Symmetry properties of quasiclassical energy levels

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The quasiclassical ground-state energies exhibit typical analytic structures, as well as covariance properties, with respect to certain symmetry transformations. Such symmetries enable us to define corresponding classes of equivalent Hamiltonians. The scaling properties of the underlying phase-space quantum have also been established. Here we shall consider spherically symmetrical Hamiltonians like $p^2/m_0+C_{n_1}/r^{n_1}+C_{n_2}/r^{n_2}$, where $n_1\neq 2$, $n_1=2$, and $C_{n_1}=0$. C_{n_i} $(i=1,2)$ denote the couplings, whereas n_i are the power exponents. Generalizations towards exact or approximate energy levels, depending on the values of n_1 and n_2, have also been performed. For $n_1=2$, this procedure leads us to reobtain the exact energy levels for $n_2=1$ and $n_2=-2$, and to propose closed estimates for the other n_2 values. The quasiclassical equivalence between the linear plus Coulomb potential and the quartic anharmonic oscillator has also been established.

I. INTRODUCTION

Proofs have been given that useful analytic expressions for the ground-state energies (GSE's) of spherically symmetrical Hamiltonians such as $H(r,p)$ can be obtained with the help of the quasiclassical minimization[1]

$$E=\min\delta H(r)\ , \tag{1.1}$$

where $\delta H(r)=H(r,\hbar d_0/r)$. Here d_0 denotes the underlying phase-space quantum (PSQ). On the other hand, the GSE's established in this manner exhibit typical analytic structures, as well as covariance properties with respect to certain symmetry transformations, which deserve further attention. Generalizations towards exact or approximate energy levels can also be performed. In what follows, we would like to concentrate on the scaling and quasiclassical symmetry properties characterizing the nonrelativistic Hamiltonian

$$H_{n_1}^{(n_2)}(r,p)=\frac{p^2}{m_0}+\frac{C_{n_1}}{r^{n_1}}+\frac{C_{n_2}}{r^{n_2}}\ , \tag{1.2}$$

where $V_{n_1}=C_{n_1}/r^{n_1}$ and $V_{n_2}=C_{n_2}/r^{n_2}$ are power potentials. Particular cases such as

$$H_n(r,p)=\frac{p^2}{m_0}+V_n\ , \tag{1.3}$$

in which $nC_n<0$ and $n<2$, as well as

$$H_2^{(n)}(r,p)=\frac{p^2}{m_0}+V_2+V_n\ , \tag{1.4}$$

where $n\neq 2$ and $C_2\neq 0$, will also be discussed in some more general terms. Above $r=|\mathbf{x}|$ and $p=|\mathbf{p}|$. Using the parametrizations $C_n=m_0\gamma(n)l_1^n$, $l_1=\hbar/m_0$, $C_2=m_0\lambda l_1^2$, and $x=r/l_1$ leads to

$$\varepsilon_2^{(n)}(x)=\frac{1}{m_0}\delta H_2^{(n)}(r)=\frac{d_0^2+\lambda}{x^2}+\frac{\gamma(n)}{x^n}\ , \tag{1.5}$$

in which $l=0$. Accordingly, the GSE of Eq. (1.5) can be parametrized as[2]

$$\varepsilon_2^{(n)}=\xi\frac{n-2}{n}F(n)[\xi(d_0^2+\lambda)]^{n/n-2}\ , \tag{1.6}$$

where[3]

$$F(n)=\left[\frac{n_0}{2}\gamma_0(n)\right]^{2/2-n}\ . \tag{1.7}$$

In addition $\mathrm{sgn}\xi(\lambda+d_0^2)=1$, where

$$\xi=\mathrm{sgn}(2-n)=\mathrm{sgn}[-n\gamma(n)]=\mathrm{sgn}(d_0^2+\lambda)\ , \tag{1.8}$$

whereas $n_0=|n|$, $\gamma_0(n)=|\gamma(n)|$, and $\lambda_0=|\lambda|$. The l dependence concerns d_0^2. However, the centrifugal barrier $l(l+1)/x^2$ will also be written down to illustrate the influence of scale factors. The dimensionless potential energy will be represented by $V(x)$ whereas $\varepsilon(x)=\delta H(r)/m_0$ (i.e., the so-called energy "dispersion") will be referred to as the Hamiltonian form.

This paper is organized as follows. In Sec. II we analyze the scaling properties of d_0^2 in terms of those of the corresponding energy levels. Symmetry transformations of the Hamiltonian forms are discussed in Sec. III. This leads us to establish the quasiclassical equivalence between the quartic anharmonic oscillator and the linear plus Coulomb potential in Sec. IV. Suitable parametrizations concerning the energy levels of Eqs. (1.3) and (1.4) will be given in Sec. V. The conclusions are presented in Sec. VI. One appendix has also been included.

II. SCALING PROPERTIES OF THE PHASE-SPACE QUANTUM

Our approach originates from the idea of extending, in a quasiclassical manner, the virial theorem towards the limiting case of singular power functions like $\psi\rightarrow\psi_a\sim r^{-id_0}$ or $\psi\rightarrow\varphi_a\sim r^{-d_0}$. This proceeds with the help of a suitable reordering of the square of the momen-

tum operator, now in terms of dilations. We recognize that the main problem is a suitable definition of d_0. In this respect a useful input is to consider that d_0 has magnitude of the order of unity. This has the meaning of a first or intermediary approximation. The understanding is that the PSQ is in many soluble cases a smoothly varying function of the underlying couplings. Then step-function approximations, i.e., the numerical description of d_0 in terms of one, two, or more constants, turn out to be quite reasonable. The advantage of such approximations is that they exhibit the convexity of the GSE with respect to the couplings.[4] Such couplings should occur linearly in the Hamiltonian. Of course, there are potentials, as, e.g., the power potentials (see below), for which d_0 takes from the very beginning constant values. In contradistinction, the PSQ of Eq. (1.4) exhibits a nontrivial square-root dependence on the C_2 coupling.[5] These results show that the functional analytic properties of d_0 are open for further discussions. Accordingly, we shall continue our previous studies by establishing a first-order differential equation for d_0, which comes from the scaling properties of the GSE.

