\{(1-\exp[i(\phi_m-\phi_n)]\}^{-1}-\text{c.c.}) \ , \tag{2b}$$

$$d|n\rangle/d\lambda = -i\sum_{m\neq n}|m\rangle V_{mn}\{1-\exp[i(\phi_n-\phi_m)]\}^{-1} \ , \tag{2c}$$

$$d\langle n|/d\lambda = i\sum_{m\neq n}\langle m|V_{mn}\{1-\exp[i(\phi_m-\phi_n)]\}^{-1} \ . \tag{2d}$$

The derivation of Eqs. (2a) and (2b) is self-evident. Equation (2c) has been derived as follows: λ derivative of Eq. (1) yields

$$-iVU|n\rangle + Ud|n\rangle/d\lambda = -ip_n\exp(-i\phi_n)|n\rangle + \exp(-i\phi_n)d|n\rangle/d\lambda \ .$$

Its left-hand side

$$= -i\sum_m |m\rangle V_{mn}\exp(-i\phi_n) + \sum_m |m\rangle\exp(-i\phi_m)\langle m|(d|n\rangle/d\lambda)$$

$$= -i|n\rangle p_n\exp(-i\phi_n) - i\sum_{m\neq n}|m\rangle V_{mn}\exp(-i\phi_n) - i\sum_{m\neq n}|m\rangle V_{mn}\{1-\exp[i(\phi_n-\phi_m)]\}^{-1}\exp(-i\phi_m) \ ,$$

where

$$\langle m|(d|n\rangle/d\lambda) = -iV_{mn}(1-\exp[i(\phi_n-\phi_m)])^{-1}$$

for $m\neq n$ together with $\langle n|(d|n\rangle/d\lambda)=0$ are used. Suppressing the common term $-i|n\rangle p_n\exp(-i\phi_n)$, we obtain Eq. (2c) together with its complex conjugate [Eq. (2d)]. [Strictly speaking, the equality $\langle n|(d|n\rangle/d\lambda)=0$ is not appropriate in general. But this problem can be resolved in a way noted in Ref. 6(b).]

In order to elucidate a completely integrable nature of Eqs. (2a)–(2c), it is convenient to rewrite these equations in a perfectly canonical form. Let us define Λ_{nm} as

$$\Lambda_{nm} = V_{nm}\{1-\exp[i(\phi_n-\phi_m)]\} \ , \tag{3}$$

which reduces to

$$\Lambda_{nm} = \langle n|V|m\rangle - \langle n|U^\dagger VU|m\rangle$$
$$= \langle n|V-U_0^\dagger VU_0|m\rangle \equiv \langle n|\Lambda|m\rangle \ .$$

The operator Λ is λ independent and proves to be Hermitian. Let x_0 be the lowest eigenvalue of Λ; then the operator $\bar{\Lambda}\equiv\Lambda - x_0 I$ becomes a λ-independent and nonnegative Hermitian. $\bar{\Lambda}$ can thereby be decomposed as $\bar{\Lambda}\equiv L^\dagger L$. Here L is an appropriate λ-independent operator which has its unique inverse L^{-1}. Λ_{nm} thus becomes

$$\Lambda_{nm} = \langle n|L^\dagger L|m\rangle + x_0\delta_{nm} \ . \tag{4}$$

$\Lambda_{nn}=0$ is self-evident from Eq. (3), which means $\langle n|L^\dagger L|n\rangle = -x_0$. By using Eqs. (3) and (4), Eq. (2b) becomes

$$dp_n/d\lambda = i\sum_{m\neq n}\langle n|L^\dagger L|m\rangle\langle m|L^\dagger L|n\rangle\,|1-\exp[i(\phi_n-\phi_m)]|^{-2}(\{1-\exp[i(\phi_m-\phi_n)]\}^{-1}-\text{c.c.})$$
$$= \tfrac{1}{4}\sum_{m\neq n}\langle n|L^\dagger L|m\rangle\langle m|L^\dagger L|n\rangle\cos[(\phi_n-\phi_m)/2]\sin^{-3}[(\phi_n-\phi_m)/2] \ . \tag{2b$'$}$$

