Dynamical threshold exponents and amplitudes in a driven single-mode laser

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As a continuation of a previous paper [J.-M. Liu, Z. Phys. B 57, 85 (1984)], the present paper is about the dynamical threshold phenomena in the system of a single-mode one-photon laser driven by an external coherent field. The values for the dynamical threshold exponents and amplitudes near the various thresholds are theoretically predicted. As shown, the characteristic time goes to infinity as the control parameters of the system approach each of these thresholds, and the influence of the existence of the absorbing atoms in the laser cavity on the dynamical threshold phenomena in the system is definite.

I. INTRODUCTION

A laser is a system far from equilibrium. Identifying the order parameter with the laser-field amplitude, the temperature with the unsaturated inversion, and the external magnetic field with the external coherent field, we discussed the critical phenomenon analogies, namely, the static threshold phenomena in the system of a single-mode one-photon laser driven by the external coherent field in Ref. 1.

We defined the static threshold exponents and amplitudes in parallel with those in critical phenomena of the equilibrium systems, and estimated the values for them. It turned out that these threshold exponents and amplitudes obey the same scaling laws in critical phenomena, and that the well-known scaling hypothesis holds for the threshold phenomena.

The results support, from a viewpoint of critical exponents, the words about a deep similarity between the phase transitions in equilibrium systems and the abrupt transitions of steady states in nonequilibrium systems.[2–4] In other words, this similarity occurs not only in their transitions, but also in their scaling behaviors.

The present paper is just the continuation of Ref. 1. It is about the dynamical threshold phenomena in the system of a single-mode one-photon laser driven by an external coherent field. We would like to investigate how the characteristic time changes with the control parameters as they approach the threshold, and how the existence of absorbing atoms in the laser cavity influences the dynamical threshold phenomena. Similarly, the dynamical threshold phenomena are shown by means of some dynamical threshold exponents and amplitudes.

The plan of this paper is as follows. In Sec. II, we first briefly recall the time-evolution equation for the system of a single-mode one-photon laser driven by an external coherent field in the presence of absorbing atoms in the laser cavity and the asymptotic (ASY) time-evolution equation near the threshold at $\bar{\sigma}=\bar{\sigma}_0$, where $\bar{\sigma}$ is the pump parameter of an absorbing atom, then make definitions for the dynamical threshold exponents and amplitudes. Section III is devoted to estimating the values for two couples of the dynamical threshold exponent and amplitude, q_a and p_a, and q_a' and p_a', while Sec. IV is devoted to estimating the values for another couple, q_α and p_α. We discuss the corresponding calculation when $\bar{\sigma}\neq\bar{\sigma}_0$ in Sec. V, omitting the details of derivation. Finally, some conclusions are drawn in Sec. VI.

II. ASYMPTOTIC TIME-EVOLUTION EQUATION FOR THE SYSTEM

Making the substitution $\kappa\langle\beta\rangle\to\kappa[\langle\beta\rangle-\alpha(t)]$, for the laser-field damping term in the semiclassical time-evolution equation for the single-mode one-photon laser in the presence of a saturable absorber,[5,6] where κ is the laser-damping constant, β and $\alpha(t)$ are the laser field and the external coherent field in the Glauber representation, respectively, ignoring distuning and performing the adiabatic elimination of the atomic variables, we have the semiclassical time-evolution equation for the system of a single-mode one-photon laser driven by the external coherent field in the presence of a saturable absorber,

$$\frac{dE}{dt}=\kappa\left[-E-\bar{\alpha}-\frac{AE}{1+SE^2}-\frac{(1-C)E}{1+\bar{S}E^2}\right],$$
$$A=\frac{N\,|\,g\,|^2\sigma}{\kappa\gamma_\perp},\quad (1-C)=\frac{\bar{N}\,|\,\bar{g}\,|^2\bar{\sigma}}{\kappa\bar{\gamma}_\perp},\qquad \sigma>0,\bar{\sigma}<0\;, \tag{1}$$
$$S\equiv\frac{4\,|\,g\,|^2}{\gamma_{\parallel}\gamma_\perp},\quad \bar{S}\equiv\frac{4\,|\,\bar{g}\,|^2}{\bar{\gamma}_{\parallel}\bar{\gamma}_\perp}\;,$$

where E and $\bar{\alpha}$ are defined by $\langle\beta\rangle=E\exp(-i\nu t)$ and $\alpha(t)=\bar{\alpha}\exp(-i\nu t)$ separately, A and C are proportional to the pump parameters σ and $\bar{\sigma}$ of an active atom and an absorbing atom, respectively.[7] The other notations in Eq. (1) are those usually used in laser theory. The notations with a bar on them denote the quantities of the absorbing atoms.