Before beginning, we would like to mention that the evaluation of the GSE in terms of quasiclassical parametrizations of energy minima has also been discussed previously.[6] For this purpose, the usual virial and Feynman-Hellman theorems have been used. However, absolute constant d_0 values have been invoked, so that further generalizations are necessary. Concrete improvements have also been proposed.[7] The main idea is to perform the energy minimization in terms of the rescaled eigenstates of a soluble Hamiltonian. This in turn relies on the general scaling-variational approach to the virial theorem.[8] Such results show that one should proceed further combining suitably the usual and quasiclassical formulations of the virial equation. For this purpose let us consider the quasiclassical minimization of

$$\varepsilon=\varepsilon_{n_1}^{(n_2)}(x)=\frac{d_0^2}{x^2}+\frac{\gamma(n_1)}{x^{n_1}}+\frac{\gamma(n_2)}{x^{n_2}}\ , \tag{2.1}$$

where $l=0$. This works in terms of the quasiclassical virial equation

$$d_0^2=-\frac{n_1}{2}\gamma(n_1)x^{2-n_1}-\frac{n_2}{2}\gamma(n_2)x^{2-n_2}\ , \tag{2.2}$$

and of the concavity condition

$$C_0\equiv n_1(n_1-2)\frac{\gamma(n_1)}{x^{n_1}}+n_2(n_2-2)\frac{\gamma(n_2)}{x^{n_2}}>0\ . \tag{2.3}$$

It is understood that Eq. (2.1) exhibits automatically the GSE as soon as Eqs. (2.2) and (2.3) are fulfilled. Going beyond the GSE, a more general quasiclassical extremization can also be considered. Coming back to Eq. (2.1) and performing the differentiation with respect to $\gamma_1=\gamma(n_1)$ and $\gamma_2=\gamma(n_2)$ gives

$$\begin{aligned}\partial_1\varepsilon&=\frac{1}{x^{n_1}}+\frac{1}{x^2}\partial_1 d_0^2>\frac{1}{x^2}\partial_1 d_0^2\ ,\\ \partial_2\varepsilon&=\frac{1}{x^{n_2}}+\frac{1}{x^2}\partial_2 d_0^2>\frac{1}{x^2}\partial_2 d_0^2\ ,\end{aligned} \tag{2.4}$$

in which $\partial_1=\partial/\partial\gamma_1$ and $\partial_2=\partial/\partial\gamma_2$. This means that d_0^2 should be a decreasing function of γ_i as soon as ε does the same. The concavity conditions

$$\begin{aligned}\partial_1^2 d_0^2&=C_0(\partial_1 x)^2+x^2\partial_1^2\varepsilon>x^2\partial_1^2\varepsilon\ ,\\ \partial_2^2 d_0^2&=C_0(\partial_2 x)^2+x^2\partial_2^2\varepsilon>x^2\partial_2^2\varepsilon\ ,\end{aligned} \tag{2.5}$$

should also be noticed. We can also make the identification $1/x^2\equiv\partial\varepsilon/\partial d_0^2$, which is tied to Eq. (2.1). Then Eq. (2.4) shows that the quasiclassical energy minimum is subject to the differential equation

$$D\varepsilon=\varepsilon+\frac{\partial\varepsilon}{\partial d_0^2}Dd_0^2\ , \tag{2.6}$$

in which D denotes the differential operator

$$D=\tfrac{1}{2}(2-n_1)\gamma_1\partial_1+\tfrac{1}{2}(2-n_2)\gamma_2\partial_2\ . \tag{2.7}$$

On the other hand, the usual virial and Feynman-Hellman theorems show that the energy eigenvalue E fulfills the scaling equation[9]

$$DE=E\ , \tag{2.8}$$

which also remains valid, of course, for the GSE. At this point we want to stress that both E and ε should exhibit definitely the same scaling behavior. Combining Eqs. (2.6) and (2.8), we then arrive at the differential equation

$$Dd_0^2=0\ , \tag{2.9}$$

which shows that the most general solution to d_0^2 has the form $d_0^2=f_1(v)$. So far f_1 is a general non-negative function of the scaling variable

$$v=|\gamma_1|^{2-n_2}|\gamma_2|^{n_1-2}\ , \tag{2.10}$$

which is defined up to a power exponent. The typical powerlike behavior of d_0^2 can also be established using the new variables $x/\gamma_1^{1/n_1}$ and $x/\gamma_2^{1/n_2}$ instead of x. Then one would have in both cases $d_0^2\sim v^\delta$ if $\delta=1/(n_1-n_2)$. This in turn leads to the typical scaling variable

$$u=|\gamma_1|^{(2-n_2)/(n_1-n_2)}|\gamma_2|^{(n_1-2)/(n_1-n_2)}\ , \tag{2.11}$$

where $n_1\neq n_2$. Thus the most general solution to d_0^2 can be expressed as $d_0^2=f_2(u)$. Further constraints referring to d_0^2 come from the concrete analytic forms of the underlying potentials. Assuming, e.g., that $2>n_1>0>n_2$, leads to the limits

$$\lim_{\gamma_1\to-\infty}d_0^2=\lim_{\gamma_2\to 0}d_0^2=f_2(\infty)=d_0^2(n_1) \tag{2.12}$$

and

$$\lim_{\gamma_1\to 0}d_0^2=\lim_{\gamma_2\to\infty}d_0^2=f_2(0)=d_0^2(n_2)\ , \tag{2.13}$$

in so far as both V_{n_1} and V_{n_2} are attractive potentials. Above $d_0(n)$ is the PSQ characterizing the GSE of Eq. (1.3). However, Eq. (2.12) [Eq. (2.13)] should be removed if V_{n_1} (V_{n_2}) would be a repulsive potential. Further requirements can also be invoked. Such requirements concern the selection of concave (convex) monotonic functions of the couplings. We may choose such qualitative

behaviors in agreement with corresponding data exhibited by Eqs. (2.12) and (2.13).

