Quantum theory of an atom near partially reflecting walls

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We consider first a dielectric medium of identical two-state atoms coupled by the radiation field to an initially excited atom outside the dielectric. From the Schrödinger equation follows a delay-differential equation describing how the atom interacts with the dielectric by virtual photon exchanges. In the macroscopic limit of a continuous distribution of atoms in the dielectric, we derive a simpler delay-differential equation in which a Fresnel reflection coefficient appears. We apply our results to a model of an atom in a multimode Fabry-Perot resonator, and obtain a general delay-differential equation for the probability amplitude of the initially excited state. This equation predicts well-known Rabi oscillations when the round-trip photon propagation time is negligible compared with the inverse of the Rabi frequency and the mirrors are highly reflective. For low mirror reflectivities we recover Purcell's prediction that the emission rate is enhanced by the cavity Q factor. When the photon bounce time is large compared with the inverse Rabi frequency, Rabi oscillations do not occur. We discuss the Ewald-Oseen extinction theorem from the standpoint of quantum mechanics.

I. INTRODUCTION

In recent years there has been considerable experimental and theoretical interest in the effects of cavity walls on atomic absorption and emission processes. Such effects include the inhibited absorption of blackbody radiation[1] and the enhancement[2] and suppression[3] of spontaneous emission rates. The theory of such cavity effects seems, by and large, well understood.[4]

Of related theoretical interest is the Jaynes-Cummings model in which a two-state atom interacts with a single mode of the electromagnetic field in the dipole and rotating-wave approximations. Jaynes and Cummings[5] showed that the same sort of "Rabi oscillations" are predicted regardless of whether the field is treated classically or quantum mechanically. Their paper was perhaps the first detailed exposition of what is now called the "dressed-atom" approach to resonant atom-field interactions. More recently Eberly *et al.*[6] have stimulated renewed interest in the Jaynes-Cummings model by describing certain "collapse and revival" effects that have now apparently been observed by Walther *et al.*[7]

Classical analogs of such effects follow from consideration of an oscillating dipole inside a cavity. The cavity walls reflect radiation back to the dipole, altering its oscillations and therefore its rate of radiation. One question we address in this paper is how such multiple reflections of radiation off the cavity walls are described quantum mechanically. For instance, if at time $t=0$ an atom is suddenly injected into a cavity, it does not "know" it is in a cavity until a time $2d/c$, where d is the distance to a cavity wall. Such an effect cannot be described within the Jaynes-Cummings model in which one begins *a priori* with a single-mode model, for retardation involves many field modes in an essential way. In our approach, therefore, the single-mode (Jaynes-Cummings) results are *derived* without the *a priori* assumption of a single-mode interaction.

Another question concerns cavity damping. Purcell[8] in 1946 argued that for a lossy cavity the spontaneous emission rate should be increased by a factor Q, the cavity quality factor. Sachdev[9] has considered this problem in the single-mode context, and has shown that Purcell's prediction is justified in the case of an overdamped cavity; in the underdamped case the Rabi oscillations are recovered, but they are damped by the factor $e^{-\gamma t}$, where γ is the field damping rate.[10] A feature of our approach here is that the field loss is not distributed but is lumped at the mirrors as a consequence of imperfect reflectivity.

Since we intend our approach to be fully quantum mechanical, we wish to show how the reflection coefficient follows from the Schrödinger equation describing the coupling of the atoms of the (dielectric) mirror to the field. This we do in Sec. II. In Sec. III we apply the results to an atom in a Fabry-Perot resonator of length L and mirror reflectivities R. We derive a delay-differential equation describing an atom in a multimode, lossy cavity. When $\Omega T \ll 1$, where Ω is the Rabi frequency and $T=2L/c$ is the photon bounce time, we recover known single-mode results. When $\Omega T \gg 1$, on the other hand, the initially excited atomic state decays exponentially with no Rabi oscillations. In Sec. IV we discuss the Ewald-Oseen extinction theorem and summarize our results. Our goal in this paper is mainly to understand how dielectric mirrors may be described in a fully quantum-mechanical way.

II. ATOM NEAR A PLANE DIELECTRIC INTERFACE: MICROSCOPIC THEORY

We consider first the situation illustrated in Fig. 1. At $\mathbf{x}_0$ is an initially excited two-state atom with transition frequency ω_0. It is near a dielectric slab of N ground-state atoms per unit volume, each of which has transition frequency $\omega \approx \omega_0$. For simplicity we assume that all the atoms have the same (real) transition dipole moment μ.