It should be noted that the laser amplitude in Eq. (1), E, is a complex variable. The system is actually a nonequilibrium system with a complex order parameter. A linear stability analysis for the dynamics of the phase of the field amplitude shows that only the in-phase stationary solutions are stable when the injected coherent field does not vanish.[8] This analysis indicates that we may treat E as a real order parameter by adding a selection rule which keeps only the in-phase stationary solutions for us.

We rewrite Eq. (1) in such a way that a family of potential functions is introduced explicitly,

$$\frac{dE}{dt}=-\frac{\partial V(E,a,c,\alpha)}{\partial E},$$
$$V(E,a,c,\alpha)=\tfrac{1}{2}\kappa\left[E^2-\frac{2\alpha E}{\kappa}-\left[\frac{A_c}{S}+\frac{2a}{S\kappa}\right]\ln(1+SE^2)-\left[\frac{1-C_c}{\bar{S}}+\frac{2c}{\bar{S}\kappa}\right]\ln(1+\bar{S}E^2)\right], \quad (2)$$
$$A_c=C_c=\bar{S}/(\bar{S}-S),\quad \bar{S}>S$$

where the transformations

$$A=A_c+\frac{2a}{\kappa},\quad C=C_c+\frac{2c}{\kappa},\quad \bar{\alpha}=\frac{\alpha}{\kappa}, \quad (3)$$

are used. $\bar{S}>S$ comes from the inequalities, $A>0$ and $C>1$, or $\sigma>0$ and $\bar{\sigma}<0$.

$V(E,a,c,\alpha)$, a family of potential functions, characterizes the system. The stationary equation for the system is

$$\frac{\partial V(E,a,c,\alpha)}{\partial E}=0 . \quad (4)$$

When the control parameters a, c, and α are fixed, the value of the order parameter of the system, E, in a steady state is given by a solution of Eq. (4), which minimizes the potential, $V(E,a,c,\alpha)$, at (a,c,α).

It is easily seen from Eq. (4) that the origin, $(a,c,\alpha)=(0,0,0)$, or equivalently, $(\sigma,\bar{\sigma},\bar{\alpha})=(\kappa\gamma_{\perp}\bar{S}/N\,|g|^2(\bar{S}-S),\ -\kappa\bar{\gamma}_{\perp}S/\bar{N}\,|\bar{g}|^2(\bar{S}-S),\ 0)$, is a threshold of the system. That means the abrupt transition between the steady states of the system will appear as the control parameters pass the origin.

The catastrophe theory (CT), first developed by Thom to understand the discontinuous phenomena in nature,[9] is powerful in discussion on threshold phenomena of nonequilibrium systems. CT not only enables us to describe qualitatively, strictly speaking, with diffeomorphic equivalent exactitude, the abrupt transitions between steady states in any nonequilibrium system having a family of potential functions, but also gives us an exact asymptotic form of the family of potential functions near a threshold.[10,1]

Just using CT, we can get the asymptotic time-evolution equation near the threshold, $(a,c,\alpha)=(0,0,0)$, for the system. That is,

$$\frac{dE}{dt}=-\frac{\partial V_{\rm asy}(E,a,c,\alpha)}{\partial E},\qquad V_{\rm asy}(E,a,c,\alpha)=\tfrac{1}{6}\kappa S\bar{S}E^6-aE^2-cE^2-\alpha E . \quad (5)$$

We refer the readers to Ref. 1 for the details of derivation.

So, letting $c=0$ in Eq. (5), we have the asymptotic time-evolution equation near the threshold, $(a,\alpha)=(0,0)$, or $(\sigma,\bar{\alpha})=(\kappa\gamma_{\perp}\bar{S}/N\,|g|^2(\bar{S}-S),0)$ for the system of single-mode one-photon laser driven by the external coherent field when $\bar{\sigma}=\bar{\sigma}_0\equiv-\kappa\bar{\gamma}_{\perp}S/\bar{N}\,|\bar{g}|^2(\bar{S}-S)$, $\bar{S}>S$,

$$\frac{dE}{dt}=-\frac{\partial V_{\rm asy}(E,a,\alpha)}{\partial E},\qquad V_{\rm asy}(E,a,\alpha)=\tfrac{1}{6}\kappa S\bar{S}E^6-aE^2-\alpha E . \quad (6)$$

Equation (6) is the starting point for our discussion of the dynamical threshold phenomena. Before discussion, we should define the dynamical threshold exponents and amplitudes. We can do this in the way of defining them in parallel with those in critical phenomena.[11] But we do not want to do so; we prefer to keep the definitions consistent with some references on this topic.[12–16]