We first recognize that constant d_0 values lead to $\partial_i \varepsilon > 0$ and $\partial_i^2 \varepsilon < 0$,[10] where $i=1,2$. This indicates that step-function approximations to d_0 mentioned above are conceivable. Setting $n_1=2$, Eq. (2.9) shows immediately that d_0^2 is independent of $\gamma(n_2)$.[5] Next let us perform the $n_1 \to n_2$ and $\gamma(n_1) \to \gamma(n_2)$ limits. Then $u \to 1$, so that d_0^2 becomes a constant. This is a nontrivial result, which shows that the PSQ characterizing Eq. (1.3) is independent of the coupling and depends on the power exponent only. Under such conditions the GSE of Eq. (1.3) reads

$$\varepsilon_n = [(n-2)/n] F(n) d_0^{2n/(n-2)} , \tag{2.14}$$

in which $d_0 = d_0(n)$. Further, one realizes immediately that the coupling independence of d_0^2 is also preserved for the logarithmic potential $V(x) = g \ln \gamma x$. This time

$$\varepsilon = g \ln \gamma + \tilde{\varepsilon} = g \ln \gamma + \left[\frac{d_0^2}{x^2} + g \ln x \right] , \tag{2.15}$$

so that the scaling equation reads

$$\frac{g}{2} + g \frac{\partial}{\partial g} \tilde{\varepsilon} = \tilde{\varepsilon} + \frac{1}{x^2} g \frac{\partial}{\partial g} d_0^2 = \tilde{\varepsilon} . \tag{2.16}$$

These results show that d_0^2 is independent of g and γ. We also observe that the above results concerning the superposition of two power potentials are subject to a straightforward generalization towards the superposition $V = \sum_i V_{n_i}$ of an arbitrary number of such potentials. In such a case Eq. (2.6) becomes

$$\mathbf{B} \cdot \partial \varepsilon = \varepsilon + \frac{\partial \varepsilon}{\partial d_0^2} \mathbf{B} \cdot \partial d_0^2 , \tag{2.17}$$

in which $\mathbf{B} \cdot \partial = \sum_i B_i \gamma_i$, $\gamma_i = \gamma(n_i)$, $\partial_i = \partial / \partial \gamma_i$, and $B_i = \frac{1}{2}(2 - n_i)\gamma_i$, whereas C_0 takes the form

$$C_0 = - \sum_i \frac{n_i}{x^{n_i}} B_i > 0 . \tag{2.18}$$

Other examples can be treated in a similar way. Just mention that the screened Coulomb and exponential potentials $V(x) = -(g/x)\exp(-\gamma x)$ and $V(x) = -g \exp(-\gamma x)$ yield the scaling equations

$$\left[g \frac{\partial}{\partial g} + \gamma \frac{\partial}{\partial \gamma} \right] d_0^2 = 0 \tag{2.19}$$

and

$$\left[g \frac{\partial}{\partial g} + \tfrac{1}{2} \gamma \frac{\partial}{\partial \gamma} \right] d_0^2 = 0 . \tag{2.20}$$

Then the typical scaling variables are $u = g/\gamma$ and $u = g/\gamma^2$, respectively.

III. SYMMETRY TRANSFORMATIONS OF HAMILTONIAN FORMS

Proofs have been given that the Schrödinger equation with Coulomb potential can be mapped into the one for the harmonic oscillator and conversely.[11] Here we shall generalize this result, thereby defining equivalence classes of Hamiltonian forms. Such classes remain invariant under the symmetry transformations which will be discussed hereafter. For this purpose, let us define a new radial coordinate y and a new state function $\varphi(y)$ as

$$y^\rho = \frac{1}{x} \tag{3.1}$$

and

$$\psi(x) = y^\omega \varphi(y) , \tag{3.2}$$

respectively. Then the radial Laplace operator preserves its form,

$$\Delta_x = \frac{\partial^2}{\partial x^2} + \frac{N-1}{x} \frac{\partial}{\partial x} \to \Delta_y = \frac{\partial^2}{\partial y^2} + \frac{N-1}{y} \frac{\partial}{\partial y} , \tag{3.3}$$

if

$$\omega = \frac{N-2}{2}(1+\rho) , \tag{3.4}$$

where N denotes the number of space dimensions. As we are concerned with radial problems we shall set $N=3$. So the originary Schrödinger equation

$$\left[-\Delta_x + \frac{l(l+1)}{x^2} + V(x) \right] \psi = E \psi \tag{3.5}$$

becomes

$$\left[-\Delta_y + \frac{\Lambda}{y^2} + \frac{l(l+1)}{y^2} \rho^2 + V\left[\frac{1}{y^\rho} \right] \frac{\rho^2}{y^{2+2\rho}} \right] \varphi = \frac{\rho^2 E}{y^{2+2\rho}} \varphi , \tag{3.6}$$

in which

$$\Lambda = - \frac{(N-2)^2}{4}(1-\rho^2) = -\tfrac{1}{4}(1-\rho^2) . \tag{3.7}$$

This means that the above symmetry transformation proceeds in terms of the mapping

$$p_x^2 \to \frac{1}{\rho^2} y^{2+2\rho} \left[p_y^2 + \frac{\Lambda}{y^2} \right] , \tag{3.8}$$

where p_x^2 and p_y^2 denote the squared radial momenta. It is now a simple matter to write down the quasiclassical counterparts of Eqs. (3.6) and (3.7). We have

$$\frac{d_0^2}{x^2} + \frac{l(l+1)}{x^2} + V(x) = \varepsilon(x) \tag{3.9}$$

and

$$\frac{\tilde{d}_0^2}{x^2} + \frac{\Lambda}{x^2} + \frac{l(l+1)}{x^2} \rho^2 + \frac{\rho^2}{x^{2+2\rho}} V\left[\frac{1}{x^\rho} \right] = \frac{\rho^2}{x^{2+2\rho}} \varepsilon(x) , \tag{3.10}$$

respectively. The latter PSQ has been denoted by $\tilde{d}_0$. Next we observe that the ρ parameter can be chosen to drop out the coordinate dependence of certain monomial terms of Eq. (3.10). This leads to the onset of new eigenvalue terms, i.e., of new Hamiltonian forms.