Let $b(t)$ be the probability amplitude for the state in which the atom at $\mathbf{x}_0$ is excited, all other atoms are in their ground states, and the field is in its vacuum state of no photons. Similarly let $b_j(t)$ be the amplitude for the state in which the atom at $\mathbf{x}_j$ ($\neq \mathbf{x}_0$) is excited, all other atoms are in their ground states, and no photons in the field. Finally let $a_n(t)$ be the amplitude for the state in which all the atoms are in their ground states and the field contains one photon in the nth mode. Then in terms of these "essential states" the Schrödinger equation becomes

$$\dot{b}(t) = -\sum_n C_n a_n(t) e^{i\mathbf{k}\cdot\mathbf{x}_0} , \tag{2.1a}$$

$$\dot{b}_j(t) = -i(\omega-\omega_0) b_j(t) - \sum_n C_n a_n(t) e^{i\mathbf{k}_n\cdot\mathbf{x}_j} , \tag{2.1b}$$

$$\dot{a}_n(t) = -i(\omega_n-\omega_0) a_n(t) + C_n b(t) e^{-i\mathbf{k}_n\cdot\mathbf{x}_0} + C_n \sum_j b_j(t) e^{-i\mathbf{k}_n\cdot\mathbf{x}_j} . \tag{2.1c}$$

We have set the energy scale such that the state with the atom at $\mathbf{x}_0$ excited has energy zero. The atom-field interaction $\hat{H}_{\text{int}}$ has been taken to be the "electric-dipole form"

$$\hat{H}_{\text{int}} = i\hbar \sum_i \sum_n C_n (\hat{\sigma}_i^\dagger \hat{a}_n e^{i\mathbf{k}_n\cdot\mathbf{x}_i} - \hat{\sigma}_i \hat{a}_n^\dagger e^{-i\mathbf{k}_n\cdot\mathbf{x}_i}) . \tag{2.2}$$

The electric field has been expanded in a complete set of free-space, plane-wave modes with associated photon annihilation operators $\hat{a}_n$. Actually, for our purposes it is very convenient to restrict our considerations to modes with wave vectors $\mathbf{k}_n$ parallel to the z axis of Fig. 1. Thus we take

$$C_n = \frac{\mu}{\hbar}(2\pi\hbar\omega_n/AL)^{1/2} , \tag{2.3}$$

where A is an effective area, L is the length along the z axis of our quantization box, and μ is the magnitude of the transition dipole moment of each two-state atom. The operators $\hat{\sigma}_j^\dagger$ and $\hat{\sigma}_j$ in (2.2) are, respectively, the raising and lowering operators for the jth two-state atom. For simplicity we take each field mode to be linearly polarized along the direction of the identical dipole moments.

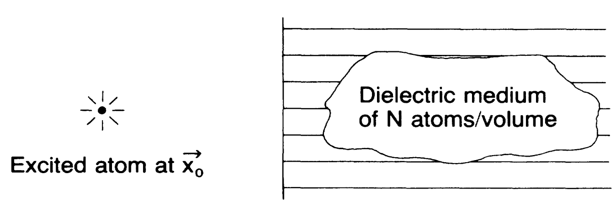

FIG. 1. Excited atom at $\mathbf{x}_0$ located a distance d from a dielectric half-space of N atoms per unit volume.

Equation (2.1c) can be used to formally eliminate the amplitudes $a_n(t)$ from (2.1a) and (2.1b). The result is the coupled set of equations

$$\dot{b}(t) = -\sum_n C_n^2 \int_0^t dt' b(t') e^{i(\omega_n-\omega_0)(t'-t)} - \sum_n \sum_j C_n^2 e^{ik_n(z_0-z_j)} \int_0^t dt' b_j(t') e^{i(\omega_n-\omega_0)(t'-t)} , \tag{2.4a}$$

$$\dot{b}_j(t) = -i(\omega-\omega_0) b_j(t) - \sum_n C_n^2 e^{ik_n(z_j-z_0)} \int_0^t dt' b(t') e^{i(\omega_n-\omega_0)(t'-t)} - \sum_n \sum_i C_n^2 e^{ik_n(z_j-z_i)} \int_0^t dt' b_i(t') e^{i(\omega_n-\omega_0)(t'-t)} . \tag{2.4b}$$

Here we have used our assumption that the atom at $\mathbf{x}_0$ is excited at $t=0$, so that $b(0)=1$, $b_j(0)=a_n(0)=0$.