The definitions are

$$\lambda=\begin{cases} p_a\,|a|^{-q_a} & \text{for } \alpha=0 \text{ and } a>0 ,\\ p_a'\,|a|^{-q_a'} & \text{for } \alpha=0 \text{ and } a<0 ,\end{cases} \quad (7)$$

and

$$\lambda=p_\alpha\,|\alpha|^{-q_\alpha}\quad \text{for } a=0 \text{ and } \alpha>0 , \quad (8)$$

where p_a, p_a', and p_α are dynamical threshold amplitudes, q_a, q_a' and q_α are dynamical threshold exponents, λ is the characteristic time defined as usual by

$$\lambda=\frac{\int_0^\infty t[-dE(t)]}{\int_0^\infty[-dE(t)]}=\frac{-\int_0^\infty t[dE(t)]}{E(0)-E(\infty)} . \quad (9)$$

III. THEORETICAL VALUES FOR q_a, q_a', p_a, AND p_a' AT $\bar{\sigma}=\bar{\sigma}_0$

In Ref. 1 we discussed the static threshold phenomena for the system and got the theoretical value for the static threshold exponents and amplitudes, calling $dE/dt=0$ in Eq. (6). But it is a different situation when we discuss the dynamical threshold phenomena: we have to deal with Eq. (6) itself. It is obviously more difficult. We first confine our attention to the case of $\alpha=0$ at $\bar{\sigma}=\bar{\sigma}_0$ in this section.

When $\alpha=0$, Eq. (6) becomes

$$-dt=dE/(\kappa S\bar{S}E^5-2aE)$$

or (10)

$$-dt=dE\left[-\frac{1}{2a}\frac{1}{E}+\frac{\kappa S\bar{S}}{2a}\frac{E^3}{\kappa S\bar{S}E^4-2a}\right].$$

The solution of Eq. (10) is

$$-t=-\frac{1}{2a}\ln E+\frac{1}{8a}\ln(\kappa S\bar{S}E^4-2a)+c(a) , \quad (11)$$

where $c(a)$ is to be determined by an initial condition.

To continue, we should separate the case where $a<0$ from the case where $a>0$.

(i) $a<0$. In accordance with the so-called steady-state condition, as $t\to\infty$, $E\to E_\infty=0$, where E_∞ means the stationary-state value of the order parameter of the system, $c(a)$ in solution (11), must have the form of

$$c(a)=\frac{1}{8|a|}\ln 2|a|+\frac{1}{8|a|}\ln c_<(a)\ , \tag{12}$$

where $c_<(a)$ is an arbitrary function of the control parameter a.

In fact, if we put $c(a)$ into the solution (11), we can get

$$\begin{aligned}-t&=\frac{1}{2|a|}\ln E-\frac{1}{8|a|}\ln(\kappa S\bar{S}E^4+2|a|)\\&\quad+\frac{1}{8|a|}\ln 2|a|+\frac{1}{8|a|}\ln c_<(a)\\&=\frac{1}{8|a|}\ln\frac{2|a|c_<(a)E^4}{\kappa S\bar{S}E^4+2a}\ ,\end{aligned}$$

or (13)

$$\exp(-8|a|t)=\frac{2|a|c_<(a)E^4}{\kappa S\bar{S}E^4+2|a|}\ .$$

So the initial condition, $t=0$, $E=E_0\neq 0$, immediately leads to

$$c_<(a)=\frac{\kappa S\bar{S}E_0^4+2|a|}{2|a|E_0^4}\ . \tag{14}$$

And from Eq. (14), it is easily seen that $c_<(a)\geq\kappa S\bar{S}/2|a|$, or equivalently

$$1\geq\frac{\kappa S\bar{S}}{2|a|c_<(a)}\ . \tag{15}$$

Equation (13) can be rewritten

$$\begin{aligned}E^4(t)&=\frac{2|a|\exp(-8|a|t)}{2|a|c_<(a)-\kappa S\bar{S}\exp(-8|a|t)}\\&=\frac{1}{c_<(a)}\frac{\exp(-8|a|t)}{1-\dfrac{\kappa S\bar{S}\exp(-8|a|t)}{2|a|c_<(a)}}\ ,\end{aligned}$$

or

$$E(t)=\left[\frac{1}{c_<(a)}\right]^{1/4}\frac{\exp(-2|a|t)}{\left[1-\dfrac{\kappa S\bar{S}\exp(-8|a|t)}{2|a|c_<(a)}\right]^{1/4}}\ . \tag{16}$$

Owing to Eq. (15), we can expand $E(t)$ as

$$\begin{aligned}E(t)=&\left[\frac{1}{c_<(a)}\right]^{1/4}\exp(-2|a|t)\\&\times\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\kappa S\bar{S}}{2|a|c_<(a)}\right]^n\exp(-8|a|nt)\right.\\&\left.\times\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]\ .\end{aligned} \tag{17}$$

On the other hand, for a large enough t, $E(t)$ has its average lifetime or characteristic time λ, which is currently defined by Eq. (9).

