Dressed-particle description of the damped harmonic oscillator

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Ullersma's model consistently describes the quantum-mechanical damped harmonic oscillator. Proper limiting processes in the initially reversible solutions lead to irreversible behavior. It is shown that the irreversible solutions can directly be derived from a field version of the discrete Ullersma Hamiltonian H ($H \rightarrow H_F$). In the ground state of the system the phonon occupation number of the oscillator does not vanish. Therefore, these "bare" phonons cannot be interpreted as "physical" quanta or particles. By suitable canonical transformations, the bare-field description is transformed into a physical description ($H_F \rightarrow H_{\text{ph}}$), which allows the definition of physical particles and which is characterized by a renormalized frequency, a screened interaction between oscillator and bath, and a symmetry of position and momentum operators. The physical description is closely related to the rotating-wave approximation of the original model. The relaxation behavior and the equilibrium properties of the different descriptions are discussed and compared.

I. INTRODUCTION

The understanding and the consistent description of the irreversible character of quantum-mechanical systems is of central importance in statistical mechanics. Especially at very low temperatures where the quantum description is indispensable. In order to understand the transition to irreversibility and to clarify many important questions in this context, it can be very useful to study simple and exactly solvable models. The most popular archetype is the damped harmonic oscillator.[1–11]

Although the pioneering studies have been made nearly 20 years ago,[1,2] lately there has been a revival of this topic.[4–11] The recent work is partly motivated by the progress of experimental low-temperature techniques. From a theoretical point of view, there are relations to the problem of quantum tunneling in dissipative systems.[12]

Usually one starts with a reversible Hamiltonian which describes the harmonic oscillator (system), the heatbath (system with many degrees of freedom), and the interaction. A proper limiting process in the initially reversible solutions, which corresponds to the transition to the infinite bath, leads to the desired irreversible behavior. The bath is often represented by a set of noninteracting harmonic oscillators which are coupled to the central oscillator; the coupling is linear in the position coordinates of system and bath [a justification of this bath modeling was given by Caldeira and Leggett; see Appendix C in Ref. 12(b)]. This same model was thoroughly analyzed by Ullersma[2] in his comprehensive work. Recently, various aspects of this model (e.g., low-temperature anomalies) were discussed by Lindenberg and West,[7] Grabert *et al.*,[5] Braun,[9] Haake and Reibold,[6] Riseborough *et al.*,[8] and the author.[11]

Ullersma's model can be related to some models in quantum field theory which describe the decay of unstable particles,[13] e.g., the decay of the excitations of a harmonically bound electron in the blackbody radiation field.[2,14] For large times ($t \rightarrow \infty$) the occupation number of the electron excitations (i.e., phonons which are described by the creation and annihilation operators $a^\dagger$ and a, respectively) should be in thermal equilibrium with the radiation field. If we initially assume a photon vacuum state (temperature $T=0$), the phonon occupation number $\langle a^\dagger a \rangle$ should vanish for $t \rightarrow \infty$. In contrast to this assumption, the calculation leads to a finite (>0) value for $\langle a^\dagger a \rangle$. Eganova and Shirokov[13(a)] have pointed out that the underlying "bare"-field description of the decay process excludes the definition of $\langle a^\dagger a \rangle$ as an expectation value of physical quanta: in the physical vacuum (i.e., in the ground state of the global system, which is reached for $t \rightarrow \infty$) physical quanta cannot be present. Eganova and Shirokov[13] developed a method which allows the definition of physical (or dressed) particles. In contrast to the bare-field description, the physical description must guarantee that the physical vacuum coincides with the ground state of the physical particles.

In the studies of Lindenberg and West[7] and Grabert *et al.*[5] it is shown that the irreversible solutions for the Ullersma model also lead to a residual population $\langle a^\dagger a \rangle$ ($T=0$, $t \rightarrow \infty$) of the oscillator. Therefore, we will transfer the argumentation and the method of Eganova and Shirokov to the Ullersma model. In this paper we want to formulate the corresponding quantum relaxation process in terms of physical phonons, i.e., in terms of quanta which fulfill the fundamental demand to be absent in the ground state of the system.

In Sec. II the discrete bath of the Ullersma Hamiltonian is rewritten in the form of a continuous Bose field ($H \rightarrow H_F$). We will prove that this field formulation directly leads to the irreversible solutions. The transition to irreversibility must no longer be performed in the reversible solutions of the Ullersma Hamiltonian[2,6,11] and no longer requires operations in the complex plane. We will discuss some important properties of the solutions.

In Sec. III we set up the canonical transformations which transform the bare-phonon description into the physical one ($H_F \rightarrow H_{\text{ph}}$). The physical Hamiltonian is characterized by a renormalized frequency of the oscillator, a screened interaction between oscillator and bath, and a symmetry of position and momentum operators. We solve the corresponding initial-value problem and we discuss the solutions of the Heisenberg equations of motion. We will see that the physical description is closely related with the rotating-wave approximation (RWA) of the bare-phonon description. In Sec. IV we compare the different solutions (bare-field, physical, and RWA description) and we discuss high- and low-temperature anomalies.

II. FIELD FORMULATION OF ULLERSMA'S MODEL

A. Diagonalization of the field Hamiltonian

The model[2] describes one central oscillator ("oscillator") which is coupled to N further harmonic oscillators ("bath"). The coupling is linear in the position coordinates[15] (momentum coordinates, p,p_ν; position coordinates, q,q_ν; frequencies, ω_0,ω_ν; masses, m_0,m_ν; coupling constant, $\sqrt{\gamma_\nu}$),

$$H=\tfrac{1}{2}(p^2/m_0+m_0\omega_0^2q^2)+\sum_{\nu=1}^{N}\tfrac{1}{2}(p_\nu^2/m_\nu+m_\nu\omega_\nu^2q_\nu^2)+\sum_{\nu=1}^{N}(\gamma_\nu m_\nu m_0)^{1/2}\omega_\nu q q_\nu \;. \tag{2.1}$$

In the following four steps, Ullersma's method[2] leads to the damped harmonic oscillator.

(i) The discrete Heisenberg equations of motion are exactly solved; i.e., the Hamiltonian (2.1) has to be diagonalized:

$$H=\sum_{n=1}^{N+1}\tfrac{1}{2}(p_n^2+z_n^2q_n^2) \;. \tag{2.2}$$

(ii) The initial conditions of the bath oscillators are characterized by thermal Bose occupation numbers (temperature T).

(iii) The discrete eigenvalues z_n^2 are replaced by a continuous set (the transition is performed in the solutions of the equations of motion and requires operations in the complex plane).

(iv) The introduction of a smooth density of states for the bath oscillators is inherently connected with step (iii):

$$\sum_\nu(\cdots)\rightarrow\int_0^\infty(\cdots)D(\omega)d\omega;\;\;\gamma_\nu\rightarrow\gamma(\omega) \;. \tag{2.3}$$

The irreversible solutions which follow from the above sketched method can be obtained more directly by a field model. The spectrum of the underlying Hamiltonian consists of one discrete eigenvalue and a set of continuous eigenvalues. A formal procedure turns the Hamiltonian (2.1) into the following desired field model.

(i) We perform the canonical transformation

$$p_\nu\rightarrow\bar{p}_\nu\sqrt{\omega_\nu m_\nu},\;\; q_\nu\rightarrow\bar{q}_\nu/\sqrt{\omega_\nu m_\nu} \;.$$

(ii) We make the transition to a continuous bath spectrum

$$\sum_\nu\rightarrow\int_0^\infty d\nu,\;\;[\bar{q}_\nu,\bar{p}_\mu]=i\hbar\delta(\nu-\mu) \;.$$

(iii) We consider ω_ν as a continuous function of ν and introduce the density of states

$$\omega_\nu=f(\nu)\Longrightarrow d\nu=D(\omega_\nu)d\omega_\nu;\;\;D(\omega_\nu)=\frac{df^{-1}(\omega_\nu)}{d\omega_\nu} \;.$$

(iv) We define the new bath variables (the continuous bath frequency is denoted by k: $\omega_\nu\rightarrow k$)

$$q_k=[D(k)]^{1/2}\bar{q}_\nu,\;\; p_k=[D(k)]^{1/2}\bar{p}_\nu \;,$$

and the corresponding canonical commutation relations

$$[q_k,p_{k'}]=i\hbar\delta(k-k') \;. \tag{2.4}$$

In the new Hamiltonian H_F one discrete (bare) oscillator is coupled to a Bose field:[16]

$$H_F=\tfrac{1}{2}(p^2/m_0+m_0\omega_0^2q^2)+\tfrac{1}{2}\int_0^\infty dk\,k(p_k^2+q_k^2)+\int_0^\infty dk\,\varepsilon(k)qq_k \;. \tag{2.5}$$

The new coupling constant $\varepsilon(k)$ is related to the old one $[\gamma_\nu\rightarrow\gamma(k)]$

$$\varepsilon(k)=[D(k)\gamma(k)km_0]^{1/2} \;. \tag{2.6}$$

Now we introduce the particle operators $a,a^\dagger$ and $b_k,b_k^\dagger$:

$$a=(2\hbar m_0\omega_0)^{-1/2}(m_0\omega_0q+ip) \;, \tag{2.7}$$

$$b_k=(2\hbar)^{-1/2}(q_k+ip_k) \;, \tag{2.8}$$

and fix the initial conditions. We assume that at $t=0$ the oscillator and the field are decoupled and that the field represents a heatbath

$$\langle b_k\rangle_0=0 \;, \tag{2.9a}$$

$$\langle b_k^\dagger b_{k'}\rangle_0=\delta(k-k')\langle n_k\rangle_{\rm th} \;, \tag{2.9b}$$

where

$$\langle n_k\rangle_{\rm th}=\left[\exp\left[\frac{\hbar k}{k_BT}\right]-1\right]^{-1} \;. \tag{2.10}$$