For the sake of discussion let us consider at the beginning that $V(x)=V_n(x)$. Then Eq. (3.10) reads

$$\frac{\tilde{d}_0^2}{x^2}+\frac{l(l+1)}{x^2}\rho^2+\frac{\Lambda}{x^2}+\frac{\rho^2\gamma(n)}{x^{2+2\rho-n\rho}}=\frac{\rho^2\varepsilon_n}{x^{2+2\rho}} . \quad (3.11)$$

Choosing the scale factor as

$$\rho=\rho_1=\frac{2}{n-2} , \quad (3.12)$$

leads to the new Hamiltonian form

$$\tilde{\varepsilon}_{n^*}=\frac{\tilde{d}_0^2}{x^2}+\frac{l(l+1)}{x^2}\rho^2+\frac{\Lambda}{x^2}+\frac{\tilde{\gamma}(n^*)}{x^{n^*}} , \quad (3.13)$$

in which

$$\Lambda=\frac{1}{4}\frac{(4-n)n}{(n-2)^2} , \quad (3.14)$$

whereas

$$\begin{aligned}\tilde{\gamma}(n^*)&=-\rho_1^2\varepsilon_n ,\\ \tilde{\varepsilon}_{n^*}&=-\rho_1^2\gamma(n) ,\end{aligned} \quad (3.15)$$

and

$$n^*=\rho_1 n=\frac{2n}{n-2} . \quad (3.16)$$

This gives $n^*\in(0,2)$ if $n\in(-\infty,0)$ and conversely. Repeating this procedure, we will come back to Eq. (3.11). This means that the above transformations exhibit the structure of a cyclic group. Indeed, the identity and inverse transformations are given by $\rho=-1$ and $\rho=1/\rho_1$, respectively. In this way, the subclass of $0<n<2$ Hamiltonians is converted into the one of $n<0$ Hamiltonians and conversely. Then the $n<2$ Hamiltonians build up an equivalence class which remains invariant under the influence of such transformations. The main point is that such symmetry transformations can help us to generalize the GSE's of Eqs. (1.3) and (1.4) towards energy levels. Such generalizations come from suitable parametrizations of the PSQ, such as Eqs. (A4) and (A5), as shown in the Appendix. The point is that these generalized expressions for the PSQ are able to work consistently in conjunction with the above symmetry transformations. This is a nontrivial "covariance" behavior, which leads to sensible results. As a matter of fact, these relationships also enable us to go beyond the starting evaluation of the GSE. Coming back to Eq. (3.15), one then finds the covariance criterion

$$\frac{4}{n^*}d_0(n^*)+\frac{4}{n}d_0(n)=1 , \quad (3.17)$$

which has a self-evident meaning [see Eq. (A9)]. This is a quite interesting result when opens the way to evaluate the $d_0(n)$'s for $0<n<2$ ($n<0$) in terms of those for $n<0$ ($0<n<2$). We also realize that Eq. (1.6) can be generalized using Eqs. (A5) and (A6). Then the consistency criterion reads

$$\tilde{d}_0^2=\frac{2}{2-n}d_0^2 , \quad (3.18)$$

thereby preserving identically Eq. (3.17).

IV. THE EQUIVALENCE BETWEEN THE LINEAR PLUS COULOMB POTENTIAL AND THE QUARTIC ANHARMONIC OSCILLATOR

The quasiclassical symmetry properties characterizing Eq. (1.2) are of further interest. Here we would like to prove that the quartic anharmonic oscillator

$$\bar{\varepsilon}_1=\frac{d_1^2}{x^2}+\mu x^2+\lambda' x^4 \quad (4.1)$$

is equivalent to the linear plus Coulomb Hamiltonian

$$\tilde{\varepsilon}_1=\frac{\tilde{d}_1^2}{x^2}-\frac{\alpha}{x}+\kappa x . \quad (4.2)$$

Above $\mu=\gamma(-2)$, $\lambda'=\gamma(-4)$, $\alpha=-\gamma(1)$, and $\kappa=\gamma(-1)$. This time general parametrization formula for d_0, such as (A4) and (A5), are not available. Then we shall prove the equivalence mentioned above using the algebraic equations[12]

$$\bar{\varepsilon}_1^3+\frac{\mu^2}{4\lambda'}\bar{\varepsilon}_1^2-\frac{9}{2}d_1^2\mu\bar{\varepsilon}_1-\frac{27}{4}\lambda' d_1^4-\frac{\mu^3}{\lambda'}d_1^2=0 \quad (4.3)$$

and

$$\tilde{\varepsilon}_1^3+\frac{\alpha^2}{4\tilde{d}_1^2}\tilde{\varepsilon}_1^2+\frac{9}{2}\alpha\kappa\tilde{\varepsilon}_1-\frac{27}{4}\kappa^2\tilde{d}_1^2+\frac{\kappa\alpha^3}{\tilde{d}_1^2}=0 , \quad (4.4)$$

which express the solutions to the quasiclassical minimization ($l=0$) of Eqs. (4.1) and (4.2). Above $d_0^2=d_1^2$ and $d_0^2=\tilde{d}_1^2$, respectively. Let us begin with Eq. (4.1). Taking into account Eq. (3.8) leads to the following Hamiltonian transforms:

$$\bar{\varepsilon}_2'=\frac{d_2'^2}{x^2}-\frac{3}{16x^2}-\frac{\bar{\varepsilon}_1}{4x}+\frac{\lambda'}{4}x=-\frac{\mu}{4} \quad (4.5)$$

and

$$\bar{\varepsilon}_3'=\frac{d_3'^2}{x^2}-\frac{2}{9x^2}+\frac{\mu}{9x^{2/3}}-\frac{\bar{\varepsilon}_1}{9x^{4/3}}=-\frac{\lambda'}{9} , \quad (4.6)$$

which are subject to cyclic transformations. Such transformations work in terms of the scale factors $\rho_{12}=-\frac{1}{2}$, $\rho_{23}=-\frac{2}{3}$, and $\rho_{13}=-\frac{1}{3}$, which are responsible for the mappings $\bar{\varepsilon}_1\to\bar{\varepsilon}_2'$, $\bar{\varepsilon}_2'\to\bar{\varepsilon}_3'$, and $\bar{\varepsilon}_1\to\bar{\varepsilon}_3'$, respectively. The inverse transformations come from the inverted ρ factors, such that $\rho_{ij}=1/\rho_{ji}$. Starting from Eq. (4.2) and proceeding quite similarly yields the transforms

$$\tilde{\varepsilon}_2'=\frac{\tilde{d}_2'^2}{x^2}+\frac{3}{4x^2}-4\tilde{\varepsilon}_1x^2+4\kappa x^4=4\alpha \quad (4.7)$$

and

$$\tilde{\varepsilon}_3'=\frac{\tilde{d}_3'^2}{x^2}-\frac{5}{36x^2}-\frac{4\alpha}{9x^{4/3}}-\frac{4\tilde{\varepsilon}_1}{9x^{2/3}}=-\frac{4}{9}\kappa . \quad (4.8)$$