Since

$$\begin{aligned}\int_0^\infty t[-dE(t)]=&\left[\frac{1}{c_<(a)}\right]^{1/4}2|a|\int_0^\infty\left[t\exp(-2|a|t)\right.\\&\left.+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\kappa S\bar{S}}{2|a|c_<(a)}\right]^n(1+4n)t\exp[-2|a|(1+4n)t]\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]dt\end{aligned}$$

and

$$\begin{aligned}E(0)-E(\infty)&=\left[\frac{1}{c_<(a)}\right]^{1/4}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\kappa S\bar{S}}{2|a|c_<(a)}\right]^n\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]\\&=\left[\frac{1}{c_<(a)}\right]^{1/4}\left[1-\frac{\kappa S\bar{S}}{2|a|c_<(a)}\right]^{-1/4}\ ,\end{aligned}$$

we have

$$\lambda=\frac{1}{2|a|}\left[1-\frac{\kappa S\bar{S}}{2|a|c_<(a)}\right]^{1/4}\left[1+\sum_{n=1}^{\infty}\frac{1}{1+4n}\frac{1}{n!}\left[\frac{\kappa S\bar{S}}{2|a|c_<(a)}\right]^n\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]\ . \tag{18}$$

Now let us rewrite Eq. (14) as

$$1-\frac{\kappa S\bar{S}}{2|a|c_<(a)}=\frac{1}{c_<(a)E_0^4}\ , \tag{19}$$

and substitute the right-hand side of Eq. (14) for $c_<(a)$ on the same side of Eq. (19). We can find

$$1-\frac{\kappa S\bar{S}}{2|a|c_<(a)}=\frac{2|a|}{\kappa S\bar{S}E_0^4+2|a|}\ . \tag{20}$$

Equation (20) gives us that $\kappa S\bar{S}/2|a|c_<(a)$ approaches 1 as $|a|$ approaches 0. So, using this asymptotic characteristic and Eq. (20) in the expression (18), we get

$$\lambda=\frac{1}{2|a|}\left[\frac{2|a|}{\kappa S\bar{S}E_0^4}\right]^{1/4}\left[1+\sum_{n=1}^{\infty}\frac{1}{1+4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]$$
$$=\left[\frac{1}{2}\right]^{3/4}\left[\frac{1}{\kappa S\bar{S}E_0^4}\right]^{1/4}$$
$$\times\left[1+\sum_{n=1}^{\infty}\frac{1}{1+4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]\left[\frac{1}{|a|}\right]^{3/4}, \tag{21}$$

as $|a|\to 0$.

Finally, comparing Eq. (21) with the definitions of the dynamical threshold exponent q_a' and amplitude p_a', we obtain

$$q_a'=\tfrac{3}{4},$$
$$p_a'=\left[\frac{1}{2}\right]^{3/4}\left[\frac{1}{\kappa S\bar{S}E_0^4}\right]^{1/4}\times\left[1+\sum_{n=1}^{\infty}\frac{1}{1+4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]. \tag{22}$$

(ii) $a>0$. When $a>0$, the steady-state condition becomes $t\to\infty$,

$$E\to E_\infty=\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}.$$

So we have

$$c(a)=\frac{1}{8a}\ln\frac{2a}{\kappa S\bar{S}}+\frac{1}{8a}\ln c_>(a) \tag{23}$$

instead of Eq. (12), where $c_>(a)$ is also a function of the control parameter a. Putting this $c(a)$ into the solution (11) again yields immediately

$$-t=\ln\left[\frac{2ac_>(a)(\kappa S\bar{S}E^4-2a)}{\kappa S\bar{S}E^4}\right]^{1/8a}, \tag{24}$$

or

$$E(t)=\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}\left[\frac{1}{1-\dfrac{\exp(-8at)}{2ac_>(a)}}\right]^{1/4}. \tag{25}$$

Using the initial condition, $E=E_0$ at $t=0$, we have further

$$E_0=\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}\left[\frac{1}{1-\dfrac{1}{2ac_>(a)}}\right]^{1/4}, \tag{26}$$

which gives

$$\frac{1}{2ac_>(a)}=\frac{\kappa S\bar{S}E_0^4-2a}{\kappa S\bar{S}E_0^4}\le 1. \tag{27}$$

The final inequality comes from $a>0$ and that we discuss only an asymptotic behavior of the system; in other words, the control parameter a is small enough.

Based on Eq. (27), we expand $E(t)$ as

$$E(t)=\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{1}{2ac_>(a)}\right]^{n}\exp(-8ant)\times\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]. \tag{28}$$

and put it into the formulation (9).