The procedure of diagonalization corresponds to the method of Eganova and Shirokov.[13(a)] We perform at first a canonical transformation which diagonalizes the momentum part of H_F and secondly an orthogonal transformation U which diagonalizes the quadratic form of the position variables. The matrix U consists of one discrete column and a set of continuous columns and satisfies the conditions of orthogonality $U^TU=UU^T=1$:

$$U=(U_{1\omega},U_{k\omega}) \;. \tag{2.11}$$

Correspondingly, the diagonalized Hamiltonian consists of a continuum of positive eigenvalues $(0<\omega<\infty)$

$$H_F^{(D)}=\tfrac{1}{2}\int_0^\infty d\omega(p_\omega^2+\omega^2q_\omega^2) \;. \tag{2.12}$$

The corresponding "particle" operators are given by

$$b_\omega=(\omega/2\hbar)^{1/2}q_\omega+i(2\hbar\omega)^{-1/2}p_\omega \;. \tag{2.13}$$

The transformation U (Ref. 17) can be represented in terms of the complex impedance $\kappa_b(\omega)$ (b represents bare) and the generalized susceptibility $\chi_b(\omega)$. The impedance is determined by the density of states and the coupling constant

$$\kappa_b'(\omega)=(\pi/2)D(\omega)\gamma(\omega) , \quad (2.14a)$$

$$\kappa_b''(\omega)=-(2/\pi)\omega P\int_0^\infty dk\frac{\kappa_b'(k)}{k^2-\omega^2} . \quad (2.14b)$$

The real and imaginary parts of κ_b are related by the Kramers-Kronig dispersion relation. Consequently, the analytically continued function $\kappa_b(z)$ $(z=\omega'+i\omega'')$ is analytical in the upper halfplane $\omega''>0$ and can only have singularities in the lower half-plane $\omega''<0$. The same statements are also valid for the generalized susceptibility (it can be shown that $\chi_b(\omega)$ is the Fourier transform of the linear response function, which is related to the perturbation Hamiltonian $H_{\rm ext}=-qF(t)$ [Ref. 2(b)])

$$\chi_b(z)=[-z^2+\Omega_b^2-iz\kappa_b(z)]^{-1} . \quad (2.15)$$

In (2.15) we have defined a renormalized frequency Ω_b:

$$\Omega_b^2=\omega_0^2-(2/\pi)\int_0^\infty d\omega\,\kappa_b'(\omega) . \quad (2.16)$$

The stability of the solution demands that Ω_b^2 must be positive.

B. The irreversible solutions

After some straightforward algebra we obtain the solutions for the particle operators $a(t)$ and $b_k(t)$ in the form[18]

$$a(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\chi_b''(\omega)\frac{\omega+\omega_0}{\omega_0}[(\omega-\omega_0)a^\dagger(0)+(\omega+\omega_0)a(0)]$$
$$+\int_0^\infty dk\left[\frac{k}{2\pi\omega_0}\kappa_b'(k)\right]^{1/2}\left[\int_{-\infty-}^{\infty-}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\omega_0+\omega}{\omega-k}2\chi_b''(\omega)b_k(0)+\int_{-\infty-}^{\infty-}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\omega_0+\omega}{\omega+k}2\chi_b''(\omega)b_k^\dagger(0)\right]$$
$$-\int_0^\infty dk\left[\frac{k}{2\pi\omega_0}\kappa_b'(k)\right]^{1/2}[(\omega_0+k)\chi_b(k)e^{-ikt}b_k(0)+(\omega_0-k)\chi_b^*(k)e^{ikt}b_k^\dagger(0)] , \quad (2.17)$$

$$b_k(t)=e^{-ikt}b_k(0)-\left[\frac{k}{2\pi\omega_0}\kappa_b'(k)\right]^{1/2}\chi_b(k)e^{-ikt}[(k+\omega_0)a(0)+(k-\omega_0)a^\dagger(0)]$$
$$+\left[\frac{2k}{\pi\omega_0}\kappa_b'(k)\right]^{1/2}\int_{-\infty-}^{\infty-}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\chi_b''(\omega)}{\omega-k}[(\omega+\omega_0)a(0)+(\omega-\omega_0)a^\dagger(0)]+\left[\frac{2k}{\pi}\kappa_b'(k)\right]^{1/2}$$
$$\times\int_0^\infty dk'\left[\frac{2k'}{\pi}\kappa_b'(k')\right]^{1/2}\int_{-\infty-}^{\infty-}\frac{d\omega}{2\pi}e^{-i\omega t}\chi_b''(\omega)\left[\frac{b_k(0)}{(\omega-k)(\omega-k')}+\frac{b_{k'}^\dagger(0)}{(\omega-k)(\omega+k')}\right]$$
$$+\left[\frac{2k}{\pi}\kappa_b'(k)\right]^{1/2}\tfrac{1}{2}P\int_0^\infty dk'\left[\frac{2k'}{\pi}\kappa_b'(k')\right]^{1/2}\left[\left[\frac{\chi_b^*(k')}{k'+k}e^{ik't}-\frac{\chi_b(k)}{k'+k}e^{-ikt}\right]b_{k'}^\dagger(0)\right.$$
$$\left.+\left[\frac{\chi_b(k')}{k-k'}e^{-ik't}-\frac{\chi_b(k)}{k-k'}e^{-ikt}\right]b_{k'}(0)\right] . \quad (2.18)$$

The solutions (2.17) and (2.18) are consistent in the sense that for all times $t\geq 0$ the canonical commutation relations are exactly valid. Of course, this is a consequence of the orthogonality conditions. All terms in (2.17) and (2.18) which include a ω integration are exponentially damped [all possible singularities in $\chi_b(\omega)$ have a negative imaginary part]. We can infer that the time-dependent nonequilibrium values of all physical quantities referring to the central oscillator approach their equilibrium values exponentially.[19] The long-time behavior of $a(t)$ is described by the last term of (2.17), which is driven by the initial values of the bath operators. Equation (2.18) reveals that for $t\to\infty$ not only the free or diagonal part is left [first term in (2.18)]. There are also contributions from all other bath oscillators [nondiagonal parts, last term in (2.18)], and the initial values of the central oscillator "are not forgotten" [second term in (2.18)]. We can return to the discrete bath in (2.17) and (2.18) by the replacement

$$\int_0^\infty dk\left[\frac{2}{\pi}\kappa_b'(k)\right]^{1/2}\to\sum_\nu\gamma_\nu$$

[additionally, we have to replace $\delta(k-k')$ by $\delta_{k,k'}$ in (2.9)]. In this discrete version the solution (2.17) coincides with Ullersma's result.[20]

C. The particle interpretation

We emphasize that $a(t)$ and $b_k(t)$ are not only determined by the initial values of the annihilation operators but also by the initial values of the creation operators: For the expansion of a and b_k we need the annihilation and the creation operators of the diagonalized Hamiltonian

$$a=\tfrac{1}{2}\int_0^\infty d\omega\,U_{1\omega}[(\sqrt{\omega_0/\omega}+\sqrt{\omega/\omega_0})b_\omega$$
$$+(\sqrt{\omega_0/\omega}-\sqrt{\omega/\omega_0})b_\omega^\dagger] , \quad (2.19)$$

$$b_k=\frac{1}{2}\int_0^\infty d\omega\, U_{k\omega}[(\sqrt{k/\omega}+\sqrt{\omega/k})b_\omega+(\sqrt{k/\omega}-\sqrt{\omega/k})b_\omega^\dagger] . \quad (2.20)$$

From (2.19) and (2.20) it follows that the ground state of the diagonalized Hamiltonian (2.12), which represents the vacuum of the "universe," i.e., the state of lowest energy ("physical" vacuum), and which is defined by the relation

$$b_\omega\,|\,\text{vac}\rangle=0 , \quad (2.21)$$

does not coincide with the ground state $|\,0\rangle$ of the operators a and b_k ($a\,|\,0\rangle=b_k\,|\,0\rangle=0$). Consequently, we find in the physical vacuum excitations of the oscillator

$$\langle \text{vac}\,|\,a^\dagger a\,|\,\text{vac}\rangle=\frac{1}{4}\int_0^\infty d\omega\, U_{1\omega}^2\frac{(\omega_0-\omega)^2}{\omega\omega_0}=\int_0^\infty\frac{d\omega}{2\pi}\frac{(\omega_0-\omega)^2}{\omega_0}\chi_b''(\omega) . \quad (2.22)$$

In the physical vacuum no physical particles (phonons) should be present and $a^\dagger a$ cannot be interpreted as a physical particle number operator.[13(a)] Therefore, we will denote $a^\dagger a$ as a bare particle number operator and we will speak of a bare-field description of the relaxation process.