One realizes immediately that the third element of the equivalence class involved in this manner is given by the superposition between the $n=\frac{2}{3}$ and $n=\frac{4}{3}$ potentials. Comparing Eqs. (4.1) and (4.7) leads to $\bar{\varepsilon}_1=4\alpha$, $d_1^2=\tilde{d}_2'^2$, $\mu=-4\tilde{\varepsilon}_1$, and $\lambda'=4\kappa$, insofar as the $n=2$ term $\frac{3}{4}x^{-2}$

would be ignored. Then Eq. (4.3) reproduces identically Eq. (4.4), provided that $\tilde{d}\,'^2_2=4\tilde{d}\,^2_1$. Similarly, Eq. (4.3) comes from Eq. (4.4) and (4.5) insofar as $d_1^{\prime 2}=4d_2^{\prime 2}$, whereas, this time, the term $-\frac{3}{16}x^{-2}$ should be suppressed. Rescaled counterparts of Eqs. (4.5)–(4.8) can also be invoked. In particular, Eqs. (4.5) and (4.7) can be rescaled by the factors 4 and $\frac{1}{4}$, respectively. This yields the fixed d_0^2 results

$$\bar{\varepsilon}_2=\frac{d_1^2}{x^2}-\frac{3}{4x^2}-\frac{\bar{\varepsilon}_1}{x}+\lambda' x=-\mu \tag{4.9}$$

and

$$\tilde{\varepsilon}_2=\frac{\tilde{d}\,_1^2}{x^2}+\frac{3}{16x^2}-\tilde{\varepsilon}_1 x^2+\kappa x^4=\alpha\ , \tag{4.10}$$

which express particular Hamiltonian forms relying on the special selection of the same d_0^2 values. It is understood that the general properties of d_0^2 under symmetry transformations are subject to further investigations, which go, however, beyond the scope of this paper. Note that the above $n=2$ terms $-\frac{3}{4}x^{-2}$ and $\frac{3}{16}x^{-2}$ can be interpreted as Langer-Kemble corrections.[13] The same remains valid for the $n=2$ terms characterizing the other Hamiltonian transforms. The mutual agreement between the signs of the couplings and the GSE's is also fulfilled in a self-consistent manner. Such results led us to say that Eqs. (4.1) and (4.2) are subject to mutual equivalence. Of course, the energy-level transforms, which are given by the right-hand sides of Eqs. (4.5)–(4.10), work in conjunction with the corresponding Hamiltonian forms. The equivalence-class affiliation of $H_{4/3}^{(2/3)}$ can be established in a similar manner.

Next let us specialize somewhat by considering some concrete examples. Starting, e.g., from the anharmonic oscillator $\bar{\varepsilon}_1=d_1^2/x^2+x^2+x^4$ yields the corrected form

$$\bar{\varepsilon}_2^{(\mathrm{cor})}=\frac{d_1^2}{x^2}-\frac{\bar{\varepsilon}_1}{x}+x=-\mu=-1\ , \tag{4.11}$$

where $\alpha=\bar{\varepsilon}_1$ and $\kappa=\lambda'=1$. Using available $\bar{\varepsilon}_1$ data,[14] one then finds immediately the corresponding d_1 estimates for several n_r and l values ($\mu=\lambda'=1$). Such d_1 estimates are presented in Table I. This, in turn, induces the α dependence of d_1. This concerns, so far, the above particular description of rescaled and corrected Hamiltonian forms.[15] The parametrization

TABLE I. The dependence of d_1 on the induced Coulomb coupling for several l and n_r values.

l	n_r	$\alpha=\bar{\varepsilon}_1$	d_1	η
0	0	4.648	1.629 672	1.129 672
10	0	54.184	11.696 143	1.196 143
4	3	59.020	12.500 154	1.142 879
0	5	60.130	12.682 433	1.107 494
50	0	375.060	51.714 501	1.214 501
20	15	418.964	56.258 025	1.153 485
0	25	427.982	57.176 426	1.111 302
100	0	914.572	101.718 274	1.218 274
50	25	1014.136	109.993 722	1.166 543
0	50	1049.208	112.860 138	1.112 476

TABLE II. The induced μ dependence of $\tilde{d}_1$ for several l and n_r values.

l	n_r	$\mu=-\tilde{\varepsilon}_1$	$\tilde{d}_1$	$\tilde{\eta}$
0	0	2.253	1.358 513	0.858 513
1	0	3.310	2.365 156	0.865 156
1	1	4.483	4.141 456	0.880 485
2	1	5.597	5.133 125	0.877 708
0	3	6.746	6.777 289	0.896 755
10	0	9.539[a]	11.367 738	0.867 738
50	25	39.444[a]	95.363 534	0.879 677

[a]These energies are WKB estimates.

$$d_1=(1+2n_r)\eta+l+\tfrac{1}{2}\ , \tag{4.12}$$

in which η is an adjustable parameter, has also been used. One sees that η takes nearly constant values, as shown in Table I. Further, the accuracy turns out to be better within selected regions such as $n_r=0$ and $l\gg 0$ ($\eta\cong 1.218$) or $l\gg 0$ and $n_r\gg 0$ ($\eta\cong 1.166$). A similar analysis can be done for the inverse transformation. So Eq. (4.10) shows that the anharmonic oscillator sets up via $\tilde{\varepsilon}_2^{(\mathrm{cor})}=\alpha$, $\mu=-\tilde{\varepsilon}_1$, and $\lambda'=\kappa=1$, where the input $\tilde{\varepsilon}_1=\tilde{d}_1^2/x^2+x-\alpha/x$ has been considered. Using energy data,[16] one then finds immediately the induced μ dependence of $\tilde{d}_1$, now for $\alpha=0.1$. These results are presented in Table II. Equation (4.12) has also been used, this time with $\tilde{\eta}$ instead of η.