In the similar way, Eqs. (2c) and (2d) become

$$d(L|n\rangle)/d\lambda = (-i/4)\sum_{m\neq n}L|m\rangle\langle m|L^\dagger L|n\rangle \times\sin^{-2}[(\phi_n-\phi_m)/2] \ , \tag{2c$'$}$$

$$d(\langle n|L^\dagger)/d\lambda = (i/4)\sum_{m\neq n}\langle n|L^\dagger L|m\rangle\langle m|L^\dagger \times\sin^{-2}[(\phi_n-\phi_m)/2] \ . \tag{2d$'$}$$

Equations (2a) and (2b$'$)–(2d$'$) describe the dynamics for both quasienergies and quasieigenfunctions and take a perfectly canonical formalism.[9] In fact, let us introduce a classical N-particle Hamiltonian with internal complex-vector space for each particle:

$$\tilde{H} = \sum_{n=1}^{N}\tfrac{1}{2}p_n^2 + \tfrac{1}{2}\sum_{n=1}^{N}\sum_{\substack{m=1\\(m\neq n)}}\langle n|L^\dagger\cdot L|m\rangle\langle m|L^\dagger\cdot L|n\rangle \times\tfrac{1}{4}\sin^{-2}[(\phi_n-\phi_m)/2] \ , \tag{5}$$

where $\langle n|L^\dagger\cdot L|m\rangle$ is now read as a scalar product of complex dual vectors. N can be either finite or infinite. Applying Poisson brackets similar to those in Ref. 6, one has canonical equations from Eq. (6) as

$$d\phi_n/d\lambda = \{\tilde{H},\phi_n\} = \partial\tilde{H}/\partial p_n \ , \tag{6a}$$

$$dp_n/d\lambda = \{\tilde{H},p_n\} = -\partial\tilde{H}/\partial\phi_n \ , \tag{6b}$$

$$d(iL|n\rangle)/d\lambda = \{\tilde{H},iL|n\rangle\} = \partial\tilde{H}/\partial\langle n|L^\dagger \ , \tag{6c}$$

$$d(\langle n|L^\dagger)/d\lambda = \{\tilde{H},\langle n|L^\dagger\} = -\partial\tilde{H}/\partial iL|n\rangle \ . \tag{6d}$$

In Eqs. (6), $\{\cdots,\cdots\}$ are Poisson brackets, giving $\{p_n,\phi_m\}=\delta_{n,m}$, $\{\langle n\,|\,L^\dagger, iL\,|\,m\rangle\}=\delta_{n,m}$, etc. (see Ref. 6). Equations (6) can be found to be exactly the same as Eqs. (2a) and (2b′)–(2d′). Equations (5) and (6) constitute a Sutherland many-particle system[10] generalized so as to include an internal complex vector space for each particle. The number of degrees of freedom for the system in Eqs. (6) is $N(N+1)$, consisting of N for the position momenta of N particles (quasienergies) and N^2 for the internal freedom (quasieigenfunctions).

We now show the astonishing fact that Eqs. (6), despite their complicated appearance, are completely integrable. Let us define $N\times N$ matrices P,Φ,Γ,L as

$$P_{nm}=\delta_{nm}p_n+(1-\delta_{nm})i\langle n\,|\,L^\dagger\cdot L\,|\,m\rangle 2^{-1}\cot[(\phi_n-\phi_m)/2]\ ,$$

$$\Phi_{nm}=\delta_{nm}\phi_n\ ,$$

$$\Gamma_{nm}=(1-\delta_{nm})i\langle n\,|\,L^\dagger\cdot L\,|\,m\rangle 2^{-2}\sin^{-2}[(\phi_n-\phi_m)/2]\ ,$$