The positive expression (2.22) corresponds to a negative value of the interaction energy

$$\left\langle \text{vac}\left|\int_0^\infty dk\,\varepsilon(k)qq_k\right|\text{vac}\right\rangle=\frac{\hbar}{2}\int_0^\infty d\omega\int_0^\infty dk\left[\frac{2}{\pi}\kappa_b'(k)\right]^{1/2}\frac{k}{\omega}U_{1\omega}U_{k\omega}$$
$$=-\hbar\int_0^\infty\frac{d\omega}{\pi}\kappa_b'(\omega)\omega\frac{2}{\pi}\int_0^\infty dk\,\chi_b''(\omega)(k+\omega)^{-1}$$
$$=-\hbar\int_0^\infty\frac{d\omega}{\pi}\chi_b''(\omega)(\omega_0^2-\omega^2) . \quad (2.23)$$

Eganova and Shirokov have pointed out that a Hamiltonian which describes the linear coupling of one harmonic oscillator (oscillator) to a harmonic Bose-field (bath), and which allows a physical particle interpretation should be symmetrical in momentum ($\tilde{p},\tilde{p}_k$) and position ($\tilde{q},\tilde{q}_k$) variables:[13(a)] $H_{\text{ph}}=f(\tilde{p},\tilde{p}_k)+f(\tilde{q},\tilde{q}_k)$. Owing to this symmetry, the particle operators have the form $\tilde{a}\sim(2\hbar)^{-1/2}(\tilde{q}+i\tilde{p})$, $\tilde{b}_k\sim(2\hbar)^{-1/2}(\tilde{q}_k+i\tilde{p}_k)$. Obviously, we can find an orthogonal transformation M which diagonalizes the momentum and the position part of H_{ph} simultaneously:

$$\begin{bmatrix}\tilde{p}\\ \tilde{p}_k\end{bmatrix}=\int d\omega\begin{bmatrix}M_{1\omega}\\ M_{k\omega}\end{bmatrix}\tilde{p}_\omega,\quad \begin{bmatrix}\tilde{q}\\ \tilde{q}_k\end{bmatrix}=\int d\omega\begin{bmatrix}M_{1\omega}\\ M_{k\omega}\end{bmatrix}\tilde{q}_\omega .$$

We obtain the diagonalized Hamiltonian $H_{\text{ph}}^{(D)}=\int d\omega\,\omega(\tilde{p}_\omega^2+\tilde{q}_\omega^2)$ and the corresponding operators $\tilde{b}_\omega\sim(2\hbar)^{-1/2}(\tilde{q}_\omega+i\tilde{p}_\omega)$. Now it is clear that $\tilde{a}$ and $\tilde{b}_k$ can solely be expressed by the annihilation operators $\tilde{b}_\omega$:

$$\tilde{a}\sim(\tilde{q}+i\tilde{p})\sim\int d\omega\,M_{1\omega}(\tilde{q}_\omega+i\tilde{p}_\omega)\sim\int d\omega\,M_{1\omega}\tilde{b}_\omega ,$$
$$b_k\sim\int d\omega\,M_{k\omega}\tilde{b}_\omega .$$

The expectation value of $\tilde{a}^\dagger\tilde{a}$ vanishes in the physical vacuum and the definition of $\tilde{a}^\dagger\tilde{a}$ as a physical particle number operator is possible. In Sec. III these considerations will lead us to the physical description of the quantum relaxation process.

D. Some equilibrium properties

For $t\to\infty$ we find the correlation function

$$[\omega\kappa_b'(\omega)\,|\,\chi_b(\omega)\,|^2=\chi_b''(\omega)]$$

(Ref. 21):

$$\langle a^\dagger(t)a(t')\rangle_\infty=\int_0^\infty\frac{dk}{2\pi}\chi_b''(k)\omega_0[(1+k/\omega_0)^2e^{ik(t-t')}\langle n_k\rangle_{\text{th}}+(1-k/\omega_0)^2e^{-ik(t-t')}\times(\langle n_k\rangle_{\text{th}}+1)] , \quad (2.24)$$

and the appropriate energy of the oscillator in thermal equilibrium

$$E_{\text{os}}^{(b)}(t\to\infty)=\hbar\omega_0[\langle a^\dagger(t)a(t)\rangle_\infty+\tfrac{1}{2}] . \quad (2.25)$$

It can easily be seen that the correlation functions $\langle a(t)a(t')\rangle_\infty$ and $\langle a^\dagger(t)a^\dagger(t')\rangle_\infty$ do not vanish.

The interaction energy ($t\to\infty$) can be derived from (2.17) and (2.18)

$$E_{\text{int}}(t\to\infty)=\int_0^\infty dk\,\varepsilon(k)\langle q(t)q_k(t)\rangle_\infty$$
$$=-\int_0^\infty\frac{d\omega}{\pi}\hbar\omega\kappa_b'(\omega)\frac{2}{\pi}\int_0^\infty d\mu\frac{\chi_b''(\mu)}{\mu+\omega}-2\int_0^\infty\frac{d\omega}{\pi}\hbar\omega\kappa_b'(\omega)\frac{2}{\pi}\int_0^\infty d\mu\frac{\chi_b''(\mu)}{\mu^2-\omega^2}(\mu\langle n_\omega\rangle_{\text{th}}-\omega\langle n_\mu\rangle_{\text{th}})$$
$$=-\hbar\int_0^\infty\frac{d\omega}{\pi}\chi_b''(\omega)2(\omega_0^2-\omega^2)(\langle n_\omega\rangle_{\text{th}}+\tfrac{1}{2}) . \quad (2.26)$$

For the calculation of (2.24)–(2.26) we have used the Heisenberg picture. The comparison of (2.24) and (2.26) with (2.22) and (2.23) (Schrödinger picture) shows us that for $T=0$ and $t\to\infty$ the system reaches its physical ground state. Therefore, the results (2.24) and (2.26) can also be derived on the assumption that at $t=0$ the diagonalized system is in thermal equilibrium [one starts with (2.19) and (2.20), uses the time dependence $b_\omega(t)$

$=b_\omega(0)\exp(-i\omega t)$ and assumes that the operators b_ω and $b_\omega^\dagger$ fulfill the bath condition (2.9)].[22] From (2.26) it can be seen that the interaction energy exactly equals the double difference of kinetic and potential energy.

We have mentioned above that for $T=0$ the positive contribution of (2.24) is made possible by the negative interaction energy (2.26). Of course, the bath cannot transfer (positive) energy to the oscillator at $T=0$. The temperature-dependent part of the interaction energy is also negative: $(\mu\langle n_\omega\rangle_{\rm th}-\omega\langle n_\mu\rangle_{\rm th})(\mu-\omega)^{-1}>0$.[23] The consideration of only the free part in (2.18) [first term in (2.18)] would have led to a complex expression (real and imaginary part) for the interaction energy. The result (2.26) crucially depends on the nondiagonal part of (2.18).

The analytical properties of $\kappa_b(\omega)$ allow us to represent the long-time solutions $(t\to\infty)$ in the form of a stochastic integro-differential equation, which is in harmony with causality:[11,24]

$$p=m_0\dot q \ , \tag{2.27}$$

$$\ddot q(t)+\Omega_b^2 q(t)+\int_{-\infty}^{t} dt'\,\bar\kappa_b(t-t')\dot q(t')=f_{\rm st}(t) \ , \tag{2.28}$$

where

$$\bar\kappa_b(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\kappa_b(\omega)e^{-i\omega t} \ . \tag{2.29}$$

$\bar\kappa_b(t)$ vanishes for $t<0$ and, consequently, $\kappa_b(\omega)$ can be described as the Laplace transform of an *a priori* given integral kernel

$$\kappa_b(\omega)=\int_0^\infty dt\,\bar\kappa_b(t)e^{i\omega t} \ . \tag{2.30}$$

The statistical properties of the stochastic operator $f_{\rm st}$ can easily be found. We insert the solution $q(t\to\infty)$ in (2.28) and use the initial condition (2.9)

$$\langle f_{\rm st}(t)\rangle=0 \ , \tag{2.31}$$

$$\langle f_{\rm st}(t)f_{\rm st}(t')\rangle =\frac{1}{m_0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}2\hbar\omega\kappa_b'(\omega)(\langle n_\omega\rangle_{\rm th}+1)e^{-i\omega(t-t')} \ . \tag{2.32}$$

The spectrum of the dissipative kernel, more exactly its real part $\kappa_b'(\omega)$, determines the spectrum of the correlation function of the stochastic operator. It has been shown by the author[11] that this fluctuation-dissipation relation represents nothing else than the well-known Callen-Welton-Kubo fluctuation-dissipation theorem.[25] For arbitrary times $t\geqq 0$ there is no simple dynamic equation which would exactly reproduce the decay process (2.17). Nevertheless, there are stochastic differential equations whose solutions approximate very well to the exact solutions in the weak damping limit.[11]

III. THE PHYSICAL DESCRIPTION

A. The transformations

In Sec. II C we have seen that the transformed Hamiltonian $H_{\rm ph}$ should represent one oscillator which is coupled to a Bose field and should be symmetrical in the new position and momentum variables. A simple canonical transformation which will succeed has the form

$$q=m_0^{-1/2}\xi\tilde q \ , \tag{3.1a}$$

$$p=m_0^{1/2}\left[\xi^{-1}\tilde p+\int_0^\infty dk\,\xi(k)p_k'\right] \ , \tag{3.1b}$$

$$q_k=k^{-1/2}[q_k'-\xi\xi(k)\tilde q] \ , \tag{3.1c}$$

$$p_k=k^{1/2}p_k' \ . \tag{3.1d}$$

ξ and $\xi(k)$ are real numbers which must be determined below. The insertion of (3.1) in (2.5) leads to a Hamiltonian which is no longer diagonal with respect to the $(p_k')^2$ terms. In the next step we diagonalize these terms by an orthogonal transformation $X=(X_{k\nu})$ and we introduce the corresponding new canonical variables $\tilde q_\nu$ and $\tilde p_\nu$. The calculation of X parallels the work of Eganova and Shirokov [see Appendix A in Ref. 13(a)].[26]

We demand that $H_{\rm ph}$ should be symmetrical with respect to the new momentum and position variables. We obtain the conditions[27]

$$(2\xi^2)^{-1}=(\xi^2/2)\left\{\omega_0^2+\int_0^\infty dk\left[\xi^2(k)-2\xi(k)\left[\frac{2}{\pi}\kappa_b'(k)\right]^{1/2}\right]\right\} =\Omega/2 \ , \tag{3.2}$$

$$(\xi\sqrt{\nu})^{-1}\int_0^\infty dk\,\xi(k)X_{k\nu} =\xi\sqrt{\nu}\int_0^\infty dk\left[\left[\frac{2}{\pi}\kappa_b'(k)\right]^{1/2}-\xi(k)\right]X_{k\nu} =(2\sqrt{\Omega m_0})^{-1}E(\nu) \ . \tag{3.3}$$

In (3.2) and (3.3) we defined a "dressed" frequency Ω and a dressed coupling function $E(\nu)$. The physical Hamiltonian has the desired structure

$$H_{\rm ph}=(\Omega/2)(\tilde p^2+\tilde q^2)+\tfrac12\int_0^\infty d\nu\,\nu(\tilde p_\nu^2+\tilde q_\nu^2) +\int_0^\infty d\nu\,E(\nu)(4m_0\Omega)^{-1/2}(\tilde q\tilde q_\nu+\tilde p\tilde p_\nu) \ . \tag{3.4}$$