V. EXPLICIT EVALUATIONS OF ENERGY LEVELS

Having now gained some experience with general properties of d_0, we are ready to discuss GSE's and energy levels of Eqs. (1.3) and (1.4) in some more details. First let us consider Eq. (1.3). We then observe immediately that Eq. (2.14) reproduces the "exact" numerical evaluations of the GSE presented in Ref. 17 for $n<0$, in terms of certain $d_0(n)$ values, as shown in the second column of Table III.

TABLE III. Numerical estimates of $d_0(n)$ for several $n<2$ values. For comparison, WKB estimates for $n<0$ have also been inserted (Ref. 17).

n	$d_0(n)$	$d_0^{\mathrm{WKB}}(n)$	n^*	$d_0(n^*)$
$-\infty$	π	0.75π	2	0.5
-14	2.165 871	1.981 507	$\frac{7}{4}$	0.708 234
-7	1.886 481	1.817 811	$\frac{14}{9}$	0.808 107
-6	1.828 466	1.778 459	1.5	0.832 116
-5	1.763 068	1.730 837	$\frac{10}{7}$	0.860 876
-4	1.688 441	1.672 492	$\frac{4}{3}$	0.896 147
-3	1.602 044	1.598 395	1.2	0.940 818
-2	1.5	1.5	1	1
-1	1.376 077	1.360 349	$\frac{2}{3}$	1.084 051
-0.5	1.302 681	1.264 466	0.4	1.142 145
0^-	1.218 629	1.140 260	0^+	1.218 629

Further,

$$\lim_{n\to-\infty}\varepsilon_n=F(-\infty)d_0^2(-\infty)\ , \tag{5.1}$$

whereas $V_n(x)$ approaches the potential well

$$\lim_{n\to-\infty}V_n(x)=\begin{cases}0, & x<1\\ \infty, & x>1\ ,\end{cases} \tag{5.2}$$

if $F(-\infty)=1$. On the other hand, the exact GSE characterizing this potential well is π^2, so that

$$\lim_{n\to-\infty}d_0(n)=d_0(-\infty)=\pi\ . \tag{5.3}$$

The WKB estimates for d_0 ($n<0$) can also be established using Eq. (5.15) of Ref. 1. This yields $d_0^{\mathrm{WKB}}(-\infty)=\frac{3}{4}\pi$, instead of Eq. (5.3). Next, we can notice, quite satisfactorily, that Eqs. (3.16) and (3.17) enable us to establish related d_0 ($0<n<2$) values. These estimates are given in the last column of Table III. In particular,

$$\lim_{n\to2}d_0(n)=\tfrac{1}{2}\ , \tag{5.4}$$

which comes from Eqs. (3.17) and (5.3). Moreover, Eq. (2.14) approaches the GSE characterizing the logarithmic potential[18] if $d_0(0^-)\cong1.218\,629$, as displayed in the last row of Table III. Highly accurate numerical estimates such as $d_0(-1)=1.376\,083\,54$ and $d_0(-4)=1.688\,432\,59$ can also be mentioned.[20] Further, the n dependence of $d_0(n)$ has been displayed in Fig. 1. This shows that $d_0(n)$ is definitely a smoothly decreasing and convex function of n. It is also interesting to notice that the GSE problem, i.e., the $d_0(n)$ problem, can be converted into the functional equation

$$S(\tau)+S(1-\tau)=1\ , \tag{5.5}$$

so that

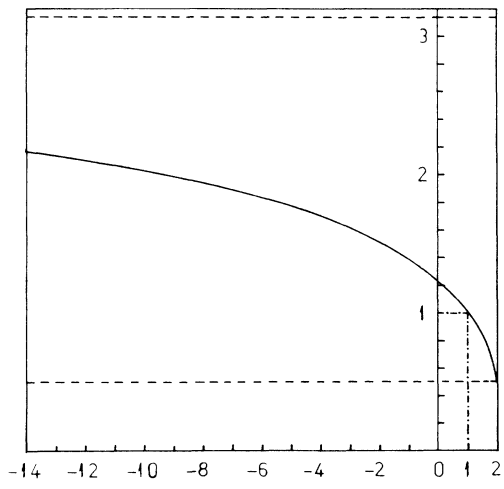

FIG. 1. n dependence of $d_0(n)$. The numerical data of Table III have been used. The dashed line from below reads $d_0(2)=\frac{1}{2}$, whereas the one from above displays the horizontal asymptote $d_0(-\infty)=\pi$. The dot-dashed line displays the Coulomb case.

$$\lim_{\tau\to\infty}\frac{S(\tau)}{2\tau}\cong1.218\,629\ , \tag{5.6}$$

where $S(\tau)=2\tau d_0(2/\tau)$ and where Eqs. (3.16) and (3.17) have been used. Here $\tau=2/n\in[1,\infty)$ and $S(\tau)\in[1,\infty)$. This time $S(\tau)$ is an increasing and convex function of τ, which is subject to the boundary conditions already mentioned above. One also has an arbitrary number of fixing conditions at several concrete points, such as $S(2)=4$, $S(3)=6d_0(\frac{2}{3})$, etc. We might notice that a useful approximation to $d_0(n)$ comes from the $1/N$-expansion method.[21] Taking $N=3$ then leads to the approximation

$$d_0^{(N)}(n)=\frac{1}{2}+\left[\frac{1}{2}-\frac{n}{4}\right]^{1/2}\ , \tag{5.7}$$

which has the virtue of reproducing exactly the limiting value at $n=2$, as well as the well-known results concerning the Coulomb potential and the harmonic oscillator. However, this approximation ceases to be valid for large negative n values, as $d_0^{(N)}(-\infty)=\infty$. This disagrees with Eq. (5.3).