In the limit in which the length L of our quantization box goes to infinity we have

$$\sum_n C_n^2 e^{i(\omega_n-\omega_0)(t'-t)} = \frac{\mu^2}{\hbar^2}\left[\frac{2\pi\hbar}{AL}\right] \sum_n \omega_n e^{i(\omega_n-\omega_0)(t'-t)} \to \frac{2\pi\mu^2}{\hbar AL}\frac{L}{2\pi}\int_{-\infty}^{\infty} dk' \omega' e^{i(\omega'-\omega_0)(t'-t)} , \tag{2.5}$$

where $|k'| = \omega'/c$. If it is assumed that only modes with frequencies $\omega' \cong \omega_0$ interact strongly with the atoms, one might replace the integration variable k' by ω'/c, allowing ω' to take on negative values. This is a good approximation for our purposes. Thus

$$\begin{aligned}\sum_n C_n^2 e^{i(\omega_n-\omega_0)(t'-t)} &\to \frac{\mu^2}{\hbar A c} e^{-i\omega_0(t'-t)} \int_{-\infty}^{\infty} d\omega' \omega' e^{i\omega'(t'-t)} \\ &= -\frac{i\mu^2}{\hbar A c} e^{-i\omega_0(t'-t)} \frac{\delta}{\delta t'} \int_{-\infty}^{\infty} d\omega' e^{i\omega(t'-t)} \\ &= -\frac{2\pi i\mu^2}{\hbar A c} e^{-i\omega_0(t'-t)} \frac{\delta}{\delta t'} \delta(t'-t) .\end{aligned} \tag{2.6}$$

In this approximation we have

$$\sum_n C_n^2 \int_0^t dt' b(t') e^{i(\omega_n-\omega_0)(t'-t)} \to -\frac{2\pi i\mu^2}{\hbar A c} \int_0^t dt' b(t') e^{-i\omega_0(t'-t)} \times \frac{\delta}{\delta t'} \delta(t'-t) \to \frac{\pi\mu^2\omega_0}{\hbar A c} b(t) . \tag{2.7}$$

We are ignoring an infinite term corresponding to a single-atom frequency shift. We have also made use of the fact that $b(t)$ is slowly varying compared with $e^{-i\omega_0 t}$. Our approach is just a variant of Weisskopf-Wigner theory, specialized to the case of field modes propagating along the z axis.

Similarly

$$\sum_n C_n^2 e^{ik_n z} e^{i(\omega_n-\omega_0)(t'-t)} \to \frac{\mu^2}{\hbar^2}\left[\frac{2\pi\hbar}{AL}\right]\left[\frac{L}{2\pi}\right]\int_{-\infty}^{\infty} dk'\omega\cos(k'z)e^{i(\omega'-\omega_0)(t'-t)}$$
$$\to \frac{\mu^2}{\hbar Ac}\int_{-\infty}^{\infty} d\omega'\omega'\cos(\omega' z/c)e^{i(\omega'-\omega_0)(t'-t)}$$
$$= \frac{-i\pi\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\frac{\delta}{\delta t'}[\delta(t'-t+z/c)+\delta(t'-t-z/c)] \tag{2.8}$$

and so

$$\sum_n C_n^2 e^{ik_n(z_0-z_j)}\int_0^t dt' b(t')e^{i(\omega_n-\omega_0)(t'-t)}$$
$$\to \frac{\pi\mu^2\omega_0}{\hbar Ac}e^{ik_0 l_j} b_j(t-l_j/c)\Theta(t-l_j/c) , \tag{2.9}$$

where Θ is the unit step function, $k_0=\omega_0/c$, and $l_j=|z_0-z_j|$.