$H_{\rm ph}$ can be expressed in terms of the physical particle operators

$$\tilde a=(2\hbar)^{-1/2}(\tilde q+i\tilde p) \ , \tag{3.5}$$

$$\tilde b_\nu=(2\hbar)^{-1/2}(\tilde q_\nu+i\tilde p_\nu): \tag{3.6}$$

$$H_{\rm ph}/\hbar=(\Omega/2)(\tilde a^\dagger\tilde a+\tilde a\tilde a^\dagger)+\tfrac12\int_0^\infty d\nu\,\nu(\tilde b_\nu^\dagger\tilde b_\nu+\tilde b_\nu\tilde b_\nu^\dagger) +\int_0^\infty d\nu(4m_0\Omega)^{-1/2}E(\nu)(\tilde a^\dagger\tilde b_\nu+\tilde a\tilde b_\nu^\dagger) \ . \tag{3.7}$$

In the following we demand that at $t=0$ the dressed field represents a heatbath:

$$\langle\tilde b_\nu\rangle_0=0 \ , \tag{3.8a}$$

$$\langle\tilde b_\nu^\dagger\tilde b_\mu\rangle_0=\delta(\nu-\mu)\langle n_\nu\rangle_{\rm th} \ . \tag{3.8b}$$

The physical Hamiltonian (3.7) is closely related to the rotating-wave approximation $(H_{\rm RWA})$ of the initial Hamiltonian H_F. If we introduce the bare particle operators

(2.7) and (2.8) in (2.5) and if we omit the "fast oscillating terms" ab_k and $a^\dagger b_k^\dagger$, we obtain the exact form of (3.7), where the renormalized quantities Ω and $E(\nu)$ are replaced by the bare quantities ω_0 and $\varepsilon(\nu)$. We emphasize that the ground state of $H_{\rm RWA}$ does not coincide with the ground state of H_F and $H_{\rm ph}$: $H_{\rm RWA}$ and H_F are not connected by a canonical transformation.

B. The dressed quantities Ω and $E(\nu)$

For the calculation of ξ and $\xi(k)$, we put $X_{k\nu}$ (Ref. 26) into (3.7). After some simple algebraic manipulations we find a nonlinear integral equation for $\xi(\nu)$

$$\xi(\nu)=(2/\pi)^{1/2}\frac{\nu\xi^2}{1+\nu\xi^2} \times \left\{ [\kappa_b'(\nu)]^{1/2} + P\int_0^\infty dk\frac{\xi(k)}{\nu^2-k^2}\{[\kappa_b'(k)]^{1/2}\xi(\nu) - [\kappa_b'(\nu)]^{1/2}\xi(k)\}\right\} . \quad (3.9)$$

From (3.9) we obtain $\xi(\nu)$ as a function of ξ. The insertion of $\xi(\nu)$ into (3.2) leads to an equation for ξ. In order to find an explicit relation between the dressed and the bare quantities, we make the following assumptions

(i) $\kappa_b'(\nu)$ should be small, so that a smallness parameter 2Γ $[2\Gamma\approx\kappa_b'(\omega_o)]$ can be attributed to $\kappa_b'(\nu)$. "Small" means that

$$\Gamma/\omega_0 \ll 1 . \quad (3.10)$$

Then (3.9) can be solved by iteration. For the zeroth and first approximation we find

$$\xi^{(0)}(\nu)=[2\kappa_b'(\nu)/\pi]^{1/2}\nu\xi^2(1+\nu\xi^2)^{-1} , \quad (3.11)$$

$$\xi^{(1)}(\nu)=\xi^{(0)}(\nu)\left[1+\frac{2}{\pi}\frac{1}{1+\nu\xi^2}\int_0^\infty dk\frac{k\kappa_b'(k)}{(k+\nu)(k+1/\xi^2)^2}\right]. \quad (3.12)$$

In the following we will limit our discussion to (3.11). The insertion of (3.11) into (3.2) leads to a transcendental equation for ξ:

$$\xi^{-4}\left[1-\frac{2}{\pi}\int_0^\infty dk\,\kappa_b'(k)(k+1/\xi^2)^{-2}\right]=\Omega_b^2 . \quad (3.13)$$

The second term in the parentheses of (3.13) is small compared to 1.

(ii) We define a cutoff parameter[28] $(\pi/2)\gamma$

$$(2/\pi)\int_0^\infty dk\,\kappa_b'(k)=2\Gamma\gamma<\omega_0^2 , \quad (3.14)$$

and we assume the inequality

$$\gamma \gg \omega_0 . \quad (3.15)$$

From (3.14) it follows that

$$\Omega_b^2=\omega_0^2-2\Gamma\gamma . \quad (3.16)$$

We point out that $2\Gamma\gamma/\omega_0^2$ need not to be very small compared with one.

(iii) $\kappa_b'(\nu)$ should be a slowly varying function of ν, so that we can replace $\kappa_b'(k)$ in (3.13) by $\kappa_b'(\omega_0)\approx 2\Gamma$.[29] The integration in (3.13) can be performed and we obtain the quadratic equation $\Omega^2-(2/\pi)\kappa_b'(\omega_0)\Omega-\Omega_b^2=0$ which yields the first-order result

$$\Omega=\Omega_b+\frac{1}{\pi}\kappa_b'(\omega_0) . \quad (3.17)$$

The consideration of γ would have led in (3.17) to additional terms of the order $\omega_0\Gamma/\gamma$. Obviously, the inequality $\Omega-\Omega_b \ll |\Omega_b-\omega_0|$ is valid.

The left-hand side of (3.3) only depends on $\xi=\Omega^{-1/2}$ and $\xi(k)$.[26] We replace $\xi(k)$ by $\xi^{(0)}(k)$ and we find the lowest-order result

$$(2\sqrt{\Omega m_0})^{-1}E(\nu)=\left[\frac{2}{\pi}\kappa_b'(\nu)\right]^{1/2}\sqrt{\nu/\Omega}(1+\nu/\Omega)^{-1} . \quad (3.18)$$

Equation (3.18) corresponds to the simple formula [see (2.6) and (2.14)]

$$E(\nu)=2\varepsilon(\nu)(1+\nu/\Omega)^{-1} . \quad (3.19)$$

The comparison of the bare and the physical relaxation process in Sec. IV will be based on the relations (3.17) and (3.18).

C. The irreversible solutions and some consequences

We have explained above that one orthogonal transformation M will lead to the diagonalized form of $H_{\rm ph}$:

$$H_{\rm ph}^{(D)}=\int_0^\infty d\omega\,\omega(\tilde p_\omega^2+\tilde q_\omega^2) . \quad (3.20)$$

The explicit form of M was calculated in Appendix B of Ref. 13(a). The definition of the quantities

$$\kappa_d(\omega)=\pi\tilde E^2(\omega)-i\omega P\int_0^\infty d\nu\frac{\tilde E^2(\nu)/\nu}{\nu-\omega} \quad (3.21)$$

("impedance") and

$$\chi_d(\omega)=[-\omega+\Omega_d-i\kappa_d(\omega)]^{-1} \quad (3.22)$$

("susceptibility"), allows the proper representation of M.[30] In (3.21) and (3.22) we introduced the coupling function $\tilde E(\nu)$:

$$\tilde E(\nu)=(2\sqrt{m_0\Omega})^{-1}E(\nu) , \quad (3.23)$$

and the renormalized frequency Ω_d

$$\Omega_d=\Omega-\int_0^\infty d\nu\,\tilde E^2(\nu)/\nu . \quad (3.24)$$

The stability of the solutions demands that $\Omega_d\geq 0$.

Above, we have placed the words impedance and susceptibility in quotation marks: The analytically continued functions $\kappa_d(z)$ and $\chi_d(z)$ need not be analytical in the upper z half-plane. The real and imaginary parts of $\kappa_d(\omega)$ do not fulfill the Kramers-Kronig dispersion relations. Furthermore, $\chi_d''(\omega)$ is not an odd function of ω.

In order to reveal the analytical structure of the problem, we introduce the new variable $x^2=\omega$ and the function $\bar{\kappa}(x)$ $(x>0)$:

$$\kappa_d(\omega=x^2)=x\bar{\kappa}(x)\ . \tag{3.25}$$

From (3.21) it follows:

$$\bar{\kappa}(x)=\pi\tilde{E}^2(\omega=x^2)/x-2ix\mathrm{P}\int_0^\infty dy\frac{\tilde{E}^2(\nu=y^2)/y}{y^2-x^2}\ . \tag{3.26}$$

We see that $\bar{\kappa}'(x)$ and $\bar{\kappa}''(x)$ are connected by the Kramers-Kronig relations and, consequently, the analytically continued function $\bar{\kappa}$ can have no singularities in the upper x half-plane. Correspondingly, the function $\bar{\chi}(x)=\chi_d(\omega=x^2)$ is analytical in the upper x half-plane.