Since

$$\int_0^\infty -t[dE(t)]$$
$$=\frac{1}{2a}\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}\left[\sum_{n=1}^{\infty}\frac{1}{4n}\frac{1}{n!}\left[\frac{1}{2ac_>(a)}\right]^{n}\times\prod_{m=1}^{n}(m-\tfrac{3}{4})\right] \tag{29}$$

and

$$E(0)-E(\infty)=\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{1}{2ac_>(a)}\right]^{n}\times\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]$$
$$=\left[\frac{2a}{\kappa S\bar{S}}\right]^{1/4}\left[\frac{1}{1-\dfrac{1}{2ac_>(a)}}\right]^{1/4}, \tag{30}$$

we have

$$\lambda=\frac{1}{2a}\left[1-\frac{1}{2ac_>(a)}\right]^{1/4}\times\left[\sum_{n=1}^{\infty}\frac{1}{4n}\frac{1}{n!}\left[\frac{1}{2ac_>(a)}\right]^{n}\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]. \tag{31}$$

Now, we notice $1-1/2ac_>(a)=2a/\kappa S\bar{S}E_0^4$ in Eq. (27) and that $1/2ac_>(a)$ approaches 1 as the control parameter a vanishes. So, the final expression for λ is

$$\lambda=\left[\frac{1}{2}\right]^{3/4}\left[\frac{1}{\kappa S\bar{S}E_0^4}\right]^{1/4}\times\left[\sum_{n=1}^{\infty}\frac{1}{4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]\left[\frac{1}{a}\right]^{3/4}, \tag{32}$$

and the values for q_a and p_a are

$$q_a=\tfrac{3}{4},$$
$$p_a=\left[\frac{1}{2}\right]^{3/4}\left[\frac{1}{\kappa S\bar{S}E_0^4}\right]^{1/4}\left[\sum_{n=1}^{\infty}\frac{1}{4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{3}{4})\right]. \tag{33}$$

IV. THEORETICAL VALUES FOR q_a AND p_a AT $\bar{\sigma}=\bar{\sigma}_0$

Now let us turn to the case of $a=0$ at $\bar{\sigma}=\bar{\sigma}_0$. At this time Eq. (6) is

$$-dt = dE/(\kappa S\bar{S}E^5 - \alpha) . \tag{34}$$

We first perform the transformation

$$\gamma = \left[\frac{\kappa S\bar{S}}{\alpha}\right]^{1/5} E , \tag{35}$$

on Eq. (34) and let it have the form of

$$-dt = -\frac{1}{\alpha}\left[\frac{\alpha}{\kappa S\bar{S}}\right]^{1/5} \frac{d\gamma}{1-\gamma^5} . \tag{36}$$

By integrating this equation, we get

$$t = \frac{1}{\alpha}\left[\frac{\alpha}{\kappa S\bar{S}}\right]^{1/5} \left[-\tfrac{1}{5}\ln(1-\gamma) + \tfrac{1}{5}\cos\left[\frac{\pi}{5}\right]\ln\left[\gamma^2 + 2\gamma\cos\left[\frac{\pi}{5}\right] + 1\right] + \tfrac{1}{5}\cos\left[\frac{3\pi}{5}\right]\ln\left[\gamma^2 + 2\gamma\cos\left[\frac{3\pi}{5}\right] + 1\right] \right.$$
$$\left. + \tfrac{2}{5}\sin\left[\frac{\pi}{5}\right]\tan^{-1}\left[\frac{\gamma + \cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right] + \tfrac{2}{5}\sin\left[\frac{3\pi}{5}\right]\tan^{-1}\left[\frac{\gamma + \cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right] + c(\alpha) \right] , \tag{37}$$

where $c(\alpha)$ is a finite function which will be determined below.

For the estimate of the characteristic time λ, it is convenient to rewrite λ defined in Eq. (9) as

$$\lambda = \frac{-\int_{E_0}^{E_s} t(E)dE}{-\int_{E_0}^{E_s} dE} = \frac{-\int_{E_0}^{E_s} t(E)dE}{E_0 - E_s} , \tag{38}$$

where $E_0 \equiv E(0)$ is an initial value of E, and $E_s \equiv E(\infty) = (\alpha/\kappa S\bar{S})^{1/5}$ is a stationary value at α when $a = 0$. Or, equivalently,

$$\gamma_0 = \left[\frac{\kappa S\bar{S}}{\alpha}\right]^{1/5} E_0 \tag{39a}$$

and

$$\gamma_s = \left[\frac{\kappa S\bar{S}}{\alpha}\right]^{1/5} E_s = 1 , \tag{39b}$$

and

$$\lambda = \frac{-\int_{\gamma_0}^{\gamma_s} t(\gamma)\left[\frac{\alpha}{\kappa S\bar{S}}\right]^{1/5} d\gamma}{E_0 - E_s} , \tag{40}$$

where $t(\gamma)$ is just given by Eq. (37).