To complete our discussion, let us now consider nonzero l and n_r values. We first realize that Eq. (A4), in conjunction with $\lambda=0$, may be taken as a useful starting point. Accordingly, we have to choose the general parametrization

$$d_0=d_0(n;l,n_r)=f_0(n_r)[d_0(n)-\tfrac{1}{2}]+l+\tfrac{1}{2}\ , \tag{5.8}$$

which exhibits covariance properties with respect to Eq. (3.8), as proved in the Appendix. In general, we have to take up the question of a suitable choice of $f_0(n_r)$. An appealing idea is to consider a linear dependence on n_r,

$$f_0(n_r)=A+Bn_r\ , \tag{5.9}$$

where A and B are constants. Of course, we are aware that further generalizations, requiring an extra bit of analysis, may be necessary. However, Eq. (5.9) exhibits the relevant attributes of the simplest possible choice. This means that, whatever the exact general form of $f_0(n_r)$ might be, the linear choice (5.9) expresses a first step towards further generalizations. This choice will then be discussed in the rest of this paper. Next let us turn to Eq. (4.12). This suggests writing the solution to d_0 as

$$d_0=(1+2n_r)[C'd_0(n)+D']+l+\tfrac{1}{2}\ , \tag{5.10}$$

where C' and D' are constants. This time one has a linear dependence of η on $d_0(n)$. Comparing Eqs. (5.9) and (5.10) then leads immediately to $A=\frac{1}{2}B=C'=-2D'$. Above l, $d_0(n)$ and n_r have been interpreted, of course, as independent variables. In consequence, one has just one independent parameter, so that $f_0(n_r)=A(1+2n_r)$. Hence, the linear description to d_0 emphasized above reads

$$d_0(n;l,n_r)=\beta_0(n,n_r)+l+\tfrac{1}{2}\ , \tag{5.11}$$

where

$$\beta_0(n,n_r)=(1+2n_r)[d_0(n)-\tfrac{1}{2}]\ , \tag{5.12}$$

whereas $A=1$ by virtue of the GSE. In practice, A and/or B can also be viewed as adjustable parameters. Restricting ourselves to $A=1$ and $B=2$ leads to

$$\varepsilon_n(l,n_r)=\frac{n-2}{n}F(n)[d_0(n;l,n_r)]^{2n/(n-2)}, \tag{5.13}$$

which reproduces identically the well-known energy levels of the Coulomb problem and of the harmonic oscillator. Such agreements are not surprising, as they have also been obtained within the WKB approach.[22] However, the main point is that Eq. (5.12) expresses a relevant approximation to the energy levels characterizing other $n<2$ cases. For this purpose, comparisons with some numerical data concerning the energy levels of the linear[23] and quartic[24] potentials are presented in Tables IV and V. The exact numerical estimates would also be produced by Eq. (5.13) in so far as one inserts $d_0^{(\mathrm{eff})}(n)$ instead of $d_0(n)$. Such effective $d_0^{(\mathrm{eff})}(n)$ values, together with the relative errors[25] to the energy levels, have been quoted in these tables for several l and n_r values. In general, $d_0^{(\mathrm{eff})}(n)$ approaches reasonably $d_0(n)$, whereas the relative errors are admissibly small. However, we observe that the relative errors become somewhat larger if $l=0$ and $n_r\neq 0$. Such cases, for which the relative errors are larger than 5%, concern, e.g., the (l,n_r) pairs (0,1), (0,5), and (0,25), as shown in Table V. We could reduce such deviations choosing

$$B=2-\Delta B(n_r)\cong 1.873, \tag{5.14}$$

instead of $B=2$, where $n=-4$ and $l=0$. This shifting is favored by the reasonable constancy of $\Delta B(n_r)$ around $\Delta B\cong 0.127$.[26] However, the understanding is that within the present minimization approach, the $l=0$ choice concerns the GSE. Accordingly, we have to take $n_r=0$ as soon as $l=0$. This also means that the quasiclassical minimization singles out a preferred solution, i.e., the one generated by $n_r=0$. This corresponds to the accurate solutions quoted above. Within a similar reasoning we can also explain the good accuracy of the $l\gg n_r$ results. As a matter of fact, the accuracy of such solutions increases with l. Such accurate solutions can then be parametrized as

$$d_0(n;l,0)=d_0(n)+n_p, \tag{5.15}$$

where $n_r=0$, in which $n_p=l=0,1,2,\ldots$, plays the role of the "principal" quantum number. Equation (5.15) can also be generalized as

$$d_0(n;l,n_r\ll l)=d_0(n)+n_p(l\gg n_r), \tag{5.16}$$

TABLE IV. Comparison of $\varepsilon_n(l,n_r)$ with some numerical energy levels of $H=p^2/2m_0+r$ $(n=-1)$.

l	n_r	$\varepsilon_n^{(\mathrm{exact})}$	ε_n	$d_0^{(\mathrm{eff})}(n)$	$r\%$
0	0	1.855 757 08	1.855 757	1.376 083	0
0	1	3.244 607 62	3.208 428	1.393 771	−1.11
1	0	2.6679	2.670 950	1.372 015	0.11
0	2	4.381 671 24	4.315 812	1.398 511	−1.50
1	1	3.8768	3.860 130	1.385 007	−0.43
2	0	3.3718	3.375 722	1.370 201	0.11

TABLE V. Comparison of $\varepsilon_n(l,n_r)$ with numerical energy levels of $H=p^2/2m_0+r^4$ $(n=-4)$.

l	n_r	$\varepsilon_n^{(\mathrm{exact})}$	ε_n	$d_0^{(\mathrm{eff})}(n)$	$r\%$
0	0	2.393 644	2.393 644	1.688 432	0
1	0	4.478 039	4.450 548	1.700 878	−0.61
0	1	7.335 730	7.724 512	1.636 951	5.30
2	0	6.830 308	6.784 800	1.706 972	−0.66
1	2	16.599 521	17.298 701	1.643 081	4.21
3	1	16.046 193	16.140 682	1.678 085	0.59
5	0	15.081 647	15.003 079	1.714 685	−0.52
0	5	35.740 315	38.545 358	1.620 456	7.85
4	3	34.980 152	35.718 076	1.659 983	2.11
10	0	31.690 628	31.580 788	1.718 909	−0.34
0	25	263.750 919	286.570 533	1.616 134	8.65
20	15	257.889 588	263.252 450	1.660 099	2.08
50	0	229.437 335	229.230 505	1.723 407	−0.09

for $l\gg n_r>0$, which works accurately for large n_p $(l\gg n_r)$ values. This time $n_p=l+2n_r[d_0(n)-\frac{1}{2}]$.