In contrast to Sec. II B we cannot extend the integration range to the negative ω axis. Nevertheless, the corresponding integrals can be evaluated by contour integrations. To begin with we shift the integration lines infinitesimally into the negative ω half-plane and express the irreversible solutions in terms of the above introduced quantities $\kappa_d(\omega)$ and $\chi_d(\omega)$:

$$\tilde{a}(t)=\tilde{a}(0)\int_0^\infty\frac{d\omega}{\pi}e^{-i\omega t}\chi_d''(\omega)+\int_0^\infty d\nu[\kappa_d'(\nu)/\pi]^{1/2}\tilde{b}_\nu(0)\int_{0-}^{\infty-}\frac{d\omega}{\pi}e^{-i\omega t}\frac{\chi_d''(\omega)}{\omega-\nu}$$
$$-\int_0^\infty d\nu\, e^{-i\nu t}[\kappa_d'(\nu)/\pi]^{1/2}\chi_d(\nu)\tilde{b}_\nu(0)\ , \tag{3.27}$$

$$\tilde{b}_\nu(t)=e^{-i\nu t}\tilde{b}_\nu(0)-[\kappa_d'(\nu)/\pi]^{1/2}\chi_d(\nu)e^{-i\nu t}\tilde{a}(0)+[\kappa_d'(\nu)/\pi]^{1/2}\int_{0-}^{\infty-}\frac{d\omega}{\pi}e^{-i\omega t}\frac{\chi_d''(\omega)}{\omega-\nu}\tilde{a}(0)$$
$$+[\kappa_d'(\nu)/\pi]^{1/2}\int_0^\infty d\mu\,\tilde{b}_\mu(0)[\kappa_d'(\mu)/\pi]^{1/2}\int_{0-}^{\infty-}\frac{d\omega}{\pi}e^{-i\omega t}\frac{\chi_d''(\omega)}{(\omega-\nu)(\omega-\mu)}$$
$$+[\kappa_d'(\nu)/\pi]^{1/2}\mathrm{P}\int_0^\infty d\mu\,\tilde{b}_\mu(0)[\kappa_d'(\mu)/\pi]^{1/2}\frac{\chi_d(\nu)e^{-i\nu t}-\chi_d(\mu)e^{-i\mu t}}{\mu-\nu}\ . \tag{3.28}$$

The terms in (3.27) and (3.28) which contain an integration over ω vanish for $t\to\infty$. In contrast to Sec. II B the physical decay process (3.27) is not strictly exponential. We close the integration line by a quarter circle from ∞ to $-i\infty$ $(t<0)$ and by the imaginary axis from $-i\infty$ to 0 to a contour C. For the sake of simplicity we assume that $\chi_d(\omega)$ has no singularities on the imaginary axis. Then we can write

$$\int_{0-}^{\infty-}d\omega\, e^{-i\omega t}f(\omega)=\oint_c d\omega\, e^{-i\omega t}f(\omega)-\int_{-i\infty}^0 d\omega\, e^{-i\omega t}f(\omega)\ . \tag{3.29}$$

Whereas the contour integration in (3.29) describes a strictly exponential decay, the second term in (3.29) can cause an algebraic decay

$$-\int_{-i\infty}^0 d\omega\, f(\omega)e^{-i\omega t}=-\frac{i}{t}\int_0^\infty dp\, f[\omega=-(ip/t)]e^{-p}\ . \tag{3.30}$$

In contrast to the bare solutions (2.17) and (2.18), the physical solutions (3.27) and (3.28) are only determined by the initial values of the *annihilation* operators. There are no physical particles in the physical vacuum.

If we replace in (3.27) and (3.28) $\kappa_d'(\nu)$ with $\kappa_b'(\nu)$ and $\chi_d(\omega)$ with

$$\chi_b^{(\mathrm{RWA})}(\omega)=\left[-\omega+\Omega_b^{(\mathrm{RWA})}-i\left[\frac{\omega}{2\omega_0}\kappa_b'(\omega)-i\frac{\omega}{2\omega_0}\frac{1}{\pi}\mathrm{P}\int_0^\infty d\nu\frac{\kappa_b'(\nu)}{\nu-\omega}\right]\right]^{-1}, \tag{3.31}$$

where

$$\Omega_b^{(\mathrm{RWA})}=\omega_0-\frac{1}{2\omega_0}\frac{1}{\pi}\int_0^\infty d\nu\,\kappa_b'(\nu)=\omega_0-\frac{\Gamma\gamma}{2\omega_0}\ , \tag{3.32}$$

we obtain exactly the solutions of the RWA version of H_F (2.5). Equation (3.31) has the same analytical properties as $\chi_d(\omega)$ and, consequently, the RWA solutions will not decay strictly exponentially.[31]

D. Nonexponential decay

We consider times for which only the nonexponentially decaying terms are essential. We obtain the asymptotic solution

$$[\tilde{a}(t)]_{\mathrm{asy}}=\tilde{a}(0)\left[-\frac{i}{t}\int_0^\infty\frac{dp}{\pi}\chi_d''\left[-\frac{ip}{t}\right]e^{-p}\right]-\int_0^\infty d\nu\, e^{-i\nu t}[\kappa_d'(\nu)/\pi]^{1/2}\chi_d(\nu)\tilde{b}_\nu(0)+\int_0^\infty d\nu[\kappa_d'(\nu)/\pi]^{1/2}\tilde{b}_\nu(0)\times\left[-\frac{i}{t}\int_0^\infty\frac{dp}{\pi}\frac{\chi_d''\left[-\frac{ip}{t}\right]}{-ip/t-\nu}e^{-p}\right]. \tag{3.33}$$

In the limit of large times we need the behavior of $\chi_d''(\omega)$ for $\omega\to0$. We assume that $\kappa_b'(\omega)=(\pi/2)D(\omega)\gamma(\omega)$ tends to $2\Gamma(\omega/\alpha)^\lambda$, $\lambda\geq0$, for $\omega\to0$. The positive quantity α denotes a characteristic frequency $(\Gamma\ll\alpha\ll\gamma)$. Then it follows from (3.18), (3.23), and (3.21): $\kappa_d'(\omega)\to4\Gamma(\omega/\Omega)(\omega/\alpha)^\lambda$ as $\omega\to0$. Now we note that $\kappa_d''(\omega)$ vanishes for $\omega\to0$. We find the result

$$\chi_d''(\omega)\to 4\Gamma\frac{\omega}{\Omega\Omega_d^2}\left[\frac{\omega}{\alpha}\right]^{\lambda}\quad \text{as } \omega\to 0\ . \tag{3.34}$$

For finite temperatures we will only discuss the special case $\lambda=0$ (Ohmic dissipation). The p integrations in (3.33) can be evaluated

$$\begin{aligned}[\tilde a(t)]_{\text{asy}}= &-\int_0^\infty d\nu[\kappa_d'(\nu)/\pi]^{1/2}\chi_d(\nu)e^{-i\nu t}\tilde b_\nu(0)\\ &-\frac{4\Gamma}{\pi\Omega}\frac{1}{(t\Omega_d)^2}\tilde a(0)\\ &+\frac{4\Gamma}{\pi\Omega\Omega_d^2}\int_0^\infty d\nu[\kappa_d'(\nu)/\pi]^{1/2}\tilde b_\nu(0)\\ &\qquad\times[\nu g(\nu t)+i\nu h(\nu t)-i/t]\ .\end{aligned} \tag{3.35}$$

In (3.35) we defined the functions

$$\begin{aligned}g(x)&=-ci(x)\cos(x)-si(x)\sin(x)\\ &\to\frac{1}{x^2}\left[1-\frac{3!}{x^2}+\cdots\right],\quad \text{as } x\to\infty\ ,\end{aligned} \tag{3.36}$$

$$\begin{aligned}h(x)&=ci(x)\sin(x)-si(x)\cos(x)\\ &\to\frac{1}{x}\left[1-\frac{2!}{x^2}+\cdots\right]\quad \text{as } x\to\infty\ .\end{aligned} \tag{3.37}$$

Equation (3.35) leads to the decay law

$$\begin{aligned}\langle\tilde a^\dagger(t)\tilde a(t)\rangle_{\text{asy}}-\int_0^\infty\frac{d\nu}{\pi}\chi_d''(\nu)\langle n_\nu\rangle_{\text{th}}
&=\left[\frac{4\Gamma}{\pi\Omega}\right]^2(t\Omega_d)^{-4}\langle\tilde a^\dagger(0)\tilde a(0)\rangle+\left[\frac{4\Gamma}{\pi\Omega\Omega_d^2}\right]^2\int_0^\infty\frac{d\nu}{\pi}\kappa_d'(\nu)\{\nu^2g^2(\nu t)+[\nu h(\nu t)-1/t]^2\}\langle n_\nu\rangle_{\text{th}}\\
&\quad-\frac{8\Gamma}{\pi\Omega\Omega_d^2}\text{Re}\int_0^\infty\frac{d\nu}{\pi}\kappa_d'(\nu)\chi_d^*(\nu)e^{i\nu t}[\nu g(\nu t)+i\nu h(\nu t)-i/t]\langle n_\nu\rangle_{\text{th}}\ .\end{aligned} \tag{3.38}$$

The terms on the right-hand side of (3.38) are of the order $0(\Gamma^2)$. Therefore, we can neglect the nonexponential contribution to the decay if we restrict the discussion to first-order results in Γ.

For $T=0$ (i.e., the bath is in the vacuum state at $t=0$) only the first term in (3.33) contributes to the decay [$\Gamma(\lambda)$ denotes the Γ function]

$$\begin{aligned}\langle\tilde a^\dagger(t)\tilde a(t)\rangle\underset{\Omega t\gg 1}{\longrightarrow}&\left[\frac{4\Gamma}{\pi\Omega}\right]^2(t\Omega_d)^{-4}(\alpha t)^{-2\lambda}\\ &\times[\Gamma(\lambda+2)]^2\langle\tilde a^\dagger(0)\tilde a(0)\rangle\ .\end{aligned} \tag{3.39}$$

It is questionable if such a small effect ($\sim\Gamma^2t^{-4-2\lambda}$) is measurable: The contribution (3.39) must be large compared with the exponential contribution

$$\langle\tilde a^\dagger(t)\tilde a(t)\rangle\approx e^{-2\Gamma t}\langle\tilde a^\dagger(0)\tilde a(0)\rangle\ . \tag{3.40}$$

Equation (3.40) is based on the assumption that the contour integrals are essentially determined by the pole $\omega=\Omega_d-i\Gamma$ in $\chi_d(\omega)$.