Quite fortunately, we can complete the integral on the right-hand side of Eq. (40), of course, at some length.

$$\lambda = \frac{1}{E_0 - E_s}\frac{1}{5\alpha}\left[\frac{\alpha}{\kappa S\bar{S}}\right]^{2/5} \left[\int_{\gamma_0}^{1} \ln(1-\gamma)d\gamma - \int_{\gamma_0}^{1} \cos\left[\frac{\pi}{5}\right]\ln\left[\gamma^2 + 2\gamma\cos\left[\frac{\pi}{5}\right] + 1\right]d\gamma \right.$$
$$- \int_{\gamma_0}^{1} \cos\left[\frac{3\pi}{5}\right]\ln\left[\gamma^2 + 2\gamma\cos\left[\frac{3\pi}{5}\right] + 1\right]d\gamma - \int_{\gamma_0}^{1} 2\sin\left[\frac{\pi}{5}\right]\tan^{-1}\left[\frac{\gamma + \cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]d\gamma$$
$$\left. - \int_{\gamma_0}^{1} 2\sin\left[\frac{3\pi}{5}\right]\tan^{-1}\left[\frac{\gamma + \cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]d\gamma - \int_{\gamma_0}^{1} 5c(\alpha)d\gamma \right] , \tag{41}$$

and

$$\frac{1}{E_0-E_\alpha}=\left[\frac{\kappa S\bar{S}}{\alpha}\right]^{1/5}\frac{1}{\gamma_0-1}\,,$$

where $c(\alpha)$ can be specified by calling $t=0$ and $\gamma=\gamma_0$ in Eq. (37), i.e.,

$$c(\alpha)=\tfrac{1}{5}\ln(1-\gamma_0)-\tfrac{1}{5}\cos\left[\frac{\pi}{5}\right]\ln\left[\gamma_0^2+2\gamma_0\cos\left[\frac{\pi}{5}\right]+1\right]-\tfrac{1}{5}\cos\left[\frac{3\pi}{5}\right]\ln\left[\gamma_0^2+2\gamma_0\cos\left[\frac{3\pi}{5}\right]+1\right]$$
$$-\tfrac{2}{5}\sin\left[\frac{\pi}{5}\right]\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]-\tfrac{2}{5}\sin\left[\frac{3\pi}{5}\right]\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]. \tag{42}$$

We integrate Eq. (41) again and can reach

$$\lambda=\frac{1}{5\alpha}\frac{1}{\gamma_0-1}\left[\frac{\alpha}{\kappa S\bar{S}}\right]^{1/5}$$
$$\times\left[(1-\gamma_0)\ln(1-\gamma_0)-(1-\gamma_0)+\cos\left[\frac{\pi}{5}\right]\left[\left[r_0+\cos\left[\frac{\pi}{5}\right]\right]\ln\left\{\left[r_0+\cos\left[\frac{\pi}{5}\right]\right]^2+\sin^2\left[\frac{\pi}{5}\right]\right\}-2\left[r_0+\cos\left[\frac{\pi}{5}\right]\right]\right.\right.$$
$$\left.+2\sin\left[\frac{\pi}{5}\right]\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]\right]$$
$$-\cos\left[\frac{\pi}{5}\right]\left[\left[1+\cos\left[\frac{\pi}{5}\right]\right]\ln\left\{\left[1+\cos\left[\frac{\pi}{5}\right]\right]^2+\sin^2\left[\frac{\pi}{5}\right]\right\}\right.$$
$$\left.-2\left[1+\cos\left[\frac{\pi}{5}\right]\right]+2\sin\left[\frac{\pi}{5}\right]\tan^{-1}\left[\frac{1+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]\right]$$
$$+\cos\left[\frac{3\pi}{5}\right]\left[\left[\gamma_0+\cos\left[\frac{3\pi}{5}\right]\right]\ln\left\{\left[\gamma_0+\cos\left[\frac{3\pi}{5}\right]\right]^2+\sin^2\left[\frac{3\pi}{5}\right]\right\}\right.$$
$$\left.\left.-2\left[\gamma_0+\cos\left[\frac{3\pi}{5}\right]\right]+2\sin\left[\frac{3\pi}{5}\right]\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]\right]\right]$$

$$-\cos\left[\frac{3\pi}{5}\right]\left[\left[1+\cos\left[\frac{3\pi}{5}\right]\right]\ln\left\{\left[1+\cos\left[\frac{3\pi}{5}\right]\right]^2+\sin^2\left[\frac{3\pi}{5}\right]\right\}\right.$$

$$\left.-2\left[1+\cos\left[\frac{3\pi}{5}\right]\right]+2\sin\left[\frac{3\pi}{5}\right]\tan^{-1}\left[\frac{1+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]\right]\Bigg]$$

$$+2\sin^2\left[\frac{\pi}{5}\right]\left\{\frac{\gamma_0+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]-\tfrac{1}{2}\ln\left[1+\left[\frac{\gamma_0+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]^2\right]\right.$$

$$\left.-\frac{1+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\tan^{-1}\left[\frac{1+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]+\tfrac{1}{2}\ln\left[1+\left[\frac{1+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}\right]^2\right]\right\}$$

$$+2\sin^2\left[\frac{3\pi}{5}\right]\left\{\frac{\gamma_0+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]-\tfrac{1}{2}\ln\left[1+\left[\frac{\gamma_0+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]^2\right]\right.$$