$$L=(L\,|\,1\rangle,L\,|\,2\rangle,\ldots,L\,|\,n\rangle,\ldots,L\,|\,N\rangle)\ ,$$

where $L\,|\,n\rangle$ are N-component column vectors. A Lax representation of Eqs. (6) is then written as $d\Phi/d\lambda=\mathrm{diag}(P)$, $dP/d\lambda=[\Gamma,P]$, $dL/d\lambda=-L\Gamma$, together with their Hermitian conjugates $dP^\dagger/d\lambda=[\Gamma,P^\dagger]$, $dL^\dagger/d\lambda=\Gamma L^\dagger$. (Note $\Gamma^\dagger=-\Gamma$.) Thus, if we would have the knowledge of quasienergies and quasieigenfunctions at $\lambda=+0$ (i.e., in the integrable limit), those at $\lambda\neq 0$ will be provided by solving the above matrix equations, e.g., by means of the inverse scattering method or of the algebraic method.

In marked contrast with autonomous systems in Ref. 6, the spectrum $\{\phi_n(\lambda)\}$ is periodic in energy with a period of 2π, as is recognized in Eq. (1). In other words, the generalized Sutherland system that has been obtained is defined on a ring chain with the length of 2π. The constants of motion in involution consist of three types: (i) $I_n=n^{-1}\mathrm{Tr}(P^n)$, (ii) $J_n(M^{(k)})=\mathrm{Tr}(P^nL^\dagger M^{(k)}L)$, (iii) $K_n=n^{-1}\mathrm{Tr}(L^\dagger L)^n$,[11] where $n=1,2,\ldots,N$, and $M^{(k)}$ are diagonal traceless constant matrices with $k=1,2,\ldots,(N-1)$ which are linearly independent. The total number of constants of motion is equal to the degree of freedom noted below Eqs. (6). The validity of these constants of motion can be checked in a periodically pulsed quantum spin system,[3,7] where Hilbert space is in finite dimensions and all the bound quasienergies are calculable in a wide range of nonintegrability (i.e., magnetic field B). The spectrum in Fig. 1 was obtained by numerical diagonalization of Eq. (4) in Ref. 3 for various μB values (μ is the Bohr magneton multiplied by g value). Energies are depicted in a fundamental "Brillouin" zone. We have recognized the absence of any accidental degeneracy in Fig. 1. Then, by examining through this figure together with numerical data for matrix elements, I_1 (total momentum), $I_2+\frac{1}{4}K_2$ (total energy), etc., are found not to show any change, irrespective of the change in the μB value. (An analogous study for a kicked quantum rotor will be made whose Hilbert space is infinite dimensional.[7] But, inevitable matrix-truncation procedures will make us confirm the presence of constants of motion less rigorously.)

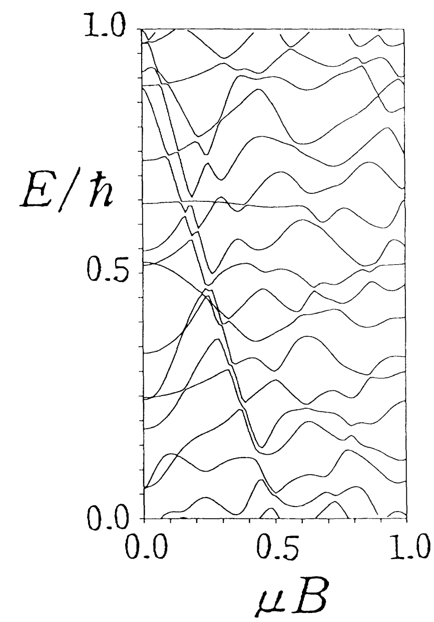

FIG. 1. Field-dependent quasienergy diagram for a pulsed-quantum spin system with spin magnitude $S=16$ in Ref. 3. Manifold with an even parity is depicted in fundamental zone ($0\leqq E/\hbar\leqq 1$). $2\pi E/\hbar$ and μB (scaled by the easy-plane anisotropy energy) should be read as ϕ and λ, respectively, in the present text.