Combining these results in Eqs. (2.4), we have the delay-differential equations[11]

$$\dot b(t) = -Kb(t) - K\sum_j e^{ik_0 l_j} b_j(t-l_j/c)\Theta(t-l_j/c) , \tag{2.10a}$$

$$\begin{aligned}\dot b_j(t) = &-i(\omega-\omega_0)b_j(t)\\ &-Ke^{ik_0 l_j} b(t-l_j/c)\Theta(t-l_j/c) - Kb_j(t)\\ &-K\sum_{i\neq j} e^{ik_0 l_{ij}} b_i(t-l_{ij}/c)\Theta(t-l_{ij}/c) ,\end{aligned} \tag{2.10b}$$

where $l_{ij}=|z_i-z_j|$ and $K\equiv\pi\mu^2\omega_0/\hbar Ac$. Equations (2.10) express the effects of atom-atom coupling in a way that displays explicitly the retarded nature of the electromagnetic interaction.[12]

Since we are interested in the case in which only the atom at $\mathbf{x}_0$ is initially excited, it is reasonable to suppose that the coupling $b_i - b_j$ of probability amplitudes for atoms inside the dielectric is small compared with the $b-b_j$ coupling involving the atom at $\mathbf{x}_0$. If $|\omega-\omega_0| \gg K$, furthermore, we may write

$$b_j(t) \cong -Ke^{ik_0 l_j}\int_0^t dt' b(t'-l_j/c)\Theta(t'-l_j/c)\times e^{i(\omega-\omega_0)(t'-t)}$$
$$\cong \frac{iK}{\omega-\omega_0}e^{ik_0 l_j} b(t-l_j/c)\Theta(t-l_j/c) . \tag{2.11}$$

This "adiabatic following" approximation is obtained by partial integration, making use of the fact that $b(t)$ is slowly varying compared with $e^{i(\omega-\omega_0)t}$, because of our assumption that $K \ll |\omega-\omega_0|$.

$$b_j(t-l_j/c)\Theta(t-l_j/c) \cong \frac{iK}{\omega-\omega_0}e^{ik_0 l_j} b(t-2l_j/c)\Theta(t-2l_j/c) \tag{2.12}$$

and, from (2.10a)

$$\dot b(t) = -Kb(t) - \frac{iK^2}{\omega-\omega_0}\sum_j e^{2ik_0 l_j} b(t-2l_j/c) \times\Theta(t-2l_j/c) . \tag{2.13}$$

This delay differential equation describes the effect of the dielectric in Fig. 1 on the probability amplitude b for the atom at $\mathbf{x}_0$ to be excited.

Now we pass to the limit in which the dielectric contains $NAdz$ atoms in the slice $[z,z+dz]$, making the replacement

$$\sum_j e^{2ik_0 l_j} b(t-2l_j/c)\Theta(t-2l_j/c)$$
$$\to NA\int_l^{\infty} dz' e^{2ik_0(z'-z_0)} \times b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c) . \tag{2.14}$$

Partial integration yields for the integral the approximate expression

$$-\frac{1}{2ik_0}e^{2ik_0 d} b(t-2d/c)\Theta(t-2d/c) , \tag{2.15}$$

where $d=l-z_0$ is the distance of the atom at $\mathbf{x}_0$ from the dielectric interface. The approximation (2.15) uses again the fact that $b(t)$ varies slowly compared with $e^{-i\omega_0 t}$, and we ignore rapidly oscillating terms. Combining this result with (2.13), we obtain

$$\dot b(t) = -Kb(t) + K\left[\frac{N\pi\mu^2/2\hbar}{\omega-\omega_0}\right]e^{2ik_0 d} b(t-2d/c) \times\Theta(t-2d/c) . \tag{2.16}$$

The refractive index of a dielectric of N two-state atoms per unit volume, each of transition frequency ω and transition dipole moment μ, is given by

$$n(\omega_0)^2-1=\frac{4\pi N\mu^2\omega/\hbar}{\omega^2-\omega_0^2} \tag{2.17}$$

if local field effects are ignored, as is permissible for a dilute medium. Of course, this is consistent with our earlier neglect of the last term in (2.10b), which corresponds to atom-atom interactions within the dielectric. For $\omega\approx\omega_0$ and $n(\omega_0)\cong 1$, therefore, we have

$$\frac{N\pi\mu^2/2\hbar}{\omega-\omega_0}\cong\frac{n-1}{n+1} \tag{2.18}$$

and

$$\dot{b}(t)\cong -K[b(t)+Re^{2ik_0d}b(t-2d/c)\Theta(t-2d/c)]\ , \tag{2.19}$$

where $R=-(n-1)/(n+1)$ is the reflection coefficient according to the Fresnel formula for normal incidence.