$$\left.-\frac{1+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\tan^{-1}\left[\frac{1+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]+\tfrac{1}{2}\ln\left[1+\left[\frac{1+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]^2\right]\right\}$$

$$+(1-\gamma_0)\left\{-\ln(1-\gamma_0)+\cos\left[\frac{\pi}{5}\right]\ln\left[\gamma_0^2+2\gamma_0\cos\left[\frac{\pi}{5}\right]+1\right]+\cos\left[\frac{3\pi}{5}\right]\ln\left[\gamma_0^2+2\gamma_0\cos\left[\frac{3\pi}{5}\right]+1\right]\right.$$

$$\left.+2\sin\left[\frac{\pi}{5}\right]\tan^{-1}\frac{\gamma_0+\cos\left[\frac{\pi}{5}\right]}{\sin\left[\frac{\pi}{5}\right]}+2\sin\left[\frac{3\pi}{5}\right]\tan^{-1}\left[\frac{\gamma_0+\cos\left[\frac{3\pi}{5}\right]}{\sin\left[\frac{3\pi}{5}\right]}\right]\right\}\Bigg]\Bigg\}, \tag{43}$$

where the expression (42) is used.

After rearrangement, a shorter expression of λ can be obtained as follows:

$$\lambda=\frac{1}{5\alpha}\left[\frac{\alpha}{\kappa S\bar{S}}\right]^{1/5}\frac{1}{\gamma_0-1}\left\{(\gamma_0-1)-2(\gamma_0-1)\cos\left[\frac{\pi}{5}\right]-2(\gamma_0-1)\cos\left[\frac{3\pi}{5}\right]\right.$$

$$+\left[\cos^2\left[\frac{\pi}{5}\right]-\sin^2\left[\frac{\pi}{5}\right]+\cos\left[\frac{\pi}{5}\right]\right]\ln\left[\frac{\gamma_0^2+2\gamma_0\cos\left[\frac{\pi}{5}\right]+1}{2+2\cos\left[\frac{\pi}{5}\right]}\right]$$

$$+\left[\cos^2\left[\frac{3\pi}{5}\right]-\sin^2\left[\frac{3\pi}{5}\right]+\cos\left[\frac{3\pi}{5}\right]\right]\ln\left[\frac{\gamma_0+2\gamma_0\cos\left[\frac{3\pi}{5}\right]+1}{2+2\cos\left[\frac{3\pi}{5}\right]}\right]$$

$$+2\sin\left[\frac{\pi}{5}\right]\left[1+2\cos\left[\frac{\pi}{5}\right]\right]\tan^{-1}\left[\frac{(\gamma_0-1)\sin\left[\frac{\pi}{5}\right]}{(1+\gamma_0)\left[1+\cos\left[\frac{\pi}{5}\right]\right]}\right]$$

$$\left.+2\sin\left[\frac{3\pi}{5}\right]\left[1+2\cos\left[\frac{3\pi}{5}\right]\right]\tan^{-1}\left[\frac{(\gamma_0-1)\sin\left[\frac{3\pi}{5}\right]}{(1+\gamma_0)\left[1+\cos\left[\frac{\pi}{5}\right]\right]}\right]\right\}. \tag{44}$$

Finally, taking $\gamma_0=(\kappa S\bar{S}/\alpha)^{1/5}E_0$ into account and remaining the highest infinite terms in the expression (44) of λ as the control parameter α approaches zero, we obtain

$$\lambda=\frac{1}{5}\left[\frac{1}{\kappa S\bar{S}}\right]^{1/5}\left[\frac{1}{\alpha}\right]^{4/5}\left[1-2\cos\left[\frac{\pi}{5}\right]-2\cos\left[\frac{3\pi}{5}\right]\right], \tag{45}$$

and then

$$p_\alpha=\frac{1}{5}\left[\frac{1}{\kappa S\bar{S}}\right]^{1/5}\left[1-2\cos\left[\frac{\pi}{5}\right]-2\cos\left[\frac{3\pi}{5}\right]\right],$$
$$q_\alpha=\tfrac{4}{5}\;. \tag{46}$$

V. THEORETICAL VALUES FOR EXPONENTS AND AMPLITUDES WHEN $\bar{\sigma}\neq\bar{\sigma}_0$

When $\bar{\sigma}$ is fixed at any value, $\bar{\sigma}<0$, but $\bar{\sigma}_0=-\kappa\gamma_\perp S/\bar{N}\,|\,\bar{g}\,|^2(\bar{S}-S)$, the system has a kind of threshold, $(a,\alpha)=(0,0)$, or equivalently, $(\sigma,\bar{\alpha})=(\sigma_c',0)$, where $\sigma_c'=\kappa\gamma_\perp(1-\bar{N}\,|\,\bar{g}\,|^2\bar{\sigma}/\kappa\bar{\gamma}_\perp)/N\,|\,g\,|^2$.