Constants of motion above will play a vital role in quantitative descriptions of quantum recurrence[7] and of other complicated dynamics. Since both Calogero-Moser's and Sutherland's systems without complex-vector space are typical examples of a completely integrable classical particle system with interparticle interactions of doubly periodic Weierstrass function type $\mathcal{P}(z\,|\,\omega,\omega')$ [note, e.g., $\mathcal{P}(z\,|\,\infty,\infty)=z^{-2}$ and $\mathcal{P}(z\,|\,0,i\pi/2)=\frac{1}{4}\sin^{-2}(z/2)$],[12] the latter system with internal complex vector space will be indicated as a more universal dynamical system which underlies the quantum chaos. Its integrability can be shown in the same way as the present procedure.

We acknowledge useful discussions and correspondence with C. Jeffries, M. Lakshmanan, and F. Haake. One of us (K.N.) thanks the Department of Physics, University of California, Berkeley, for its hospitality, and another of us (H.J.M.) thanks the Japan Society for Promotion of Science for financial support for his stay at Fukuoka Institute of Technology.

RAPID COMMUNICATIONS

[1](a) See papers in *Chaotic Behavior in Quantum Mechanics,* edited by G. Casati (Plenum, New York, 1985); (b) J. Ford, in *The New Physics,* edited by S. Capelin (Cambridge Univ. Press, Cambridge, in press).

[2]M. Feingold, S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. B **31**, 6852 (1985); F. M. Izrailev, Phys. Rev. Lett. **56**, 541 (1986).

[3]K. Nakamura, Y. Okazaki, and A. R. Bishop, Phys. Rev. Lett. **57**, 5 (1986).

[4]F. J. Dyson, J. Math. Phys. **3**, 1191 (1962).

[5]P. Pechukas, Phys. Rev. Lett. **51**, 943 (1983); M. V. Berry, in Ref. 1(a); T. Yukawa, Phys. Rev. Lett. **54**, 1883 (1985).

[6](a) K. Nakamura and M. Lakshmanan, Phys. Rev. Lett. **57**, 1661 (1986); (b) *ibid.* **57**, 2772 (1986).

[7]For a kicked rotor, see G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems,* edited by G. Casati and J. Ford (Springer, Berlin, 1979); T. Hogg and B. A. Hurberman, Phys. Rev. Lett. **48**, 711 (1982); S. Fishman, D. R. Grempel, and R. E. Prange, *ibid.* **49**, 509 (1982). For a pulsed spin system, see F. Haake, M. Kus, and R. Scharf (unpublished) and H. Frahm and H. J. Mikeska (unpublished), together with Ref. 3.

[8]J. von Neumann and E. P. Wigner, Phys. Z. **30**, 467 (1929).

[9]One might construct an equation of motion for $\overline{\Lambda}_{nm}$ $(\equiv\langle n|L^{\dagger}\cdot L|m\rangle)$ by using Eqs. (2c′) and (2d′) and thereby be tempted to replace the latter equations by the former one. Such a replacement will inevitably destroy knowledge of quasieigenfunctions and cannot be fruitful at all.

[10]B. Sutherland, Phys. Rev. A **4**, 2019 (1971). A classical version of the system in this reference is related to the present findings.

[11]The constants $\{\langle n|L^{\dagger}L|n\rangle\}$, which are excluded in Eqs. (5) and (6), should not be included in a family of nontrivial constants of motion in involution. The former constants used in Ref. 6 should be replaced by $\{K_n\}$ in the present text.

[12]J. Gibbons and T. Hermsen, Physica D **11**, 337 (1984). See also, M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71**, 313 (1981).