The solution of (2.19) with the initial condition $b(0)=1$ is

$$b(t)=\sum_{n=0}^{\infty}\frac{(-KRe^{2ik_0d})^n}{n!}(t-2nd/c)^n \times e^{-K(t-2nd/c)}\Theta(t-2nd/c)\ . \tag{2.20}$$

Similar solutions were discussed some time ago for the resonant two-atom interaction.[12] When $2d/c\rightarrow 0$ (2.20) reduces to

$$b(t)=e^{-K(1+Re^{2ik_0d})t}\ , \tag{2.21}$$

which also could have been deduced from (2.19). From this result we can see that $K[1+R\cos(2k_0d)]$ and $KR\sin(2k_0d)$ represent a decay rate and frequency shift, respectively, of the atom near the dielectric, although these expressions are unrealistic in the sense that we have only included modes propagating parallel to the z axis in our analysis. The more realistic expressions including all field modes are easily derived.[13] Our point here is that (2.20) describes how the atom-dielectric interaction builds up, as it were, by "virtual photon exchanges." The restriction to modes propagating parallel to the z axis becomes somewhat more meaningful in Sec. III.

Note that in effect a layer of atoms of depth $\approx k_0^{-1}\approx\lambda_0$ gives rise to the reflection coefficient. That is, the reflection coefficient arises mainly from atoms near the surface of the dielectric interface. This is seen from the approximation (2.15) to the integral appearing in (2.14). From a rigorous point of view, however, the reflection coefficient has contributions from all the atoms comprising the dielectric. We discuss this point further in Sec. IV.

III. ATOM IN A FABRY-PEROT RESONATOR

We first consider an atom between two perfectly conducting walls (Fig. 2). Then we will generalize to the case of mirror reflectivities $R\neq -1$. We could proceed as in the preceding section, but instead we will follow a slightly different approach, expanding the field in mode functions $\sin(k_nz)$, $k_n=n\pi/L$. The Schrödinger equation now takes the form

$$\dot{b}(t)=-D\sum_n\omega_n^{1/2}b_n(t)\sin(k_nz_0)\ , \tag{3.1a}$$

$$\dot{b}_n(t)=-i(\omega_n-\omega_0)b_n(t)+D\omega_n^{1/2}b(t)\sin(k_nz_0)\ , \tag{3.1b}$$

where $D=(4\pi\mu^2/\hbar AL)^{1/2}$ and again we have employed the essential-states approximation, $b(t)$ being the amplitude for the state in which the atom is excited and there are no photons in the field, and $b_n(t)$ the amplitude for one photon in mode n and the atom deexcited. Using the formal solution of the second equation in the first, we have $[b_n(0)=0,\ b(0)=1]$

$$\dot{b}(t)=-D^2\int_0^t dt'b(t')\sum_{n=0}^{\infty}\omega_n\sin^2(k_nz_0)\times e^{i(\omega_n-\omega_0)(t'-t)}\ , \tag{3.2}$$

where $\omega_n=n\Delta$, $k_n=\omega_n/c$, and $\Delta=\pi c/L$ is the mode (angular) frequency separation. Now by the same sort of argument used in Sec. II we let the ω_n take on negative values, arguing that frequencies far removed from the atomic resonance frequency cannot make a significant contribution. Thus we replace (3.2) by

$$\begin{aligned}\dot{b}(t)&=iD^2\int_0^t dt'b(t')e^{-i\omega_0(t'-t)}\frac{\delta}{\delta t'}\sum_{n=-\infty}^{\infty}\sin^2(k_nz_0)e^{in\Delta(t'-t)}\\&=\frac{i}{2}D^2\int_0^t dt'b(t')e^{-i\omega_0(t'-t)}\frac{\delta}{\delta t'}\left[\sum_{-\infty}^{\infty}e^{in\Delta(t'-t)}-\tfrac{1}{2}\sum_{-\infty}^{\infty}e^{in\Delta(t'-t+2z_0/c)}-\tfrac{1}{2}\sum_{-\infty}^{\infty}e^{in\Delta(t'-t-2z_0/c)}\right]\ .\end{aligned} \tag{3.3}$$