E. Some equilibrium properties

For $t\to\infty$ we obtain the correlation function [see (3.27)]

$$\langle\tilde a^\dagger(t)\tilde a(t')\rangle_\infty=\int_0^\infty\frac{d\nu}{\pi}\chi_d''(\nu)e^{i\nu(t-t')}\langle n_\nu\rangle_{\text{th}}\ , \tag{3.41}$$

and the corresponding energy

$$E_{\text{os}}^{(\text{ph})}(t\to\infty)=\hbar\Omega[\langle\tilde a^\dagger(t)\tilde a(t)\rangle_\infty+\tfrac{1}{2}]\ . \tag{3.42}$$

In contrast to the bare-field description, the correlation functions $\langle\tilde a(t)\tilde a(t')\rangle_\infty$ and $\langle\tilde a^\dagger(t)\tilde a^\dagger(t')\rangle_\infty$ vanish.

From (3.27) and (3.28) the interaction energy can easily be evaluated

$$\begin{aligned}E_{\text{int}}^{(\text{ph})}(t\to\infty)&=\int_0^\infty d\nu\tilde E(\nu)\langle\tilde a^\dagger\tilde b_\nu+\tilde a\tilde b_\nu^\dagger\rangle_\infty\\ &=-\hbar\int_0^\infty\frac{d\nu}{\pi}\kappa_d'(\nu)\\ &\quad\times\text{P}\left[\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\pi}\frac{\chi_d''(\omega)}{\omega-\nu}(\langle n_\nu\rangle_{\text{th}}-\langle n_\omega\rangle_{\text{th}})\right]\\ &=-\hbar\int_0^\infty\frac{d\nu}{\pi}\chi_d''(\nu)2(\Omega-\nu)\langle n_\nu\rangle_{\text{th}}\ .\end{aligned} \tag{3.43}$$

For $T=0$ and $t\to\infty$ the system is in its vacuum state $|\text{vac}\rangle$. In this state there are no physical particles and the interaction energy must vanish.

The results (3.41)—(3.43) can also be derived on the assumption that at $t=0$ the diagonalized system is in thermal equilibrium [one uses the time dependence of the bath operators $b_\omega(t)=b_\omega(0)e^{-i\omega t}$ and the initial conditions (2.9) with respect to the bath operators b_ω]. Of course, the interaction energy (3.43) is negative for all temperatures $[(\langle n_\nu\rangle_{\text{th}}-\langle n_\omega\rangle_{\text{th}})(\omega-\nu)^{-1}>0]$. As above, the result (3.43) depends essentially on the consideration of the off-diagonal contributions in (3.28) [last term in (3.28)].

Now we pose the question, if the solution (3.27) for $t\to\infty$ can be determined as the exact solution of a causal dynamical equation. It can easily be shown that with the definition

$$\kappa_d(\nu)=\int_0^\infty d\tau\tilde\kappa_d(\tau)e^{i\nu\tau}\ , \tag{3.44}$$

$\tilde a(t\to\infty)$ fulfills the dynamic equation

$$\left[-i\frac{d}{dt}+\Omega_d\right]\tilde{a}(t)-i\int_{-\infty}^{t}d\tau\tilde{\kappa}_d(t-\tau)\tilde{a}(\tau)=-\int_0^\infty d\nu\left[\frac{1}{\pi}\kappa_d'(\nu)\right]^{1/2}\tilde{b}_\nu(0)e^{-i\nu t}\ . \quad (3.45)$$

The right-hand side of (3.45) can be interpreted as a stochastic "Gaussian" operator $g_{st}(t)$ with the properties

$$\langle g_{st}(t)\rangle=\langle g_{st}^\dagger(t)\rangle=0\ , \quad (3.46a)$$

$$\langle g_{st}(t)g_{st}(t')\rangle=\langle g_{st}^\dagger(t)g_{st}^\dagger(t')\rangle=0\ , \quad (3.46b)$$

$$\langle g_{st}(t)g_{st}^\dagger(t')\rangle=\int_0^\infty\frac{d\nu}{\pi}\kappa_d'(\nu)e^{-i\nu(t-t')}(\langle n_\nu\rangle_{th}+1)\ , \quad (3.46c)$$

$$\langle g_{st}^\dagger(t)g_{st}(t')\rangle=\int_0^\infty\frac{d\nu}{\pi}\kappa_d'(\nu)e^{i\nu(t-t')}\langle n_\nu\rangle_{th}\ . \quad (3.46d)$$

In order to construct the equation (3.45) we have to prescribe the way from an *a priori* given function $\kappa_d(\nu)$ to the function $\tilde{\kappa}_d(\tau)$; i.e., we have to invert (3.44). On the one hand we know that in general $\kappa_d(\nu)$ can have singularities in the upper ν half-plane. On the other hand, the analytical continuation of (3.44) leads to a function which must be analytical in the upper half-plane [we assume that $\tilde{\kappa}_d(\tau)$ is a reasonable linear response function]. Consequently, we cannot find a reasonable function $\tilde{\kappa}_d(\tau)$ in general. There is only the possibility that we abandon the causality and that we extend the τ integration range in (3.44) to minus infinity and in (3.45) to plus infinity, respectively. In this case, $\tilde{\kappa}_d(\tau)$ represents the inverse Fourier transform of $\kappa_d(\nu)$.

IV. COMPARISON AND DISCUSSION

A. The distributions

In the following we refer to the approximation (3.18) and the corresponding relation between $\kappa_d(\omega)$ and $\kappa_b(\omega)$:

$$\kappa_d'(\omega)=(2\omega/\Omega)(1+\omega/\Omega)^{-2}\kappa_b'(\omega)\ , \quad (4.1a)$$

$$\kappa_d''(\omega)=-2i\frac{\omega}{\Omega\pi}\mathrm{P}\int_0^\infty d\nu\frac{\kappa_b'(\nu)}{(\nu-\omega)(1+\nu/\Omega)^2}\ . \quad (4.1b)$$

For the calculation of physical quantities we need the distributions (2.15), (3.31), and (3.22):

$$\chi_b(\omega)=\left[-\omega^2+\left[\omega_0^2-2\Gamma\gamma-\frac{2}{\pi}\omega^2\mathrm{P}\int_0^\infty d\nu\frac{\kappa_b'(\nu)}{\nu^2-\omega^2}\right]-i\omega\kappa_b'(\omega)\right]^{-1}\ , \quad (4.2)$$

$$\chi_b^{(\mathrm{RWA})}(\omega)=\left[-\omega+\left[\omega_0-\frac{\Gamma\gamma}{2\omega_0}-\frac{\omega}{2\omega_0\pi}\mathrm{P}\int_0^\infty d\nu\frac{\kappa_b'(\nu)}{\nu-\omega}\right]-i\frac{\omega}{2\omega_0}\kappa_b'(\omega)\right]^{-1}\ , \quad (4.3)$$

$$\chi_d(\omega)=\left[-\omega+\left[\Omega-\frac{2\Omega}{\pi}\int_0^\infty d\nu\frac{\kappa_b'(\nu)}{(\nu+\Omega)^2}-\frac{\omega}{2\Omega\pi}\mathrm{P}\int_0^\infty d\nu\frac{\kappa_b'(\nu)}{\nu-\omega}\frac{4}{(1+\nu/\Omega)^2}\right]-i2\frac{\omega}{\Omega}\frac{\kappa_b'(\omega)}{(1+\omega/\Omega)^2}\right]^{-1}\ . \quad (4.4)$$

We will only discuss first-order results in Γ and, therefore, we can replace ω by ω_0 or Ω in the additional ω-dependent shift terms. Furthermore, the assumption that κ_b' does not change appreciably in the range in which the function $(1+\omega/\Omega)^2$ is different from zero (note that $\gamma\gg\Omega$) allows us to replace $\kappa_b'(\omega)$ by 2Γ in (4.4) and, accordingly, the cutoff parameter γ is explicitly no longer present in $\chi_d(\omega)$. The structure of the bare interaction is screened by the function $4\omega/\Omega(1+\omega/\Omega)^{-2}$.

For $\omega\to0$ the imaginary parts of (4.2)–(4.4) show the same qualitative behavior (see Sec. III E):

$$\chi_b''(\omega\to0)\to2\Gamma\omega_0^{-4}\omega\left[\frac{\omega}{\alpha}\right]^\lambda\ , \quad (4.5a)$$

$$\chi_d''(\omega\to0)\to4\Gamma\Omega^{-3}\omega\left[\frac{\omega}{\alpha}\right]^\lambda\ , \quad (4.5b)$$

$$\chi_b^{(\mathrm{RWA})''}(\omega\to0)\to\Gamma\omega_0^{-3}\omega\left[\frac{\omega}{\alpha}\right]^\lambda\ . \quad (4.5c)$$

With respect to (4.2) and (4.3) we will sometimes refer to the Drude model ($\to\lambda=0$) which fulfills the above demanded and assumed properties

$$\kappa_b(\omega)=2\Gamma\gamma(\gamma-i\omega)^{-1}\ , \quad (4.6a)$$

where

$$\Gamma\ll\omega_0\ll\gamma\ . \quad (4.6b)$$

The shift terms in (4.2) and (4.3) are to be related to the bare frequency ω_0 and, therefore, the contributions of the principal value integrals in (4.2) and (4.3) ($\omega\to\omega_0$) can be neglected compared to $2\Gamma\gamma$ and $\Gamma\gamma/2\omega_0$, respectively. These considerations lead to the simplified versions of (4.2)–(4.4):

$$\chi_b(\omega)=\left[-\omega^2+(\omega_0^2-2\Gamma\gamma)-i2\Gamma\omega\frac{\gamma^2}{\omega^2+\gamma^2}\right]^{-1}\ , \quad (4.7)$$

$$\chi_b^{(\mathrm{RWA})}(\omega)=\left[-\omega+\left[\omega_0-\frac{\Gamma\gamma}{2\omega_0}\right]-i\Gamma\frac{\omega}{\omega_0}\frac{\gamma^2}{\omega^2+\gamma^2}\right]^{-1}\ , \quad (4.8)$$

$$\chi_d(\omega)=\left[-\omega+\left[\Omega-\frac{2\Gamma}{\pi}\right]-i\Gamma\frac{\omega}{\Omega}\frac{4}{(1+\omega/\Omega)^2}\right]^{-1}\ . \quad (4.9)$$

With increasing frequency ($\omega\gtrsim\Omega$) the "width" in $\chi_d(\omega)$ decreases faster than the "widths" in χ_b and $\chi_b^{(\mathrm{RWA})}$; e.g., for $\omega=\gamma$ we have $\chi_d''(\omega=\gamma)\approx4\Omega\Gamma/\gamma^3$ and $\chi_b^{(\mathrm{RWA})''}(\omega=\gamma)\approx\Gamma/(2\omega_0\gamma)$. The right flank of $\chi_d''(\omega)$ is steeper than the right flank of $\chi_b^{(\mathrm{RWA})''}$ and $\chi_b''(\omega)$. This effect will be negligible if the contributions to physical quantities can be neglected for $\omega \gg \omega_0$.