And the asymptotic time-evolution equation for the system near each of these thresholds is

$$\frac{dE}{dt}=-\frac{\partial V_{\rm asy}(E,a,\alpha)}{\partial E}\;,$$
$$V_{\rm asy}(E,a,\alpha)=RE^4-aE^2-\alpha E\;,$$

where $a=\kappa(\sigma-\sigma_c')/2$, $\alpha=\kappa\bar{\alpha}$, and $R=S+\bar{N}\,|\,\bar{g}\,|^2(\bar{S}-S)\bar{\sigma}/4\bar{\gamma}_\perp$.

Starting with this equation and in a way similar to that in Secs. III and IV, we can estimate theoretically the values for the dynamical threshold exponents and amplitudes at each of these thresholds. To avoid the repeat, we only list the results as follows:

$$q_a=q_a'=\tfrac{1}{2}\;,$$
$$p_a=\left[\frac{1}{8R}\right]^{1/2}\frac{1}{E_0}\left[\sum_{n=1}^{\infty}\frac{1}{2n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{1}{2})\right],$$
$$p_a'=\left[\frac{1}{8R}\right]^{1/2}\frac{1}{E_0}\left[1+\sum_{n=1}^{\infty}\frac{1}{1+2n}\frac{1}{n!}\prod_{m=1}^{n}(m-\tfrac{1}{2})\right], \tag{47}$$

$$q_\alpha=\tfrac{2}{3}\;,$$
$$p_\alpha=\tfrac{1}{3}\left[\frac{1}{4R}\right]^{1/3}.$$

TABLE I. The theoretical values for the dynamical threshold exponents and amplitudes.

	$\bar{\sigma}=\bar{\sigma}_0$	$\bar{\sigma}\neq\bar{\sigma}_0$
q_a	$\frac{3}{4}$	$\frac{1}{2}$
p_a	$\left[\frac{1}{2}\right]^{3/4}\left[\frac{1}{\kappa S\bar{S}}\right]^{3/4}\frac{1}{E_0}\left[\sum_{n=1}^{\infty}\frac{1}{4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\frac{3}{4})\right]$	$\left[\frac{1}{8R}\right]^{1/2}\frac{1}{E_0}\left[\sum_{n=1}^{\infty}\frac{1}{2n}\frac{1}{n!}\prod_{m=1}^{n}(m-\frac{1}{2})\right]$
q_a'	$\frac{3}{4}$	$\frac{1}{2}$
p_a'	$\left[\frac{1}{2}\right]^{3/4}\left[\frac{1}{\kappa S\bar{S}}\right]^{3/4}\frac{1}{E_0}\left[1+\sum_{n=1}^{\infty}\frac{1}{1+4n}\frac{1}{n!}\prod_{m=1}^{n}(m-\frac{3}{4})\right]$	$\left[\frac{1}{8R}\right]^{1/2}\frac{1}{E_0}\left[1+\sum_{n=1}^{\infty}\frac{1}{1+2n}\frac{1}{n!}\prod_{m=1}^{n}(m-\frac{1}{2})\right]$
q_α	$\frac{4}{5}$	$\frac{2}{3}$
p_α	$\frac{1}{5}\left[\frac{1}{\kappa S\bar{S}}\right]^{1/5}\left[1-2\cos\left[\frac{\pi}{5}\right]-2\cos\left[\frac{3\pi}{5}\right]\right]$	$\frac{1}{3}\left[\frac{1}{4R}\right]^{1/3}$

VI. CONCLUSIONS AND DISCUSSIONS

Let us collect our results in Table I.

From the table, it is easy to see that there is slowing down in the dynamical threshold phenomena of the system. In other words, when the control parameters a and α approach the threshold of either $\bar{\sigma}=\bar{\sigma}_0$ or $\bar{\sigma}\neq\bar{\sigma}_0$, the characteristic time goes to infinite.

The critical slowing down in some other nonequilibrium systems was pointed out both theoretically[12–14] and experimentally[15,16] several years ago. Our results indicate again that critical slowing down is rather a universal characteristic of nonequilibrium systems as well as of equilibrium systems.

It is also easily seen that the influence of the existence of absorbing atoms in the laser cavity on the dynamical threshold phenomena is definite, just as well as on the static threshold phenomena.[1] A laser system is a real device in which a lot of theoretical predictions can be experimentally checked. So we expect for an experimental check of the influence of the absorbing atoms on the dynamical threshold exponents and amplitudes.

Finally, we would like to stress two things. The first is that the method used in the context can be applied to other nonequilibrium systems, especially to estimating the switching time in the optical bistability systems.[17] Secondly, because the starting point is the same for calculating the dynamical threshold exponents and amplitudes and the static ones, the expectation that there are some scaling relations between the static and dynamical threshold exponents and amplitudes would not be futile. We shall discuss this problem in another paper.

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