From the Poisson summation formula

$$\sum_{-\infty}^{\infty}e^{inx}=\sum_{-\infty}^{\infty}\delta(x/2\pi-n)\ , \tag{3.4}$$

therefore, we obtain after some straightforward manipulations as above the delay-differential equation

$$\dot{b}(t)=-\tfrac{1}{2}\Omega^2T\left[\tfrac{1}{2}b(t)+\sum_{n=1}^{\infty}e^{in\omega_0T}b(t-nT)\Theta(t-nT)-\tfrac{1}{2}e^{2ik_0z_0}\sum_{n=0}^{\infty}e^{in\omega_0T}b(t-T_0-nT)\Theta(t-T_0-nT)\right.$$
$$\left.-\tfrac{1}{2}e^{-2ik_0z_0}\sum_{n=1}^{\infty}e^{in\omega_0T}b(t+T_0-nT)\Theta(t+T_0-nT)\right], \tag{3.5}$$

where $\Omega^2=D^2\omega_0=4\pi\mu^2\omega_0/\hbar AL$, $k_0=\omega_0/c$, $T=2L/c$, and $T_0=2z_0/c$. This equation displays retardation effects due to the presence of *two* walls, and as such has a much more complicated delay-time structure than (2.19).

Writing $\omega_0=\omega+\Delta_0$, where ω is the frequency of the cavity mode closest to the atomic frequency ω_0, we have

$$e^{i\omega_0T}=e^{i(\omega+\Delta_0)(2L/c)}=e^{i\Delta_0T} \tag{3.6}$$

as a result of the mode condition $\sin(\omega L/c)=0$. Therefore we may replace $e^{in\omega_0T}$ by $e^{in\Delta_0T}$ in (3.5).

To account for mirror reflectivities $|R|<1$ we modify (3.5) based on the results of Sec. II. Assuming both mirrors have the same reflectivity $R=-(n-1)/(n+1)$, and associating with each "bounce" off a mirror a factor R, therefore, we replace (3.5) by

$$\dot{b}(t)=-\tfrac{1}{2}\Omega^2T\left[\tfrac{1}{2}b(t)+\sum_{n=1}^{\infty}(R^2e^{i\Delta_0T})^nb(t-nT)\theta(t-nT)+\tfrac{1}{2}\mathrm{R}e^{2ik_0z_0}\sum_{n=0}^{\infty}(R^2e^{i\Delta_0T})^nb(t-T_0-nT)\theta(t-T_0-nT)\right.$$
$$\left.+\tfrac{1}{2}R^{-1}e^{-2ik_0z_0}\sum_{n=1}^{\infty}(R^2e^{i\Delta_0T})^nb(t+T_0-nT)\theta(t+T_0-nT)\right]. \tag{3.7}$$

Equation (3.7) is quite general in that it includes possible effects of *all* longitudinal modes, as well as mirror reflectivities $R\neq-1$. In general, however, the time dependence prescribed by (3.7) is rather complicated. For this reason it is worthwhile to focus our attention on some special cases, and show that some well-known results can be recovered from (3.7).

A. Rabi oscillations

For perfectly reflecting mirrors, $R=-1$. For $\Omega T\ll1$ and $t\gg T$ we have in this case

$$\dot{b}(t)\cong-\tfrac{1}{2}\Omega^2e^{i\Delta_0t}\left[T\sum_{n=1}^{\infty}e^{-i\Delta_0(t-nT)}b(t-nT)\Theta(t-nT)-\tfrac{1}{2}e^{2ik_0z_0}T\sum_{n=0}^{\infty}e^{-i\Delta_0(t-nT)}b(t-T_0-nT)\Theta(t-T_0-nT)\right.$$
$$\left.-\tfrac{1}{2}e^{-2ik_0z_0}T\sum_{n=1}^{\infty}e^{-i\Delta_0(t-nT)}b(t+T_0-nT)\Theta(t+T_0-nT)\right]$$
$$\cong-\tfrac{1}{2}\Omega^2e^{i\Delta_0t}(1-\tfrac{1}{2}e^{2ik_0z_0}-\tfrac{1}{2}e^{-2ik_0z_0})\int_0^t dt'b(t')e^{-i\Delta_0t'}$$
$$=-\Omega^2\sin^2(k_0z_0)e^{i\Delta_0t}\int_0^t dt'b(t')e^{-i\Delta_0t'} \tag{3.8}$$

or

$$\ddot{b}(t)-i\Delta_0\dot{b}(t)+\frac{\lambda^2}{4}b(t)\cong0 \tag{3.9}$$

with $\lambda=2\Omega\sin(k_0z_0)$. Thus

$$b(t)\cong e^{i\Delta_0t}\left[\cos[\tfrac{1}{2}(\Delta_0^2+\lambda^2)^{1/2}t]-\frac{i\Delta_0}{(\Delta_0^2+\lambda^2)^{1/2}}\sin[\tfrac{1}{2}(\Delta_0^2+\lambda^2)^{1/2}t]\right]. \tag{3.10}$$