B. The decay laws

Whereas the bare-field description led to a strictly exponential decay law, the physical description additionally showed an algebraic decay. If we restrict ourselves to results of first order in Γ, the algebraic decay can be neglected and the decay laws are characterized by the negative imaginary parts of the poles in (4.2)–(4.4). In (4.3) and (4.4) only poles with positive real parts contribute to the decay. Now, we specialize to the Drude model (4.7)–(4.9). In $\chi_b(\omega)$ we find three poles with negative imaginary parts (we restrict to lowest-order results in Γ and γ^{-1}):

$$\omega_{b1} = -i\gamma \ , \tag{4.10}$$

$$\omega_{b2,3} = \pm(\omega_0 - \Gamma\gamma/\omega_0) - i\Gamma \ . \tag{4.11}$$

In $\chi_b^{(\mathrm{RWA})}(\omega)$ there are two poles with positive real parts:

$$\omega_{b1}^{(\mathrm{RWA})} = -i\gamma + \Gamma\gamma/2\omega_0 \ , \tag{4.12}$$

$$\omega_{b2}^{(\mathrm{RWA})} = (\omega_0 - \Gamma\gamma/2\omega_0) - i\Gamma \ . \tag{4.13}$$

In $\chi_d(\omega)$ only one pole with a positive real part exists:

$$\omega_d = \left[\Omega - \frac{2\Gamma}{\pi}\right] - i\Gamma \ . \tag{4.14}$$

For not too small times, i.e., for $t\gamma \gg 1$, in all three cases the decay of energylike quantities is characterized by the exponential factor $\exp(-2\Gamma t)$.

C. Some equilibrium properties

1. Anomalous low-temperature effects ($k_B T \ll \hbar\omega_0$)

It can be seen from (2.24) and (4.6) that the $T=0$ contribution to the occupation number $\langle a^\dagger a \rangle_\infty$ crucially depends on the cutoff parameter γ: If we would replace $\gamma^2(\omega^2+\gamma^2)^{-1}$ with one in (4.7), the integrand in (2.24) ($\langle n_k \rangle_{\mathrm{th}} = 0$) would behave as ω^{-1} for $\omega \to \infty$. Therefore, the convergency requires a finite cutoff parameter γ. Lindenberg and West[7] were the first who calculated this contribution (Drude model)

$$\langle a^\dagger a \rangle_\infty^{T=0} = \frac{\Gamma}{\pi\omega_0}\left[\ln\left[\frac{\gamma}{\omega_0}\right] - 1\right] . \tag{4.15}$$

Energetically, the contribution (4.15) is made possible by a corresponding negative contribution of the interaction energy(2.26)

$$[E_{\mathrm{int}}(t\to\infty)]^{T=0} = -\frac{\hbar\Gamma\gamma}{\omega_0} + \frac{2}{\pi}\hbar\Gamma\ln\left[\frac{\gamma}{\omega_0}\right] . \tag{4.16}$$

The positive difference $\hbar\omega_0\langle a^\dagger a\rangle_\infty^{T=0} - [E_{\mathrm{int}}(t\to\infty)]^{T=0}$ is transferred to the bath. The occupation number of the physical particles and the occupation number which is based on the RWA approximation vanishes for $T=0$.

In the low-temperature limit ($k_B T \ll \hbar\omega_0$) the asymptotic behavior (4.5b) causes an anomalous T dependence of the occupation number [$\zeta(\lambda)$ denotes the Riemann ζ function]

$$\langle \tilde{a}^\dagger(t)\tilde{a}(t)\rangle_\infty^{T\to 0} = \frac{4\Gamma}{\pi\Omega}\frac{(k_B T)^{2+\lambda}}{(\hbar\Omega)^2(\hbar\alpha)^\lambda}\Gamma(\lambda+2)\zeta(\lambda+2) \ . \tag{4.17}$$

Of course, the bare and the RWA description have the same qualitative behavior. For $\lambda=0$ [$\Gamma(\lambda+2)\zeta(\lambda+2)\to\pi^2/6$] the corresponding formulas have been given by Lindenberg and West.[7]

Whereas the kinetic and potential energy of the physical particles are equal, the bare-field description leads to a well-known asymmetry. In the low-temperature regime we find the result (Drude model)

$$E_{\mathrm{kin}}^{(b)} - E_{\mathrm{pot}}^{(b)} = -\frac{\hbar\Gamma\gamma}{2\omega_0}\left[1 - \frac{2\omega_0}{\gamma}\ln\left[\frac{\gamma}{\omega_0}\right]\right] . \tag{4.18}$$

For $|t-t'| \gg \hbar/k_B T$ the correlation function $\langle \tilde{a}^\dagger(t)\tilde{a}(t')\rangle_\infty$ reveals an anomalous time dependence (in the "normal" case one would expect the exponential time dependence $\sim \exp(-\Gamma|t-t'|)\exp[i\omega_0(t-t')]$). We insert (4.5b) in (3.41)

$$\langle \tilde{a}^\dagger(\tau)\tilde{a}(0)\rangle_\infty = \frac{4\Gamma}{\pi\Omega}\left[\frac{k_B T}{\hbar\Omega}\right]\left[\frac{k_B T}{\hbar\alpha}\right]^\lambda \Gamma(\lambda+2) \times\zeta(\lambda+2, 1-i(k_B T/\hbar)\tau) \ . \tag{4.19}$$

The expansion of the generalized ζ function for $(k_B T/\hbar)\tau \gg 1$ leads to an algebraic relaxation behavior. For $\lambda=0$ (Ohmic dissipation) we find

$$\mathrm{Re}\langle \tilde{a}^\dagger(\tau)\tilde{a}(0)\rangle_\infty = \frac{2\Gamma}{\pi\Omega}(\Omega\tau)^{-2} \ . \tag{4.20}$$

These considerations can also be transferred to the bare-field and the RWA description [see formula (3.23) in Ref. 5].

2. The Bose distribution for $k_B T \gtrsim \hbar\Omega$

For $k_B T \gtrsim \hbar\Omega$ we can assume that the function $\langle n_\omega \rangle_{\mathrm{th}}$ does not vary over the width of $\chi_d''(\omega)$ and that the contribution to (3.41) in the frequency range $0<\omega<\Omega-\Gamma$ can be neglected in comparison to the "peak" contribution $\Omega-\Gamma<\omega<\Omega+\Gamma$. Consequently, the Bose factor $\langle n_\omega\rangle_{\mathrm{th}}$ can be extracted from the integral at the position $\omega=\Omega-(2\Gamma/\pi)$

$$\langle \tilde{a}^\dagger\tilde{a}\rangle_\infty = \langle n_{\Omega-2\Gamma/\pi}\rangle_{\mathrm{th}}\int_0^\infty \frac{d\nu}{\pi}\chi_d''(\nu) \ .$$

The orthogonality condition allows us the representations

$$\langle \tilde{a}^\dagger\tilde{a}\rangle_\infty = \langle n_\Omega\rangle[1+0(\Gamma/\Omega)] \tag{4.21}$$

and

$$E_{\mathrm{os}}^{(\mathrm{ph})}(t\to\infty) = \hbar\Omega\{\langle n_\Omega\rangle[1+O(\Gamma/\Omega)]+\tfrac{1}{2}\} \ . \tag{4.22}$$

The deviation from the ideal behavior (free phonons with the frequency Ω) is of the order $O(\Gamma/\Omega)$. In contrast to this result, the corresponding considerations for the bare-field and the RWA description lead to deviations of the order $O(\gamma\Gamma/\omega_0^2)$. In these cases we have to refer to the bare frequency ω_0 [see (2.25)].

3. High-temperature expansions

In the classical regime $k_B T \gg \hbar\Omega$, the physical description shows deviations from the classical value $k_B T/2$, which are of the order $O(\Gamma/\Omega)$

$$E_{\text{kin}}^{(\text{ph})} = E_{\text{pot}}^{(\text{ph})} = \frac{k_B T}{2}\left[1+\frac{2\Gamma}{\pi\Omega}\right] . \tag{4.23}$$

Equation (4.23) is to be confronted with the results of the bare-field description

$$E_{\text{kin}}^{(b)} = \tfrac{1}{2} k_B T , \tag{4.24}$$

$$E_{\text{pot}}^{(b)} = \tfrac{1}{2} k_B T (1+2\Gamma\gamma/\omega_0^2) . \tag{4.25}$$

We emphasize that $\Gamma\gamma/\omega_0^2$ need not be small compared with 1.

D. Conclusions

The dressed description of the quantum relaxation process allowed the definition of physical particles, i.e., particles which are absent in the ground state of the system ($t \to \infty$, $T=0$). For not too low temperatures the exponential decay of the energylike quantities leads to equilibrium properties which nearly correspond to a free-phonon system (frequency Ω). The deviations are of the order $O(\Gamma/\Omega)$ and, therefore, they are small compared with the analogous deviations of the bare-field description (frequency ω_0), which are of the order $\gamma\Gamma/\omega_0^2$.

One can visualize a physical particle (phonon) as a bare particle which is surrounded by a "cloud" of bare-field particles, so that there is only a screened interaction between the physical oscillator and the physical bath. Because of this screening, the cutoff parameter γ does not appear explicitly in the physical description.

The low-temperature anomalies of Sec. IV C 1 are present in all descriptions. The algebraic temperature and time dependences in (4.17) and (4.19) are due to the asymptotic formulas (4.5). If the impedance $\kappa_b'(\omega)$ would be zero between $\omega=0$ and $\omega=u$ $(0<u\ll\omega_0)$ we would find a normal exponential behavior in the low-temperature regime (e.g., one could relate a lower cutoff frequency u with a finite extension of the bath).

V. SUMMARY

Ullersma's model could be mapped onto a field formulation which directly led to the solution for the damped harmonic oscillator. Because the occupation number of the oscillator, i.e., the number of bare particles, did not vanish in the physical vacuum, this relaxation process could not be interpreted as the decay of physical quanta. The postulate that a quantum-mechanical relaxation process should be described "physically" required a description in terms of dressed particles, i.e., particles whose vacuum state must coincide with the physical vacuum. The physical decay process represented a more ideal process than the bare one; more ideal in the sense that the physical description implied a screening of the bare interaction.

Two essential virtues of the physical description are implied by the above statements (i) the decay process leads for $T=0$ to a *pure state* (ground state or physical vacuum) with respect to the *physical particles* and (ii) the effective or physical interaction is explicitly independent of a cutoff parameter. On the contrary, the physical vacuum must be interpreted as a *mixed state* with respect to the *bare particles.* An essential drawback of the physical description was the nonexistence of a "macroscopic" dynamic equation which satisfies causality and which reproduces the exact physical solution for $t\to\infty$. It was shown that the physical description is closely related to the rotating-wave approximation of the bare description.

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[1](a) I. R. Senitzky, Phys. Rev. **119**, 670 (1960); (b) **124**, 642 (1961); (c) M. Lax, *ibid.* **129**, 2342 (1962); (d) G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965); (e) M. Lax, Phys. Rev. **145**, 110 (1966).

[2](a) P. Ullersma, Physica (Utrecht) **32**, 27 (1966); **32**, 56 (1966); **32**, 74 (1966); (b) **32**, 90 (1966).

[3](a) E. Braun, Physica (Utrecht) **33**, 528 (1967); (b) G. S. Agarwal, Phys. Rev. A **4**, 739 (1971); (c) W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973); (d) M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974); (e) H. Haken, Rev. Mod. Phys. **47**, 67 (1975); (f) W. Eckhardt, Opt. Commun. **14**, 95 (1975); (g) E. Braun and S. V. Godoy, Physica (Utrecht) **86A**, 337 (1977); (h) H. Dekker, Phys. Rep. **80**, 1 (1981); (i) R. Benguria and M. Kac, Phys. Rev. Lett. **46**, 1 (1980).

[4](a) H. Grabert and P. Talkner, Phys. Rev. Lett. **50**, 1335 (1983); (b) K. Lindenberg and B. J. West, *ibid.* **51**, 1370 (1983); (c) P. Ruggiero and M. Zannetti, Phys. Rev. A **28**, 987 (1983).

[5]H. Grabert, U. Weiss, and P. Talkner, Z. Phys. B **55**, 87 (1984).

[6]F. Haake and R. Reibold, Phys. Rev. A **32**, 2462 (1985).

[7]K. Lindenberg and B. J. West, Phys. Rev. A **30**, 568 (1984).

[8]P. S. Riseborough, P. Hänggi and U. Weiss, Phys. Rev. A **31**, 471 (1985).

[9]E. Braun, Physica (Utrecht) **129A**, 262 (1985).

[10](a) H. Dekker, Phys. Rev. A **31**, 1067 (1985); (b) **33**, 2140 (1986).

[11]W. Eckhardt, Physica (Utrecht) **141A**, 81 (1987).

[12](a) A. Schmid, J. Low. Temp. Phys. **49**, 609 (1982); (b) A. O. Caldeira and A. J. Leggett, Ann. Phys. (N. Y.) **149**, 374 (1983); (c) Physica (Utrecht) **121A**, 587 (1983); (d) W. Zwerger, Z. Phys. B **51**, 301 (1983); (e) H. Grabert and U. Weiss, *ibid.* **56**, 171 (1984).

[13](a) I. A. Eganova and M. I. Shirokov, Yad. Fiz. **9**, 1097 (1969) [Sov. J. Nucl. Phys. **9**, 642 (1969)]; (b) *ibid.* **21**, 674 (1975) [*ibid.* **21**, 347 (1975)].

[14](a) N. G. Van Kampen, Dan Mat. -Fys. Medd. **26** (1951); (b) W. Żakowicz and K. Rzążewski, J. Phys. A **7**, 869 (1974); (c) K. Rzążewski and W. Żakowicz, *ibid.* **9**, 1159 (1976); (d) H. Dekker, Physica (Utrecht) **133A**, 1 (1985); (e) W. Eckhardt, Phys. Lett. **115A**, 307 (1986); (f) W. Eckhardt, Z. Phys. B **64**,

515 (1986).

[15]In Ullersma's formulation it was assumed that $m_0 = m = 1$.

[16]The Hamiltonian which was discussed by Eganova and Shirokov [formula (10) in Ref. 13(a)] differs from H_F by the additional term

$$(2m_0)^{-1}\left[\int_0^\infty dk\, \varepsilon(k) p_k\right]^2 .$$

The coupling in Ref. 13(a) is given by the fine-structure constant. In order to obtain finite results in the dipole approximation, a suitable high-frequency cutoff procedure has to be performed.

[17]

$$U_{1\omega} = \left[\frac{2}{\pi}\kappa_b'(\omega)\right]^{1/2} \omega\, |\chi_b(\omega)| ,$$

$$U_{k\omega} = U_{1\omega} k \left[\frac{2}{\pi}\kappa_b'(k)\right]^{1/2} \mathrm{P}\frac{1}{\omega^2 - k^2} - 2\omega \frac{\chi'(\omega)}{|\chi(\omega)|}\delta(\omega^2 - k^2) .$$

[18]In contrast to the exact solution (2.17), $a(t)$ in Ref. 7 [formula (3.14)] represents no exact solution of the initial-value problem. This point is extensively discussed in Ref. 11.

[19]The statement is not in contradiction to the well-known theorem that for $t \to \infty$ the decay probability must decrease slower than an exponential because for $t \to \infty$ the finite value (2.28) is left. For this theorem, see, e.g., R. E. A. Paley and N. Wiener, *Fourier Transform in the Complex Domain,* (A.M.S., Providence, 1932); M. V. Terent'ev, Ann. Phys. (N.Y.) **74**, 1 (1972).

[20]The solution (2.17) can be derived from Ullersma's solutions for the momentum and position operators. The solutions for the bath operators were not explicitly calculated in Ref. 2(a).

[21]The linear response theory $[\delta q(\omega)/\delta F(\omega) = \chi_b(\omega), H_1 = -qF(t)]$ and the fluctuation-dissipation theorem also lead to the correlation function (2.24). This method was applied by Grabert *et al.* (Ref. 5) for the calculation of the position and momentum fluctuations in thermal equilibrium.

[22]Haake and Reibold (Ref. 6) have also discussed these two different initial conditions (Secs. III and IV in Ref. 6: "thermal equilibrium" and "partial equilibrium"). Furthermore, they defined a "constrained equilibrium," which roughly corresponds to the "conditional thermal average" of Lindenberg and West (Ref. 7). Of course, the various initial conditions are irrelevant for $t \to \infty$.

[23]A qualitative discussion of this point is given in Sec. V B of Ref. 7.

[24]The solution of the initial-value problem in Ref. 7 is based on the application of the Laplace-transformation method to Eq. (2.34), in which the lower limit in the memory term is zero.

[25]See, e.g., H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951); R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).

[26]The "continuous" matrix X is defined by the elements

$$\chi_{k\nu} = \eta(\nu)\left[\mathrm{P}\xi(k)(\nu^2 - k^2)^{-1} + \xi(\nu)^{-1}\left[1 + \mathrm{P}\int_0^\infty dk'\, \xi^2(k')(k'^2 - \nu^2)^{-1}\right] \times \delta(k - \nu)\right] ,$$

where

$$\eta(\nu) = \int_0^\infty dk\, \xi(k) X_{k\nu} = \xi(\nu)\left[[\pi\xi^2(\nu)/2\nu]^2 + \left[1 + \mathrm{P}\int_0^\infty dk\, \xi^2(k)\,(k^2 - \nu^2)^{-1}\right]^2\right]^{-1/2} .$$

[27]The conditions (3.2) and (3.3) differ from the corresponding conditions in Ref. 13(a) [formulas (23) and (24)]. In our model the renormalized frequency Ω becomes smaller than ω_0 [see (3.16) and (3.17)], whereas the renormalized frequency of the harmonically bound electron becomes greater than ω_0. This different behavior is due to the A^2 term in the minimal coupling Hamiltonian.

[28]With respect to the electron-photon coupling in the dipole approximation, the high-frequency cutoff γ can be related to a finite extension of the electron.[2,14(d)]

[29]In Ref. 8 an example is discussed for which the conditions (ii) and (iii) are no longer valid.

[30]

$$M_{1\omega} = [\kappa_d'(\omega)/\pi]^{1/2}\, |\chi_d(\omega)| ,$$

$$M_{\nu\omega} = |\chi_d(\omega)|\, \{[\kappa_d'(\omega)\kappa_d'(\nu)/\pi]^{1/2}\mathrm{P}\frac{1}{\omega - \nu} - \chi_d'(\omega)\, |\chi_d(\omega)|^{-2}\delta(\omega - \nu)\} .$$

[31]In Ref. 7 the solution for $a(t)$ in the rotating-wave approximation is represented as an inverse Laplace transformation. The corresponding decay process is strictly exponential. For example, the formula (4.22) in Ref. 7 leads for $T = 0$ to the exponential decay ($\mathrm{Re}\, s_j < 0$)

$$\langle a^\dagger(t) a(t)\rangle^{(\mathrm{RWA})} = \langle a^\dagger(0) a(0)\rangle \sum_{j,k} \exp[(s_j^* + s_k)t] B_j^* B_k .$$