which displays the well-known Rabi oscillations for an atom interacting with a single field mode. (In this case they may be termed "vacuum" Rabi oscillations.) For exact resonance, $\Delta_0=0$, we have $b(t)=\cos(\tfrac{1}{2}\lambda t)$, and the population difference

$$|b(t)|^2-(1-|b(t)|^2)=\cos(\lambda t). \tag{3.11}$$

Thus our general delay-differential equation (3.7) predicts Rabi oscillations when $\Omega T\ll1$, i.e., when the photon bounce frequency $T^{-1}=c/2L$ is much greater than the Rabi frequency. Note that when this condition is satisfied we also have $\Omega^2T\ll T^{-1}=\Delta$, i.e., the cavity-mode spacing is large compared with the spontaneous decay rate and therefore the natural linewidth.[14] Furthermore $\Omega^2T\ll\Omega$ means that the spontaneous emission rate is small compared with the Rabi frequency. Thus it is not surprising that Rabi oscillations occur in this limit. This

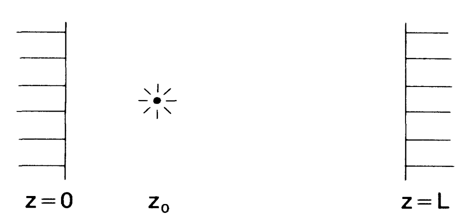

FIG. 2. Excited atom between two dielectric walls, located a distance z_0 from the nearest wall.

single-mode limit is the same limit for which retardation effects are negligible.

B. Damped single-mode cavity

Now let us consider the same single-mode limit for $R \neq -1$. Writing

$$R^2 e^{i\Delta_0 T} = e^{(i\Delta_0 + T^{-1}\ln R^2)T} = e^{i(\Delta_0 + i\gamma)T} \tag{3.12}$$

with $\gamma = -(c/2L)\ln R^2 > 0$ the field damping rate due to imperfectly reflecting walls, we can perform the same manipulations that led to (3.9) to obtain

$$\ddot{b}(t) - i(\Delta_0 + i\gamma)\dot{b}(t) + \frac{\lambda^2}{4} b(t) \cong 0 \ . \tag{3.13}$$

For $\Delta_0 = 0$ and $\lambda \gg \gamma$, for instance, we have

$$|b(t)|^2 \simeq e^{-\gamma t}\cos^2(\tfrac{1}{2}\lambda t) \ . \tag{3.14}$$

For $\gamma \gg \lambda$, on the other hand,

$$|b(t)|^2 \cong e^{-\lambda^2 t/2\gamma} \ . \tag{3.15}$$

These results agree with those of Purcell[8] and Sachdev.[9] In fact, the solution of (3.13) with $\Delta_0 = 0$ is exactly equivalent to Sachdev's general solution (5.2) obtained by a different approach. We therefore refer the reader to Ref. 9 for a discussion of the single-mode case with cavity damping. (See also Ref. 10.)

C. Damped multimode cavity

If T is increased the solution of (3.7) has a complicated delay-time structure, as discussed earlier for a similar but somewhat simpler quantum delay-differential system.[15] In particular, if T is large then the Rabi frequency is large compared with the photon bounce frequency. The atom can emit spontaneously before feeling the presence of the cavity, and later it can absorb the emitted photon, reemit, etc., without any coherent Rabi oscillations associated with the single-mode limit $\Omega T \ll 1$. Since the solutions in this case resemble those shown in Ref. 15, we will not take the time here to display them graphically.

IV. REMARKS ON THE EWALD-OSEEN EXTINCTION THEOREM

According to the classical Ewald-Oseen extinction theorem,[16] the polarization induced in a dielectric medium produces in the medium a field that consists of two parts. One part exactly cancels the incident field inside the medium, whereas the other propagates inside the medium at the phase velocity c/n. The field radiated out from the medium is just the reflected field, with amplitude determined by the Fresnel reflection coefficient.

In Sec. II we obtained, starting from the Schrödinger equation, the correct Fresnel reflection coefficient for normal incidence, assuming $n \cong 1$. The approximation (2.15) to the integral appearing in (2.14) indicates that a number of atoms $\approx NA/k_0 \approx NA\lambda_0$ contributes to the reflected field. That is, atoms in a layer of depth $\approx \lambda_0$ at the surface of the dielectric cooperate to produce the reflected field. This result is consistent with classical arguments.[17,18]

To describe the extinction of the incident field inside the medium, consider the steady-state solution of Equation (2.10b),

$$b_j = -(iK/\Delta_0) b e^{ik_0 l_j} - (iK/\Delta_0)\sum_{i\neq j} b_i e^{ik_0 l_{ij}} \ . \tag{4.1}$$

Here we have again used the nonresonance assumption $|\Delta_0| = |\omega_0 - \omega| \gg K$ to replace $\Delta_0 + iK$ by Δ_0. The steady-state assumption is useful in order to focus our attention on a single-frequency component at a time (as in the classical Ewald-Oseen extinction theorem). If the second term on the right side of (4.1) were absent, we would have

$$|b_j|^2 = (K^2/\Delta_0^2)|b|^2 \tag{4.2}$$

for the probability of exciting some atom j in the medium. We now ask how this result is modified by the presence of the last term in (4.1). In other words, what is the probability that the initially excited atom can excite an atom within an entire dielectric medium of atoms?

In the continuum limit we replace the summation in (4.1) by an integral over z, as in Sec. II. Writing $b(z)$ instead of b_j, we then have the following integral equation for $b(z)$:

$$\begin{aligned} b(z) &= -(iK/\Delta) b e^{ikz} - (iK/\Delta_0) NA \int_0^\infty dz' b(z') e^{ik|z'-z|} \\ &= -(iK/\Delta_0) b e^{ikz} - (iK/\Delta_0) NA \left[\int_0^z dz' b(z') e^{ik(z-z')} + \int_z^\infty dz' b(z') e^{ik(z'-z)} \right] , \end{aligned} \tag{4.3}$$

where we have written k for k_0. To solve this equation we seek a solution of the form

$$b(z) = Ce^{ik'z} , \tag{4.4}$$

where C and k' are constants to be determined. Using this form in (4.3), and performing the integrals, we obtain

$$Ce^{ik'z} = -(iK/\Delta_0) b e^{ikz} - (KNAC/\Delta_0)\left[\frac{e^{ik'z} - e^{ikz}}{k'-k} - \frac{e^{ik'z}}{k'+k} \right] \tag{4.5}$$

and, equating coefficients of e^{ikz} and $e^{ik'z}$,

$$k'^2 - k^2 = -2kKNA/\Delta_0 , \tag{4.6a}$$

$$C = i(k'-k)/NA \ . \tag{4.6b}$$

Equation (4.6a) gives $k' = kn$, where

$$n^2 - 1 = -2KNA/k\Delta_0 \tag{4.7a}$$

or

$$n \cong 1 + \pi\mu^2 N/\hbar(\omega - \omega_0) \tag{4.7b}$$

for $n-1 \ll 1$. n is just the refractive index for light of frequency ω_0 in a medium of N two-state atoms per unit volume, each with transition dipole moment μ and transition frequency ω. [Equation (2.18).]

Equation (4.6b) is the condition that the incident field in the medium is exactly cancelled by the part of the dipole field in the medium that varies as $\exp(ikz)$. [See Eq. (4.5).] Using our result for k', (4.6b) gives

$$C=\frac{2}{n+1}(-iKb/\Delta_0)\cong\frac{2n}{n+1}(-iKb/\Delta_0) . \tag{4.8}$$

From Eqs. (4.2) and (4.5) we see that the amplitude to excite any atom in the *medium* is a factor $2n/(n+1)$ times the amplitude to excite that atom if it were in *free space.* This factor is just the Fresnel transmission coefficient (for normal incidence, because in our model we only allow plane-wave modes propagating normally to the dielectric interface).

These results provide a fully *microscopic,* quantum-mechanical basis for the Ewald-Oseen extinction theorem, starting from the Schrödinger equation. Equation (4.3) is a quantum analogue, in terms of probability amplitudes, of the classical superposition principle for the field.

The extinction theorem is a nonlocal boundary condition that the field must satisfy.[19] Physically, the cancellation of the incident field is often regarded as "caused by the dipoles on the *boundary* of the medium,"[20] since in the classical macroscopic approach the term that cancels the incident field can be cast in the form of a surface integral over dipole sources. Our approach shows how all the dipole contributions add up in such a way that the cancellation is effectively due to dipoles within a depth approximately equal to λ at the surface. In particular, "The (reflected) radiation comes from everywhere in the interior, but it turns out that the total effect is equivalent to a reflection from the surface."[18]

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