Coming back to Eqs. (2.12) and (2.13), further comments are in order. First, $\frac{1}{2}<d_0(n_1)<d_0(n_2)$ if $2>n_1>n_2$. Next we take account of the horizontal asymptotes $d_0=d_0(n_1)$ and $d_0=d_0(n_2)$. Accordingly, we have to choose d_0 as an increasing (decreasing) and convex (concave) function of $\gamma(n_2)$ $[-\gamma(n_1)]$. These latter couplings have the general form $-\gamma(n)\mathrm{sgn}n$. Such proposals are able to be supported by available data. Considering as an example the linear plus Coulomb potential, one finds that d_0 is actually an increasing and convex function of $\kappa=\gamma(-1)$, as displayed in Fig. 2. (See Table VI.) These d_0 values are solutions to Eq. (4.4) obtained for some available GSE data.[27] Invoking general reliability grounds, leads to consider that d_0 should be, correspondingly, a decreasing and concave function of α. Another interesting observation is that Eqs. (2.12) and (2.13) can be combined with the algebraic equations for $\varepsilon_{n_1}^{(n_2)}$ to build up accurate asymptotic formula for the GSE or for the energy levels. In particular, Eq. (4.4) works accurately in terms of $\tilde{d}_1=d_0(1;l,n_r)$ $[\tilde{d}_1=d_0^{(\mathrm{exact})}(-1;l,n_r)]$ if

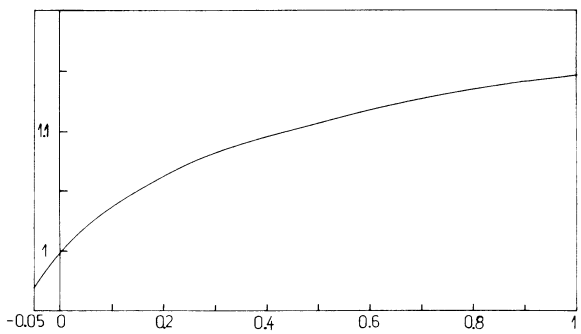

FIG. 2. κ' dependence of d_0 for the Hamiltonian form $d_0^2/2x^2-1/x+\kappa' x$.

TABLE VI. Numerical d_0 estimates for $n_1=1$, $n_2=-1$, and $\gamma(-1)=2\kappa'$ ($\alpha=2$).

κ'	$d_0(\kappa')$	κ'	$d_0(\kappa')$
1	1.147 059	0.05	1.021 662
0.8	1.134 654	0	1
0.6	1.118 732	−0.02	0.989 279
0.3	1.082 438	−0.03	0.983 273
0.2	1.063 851	−0.04	0.976 651

$\alpha \gg |\kappa|$ ($\kappa \gg |\alpha|$). The same remains valid for Eq. (4.3), this time for $\mu \gg \lambda_0' = |\lambda'|$ ($\lambda' \gg \mu_0 = |\mu|$) and $d_1 = d_0(-2;l,n_r)$ [$d_1 = d_0^{(\text{exact})}(-4;l,n_r)$]. Concrete manifestations of such convergences have also been discussed before.[28]

The Hamiltonian (1.5) can be treated in a similar way. This time Eqs. (A5) and (A6) have to be used. We then find the energy levels

$$\varepsilon_2^{(n)}(l,n_r) = \frac{n-2}{n} F(n)[d_0^2(n;l,n_r)]^{2n/(n-2)} , \tag{5.17}$$

if $n<2$, in which $n\gamma(n)<0$ and[29]

$$d_0^2(n;l,n_r) = \beta_0(n,n_r) + [(l+\tfrac{1}{2})^2+\lambda]^{1/2} , \tag{5.18}$$

by virtue of the linear parametrization of $f_0(n_r)$ discussed above. Taking $n=1$ and -2, one sees immediately that Eqs. (5.17) and (5.18) reproduce identically the exact energy levels established previously.[30] This is a nontrivial generalization of the previous agreement concerning the Coulomb problem and the harmonic oscillator. We also realize that the accurate d_0^2 parametrizations are given, in general, by $d_0^2 = d_0^2(n;l,0)$ and $d_0^2 = d_0^2(n;l,n_r \ll l)$. If $\lambda<0$, the angular momentum is subject to the condition

$$\lambda_0 \le (l+\tfrac{1}{2})^2 . \tag{5.19}$$

This leads to the onset of the stability condition $\lambda_0 \le \frac{1}{4}$. Indeed, Eqs.(5.17) and (5.18) produce the $l=0$ resonance[31] characterizing Eq. (1.4) for $n=1$, $\gamma(1)>0$, and $\lambda<0$, insofar as supercritical $\lambda_0 > \frac{1}{4}$ values would be considered.[32] The case $\lambda<0$, $n>2$, and $\xi=-1$ is of a special interest. Now the only candidate satisfying the condition $\lambda_0 > d_0^2$ is[5]

$$d_0^2 = d_0^2(n>2) = \tfrac{1}{2} - \lambda_0 - [(l+\tfrac{1}{2})^2 - \lambda_0]^{1/2} > 0 , \tag{5.20}$$

provided that it takes positive values needed. This condition is fulfilled immediately if $l=0$. Within the present stage of our calculations one has, again $d_0(n)=1$. This has the meaning of an intermediary symmetry condition, as one might expect. Indeed, $\varepsilon_2^{(n)}$ can be rewritten as

$$\varepsilon_2^{(n)} = q_0 \tilde{\varepsilon}_2^{(n')} = q_0 \left[\frac{d_0^2(n'<2)}{y^2} - \frac{\lambda_0}{y^2} + V_{n'}'(y) \right] , \tag{5.21}$$

in which $l=0$, $n'=4/n \in (0,2)$, and $y = x^{n/2}$, whereas

$$q_0 = \frac{\gamma_0(4/n')}{g_0(\lambda_0)} [\lambda_0 - d_0^2(4/n')] \tag{5.22}$$

and

$$V_{n'}'(y) = -\frac{g_0(\lambda_0)}{\gamma_0(4/n')} \frac{1}{y^{n'}} . \tag{5.23}$$

Above $q_0>0$. The same is valid for $g_0(\lambda_0)$, since

$$g_0(\lambda_0) = [d_0^2(n'<2) - \lambda_0] \left[\lambda_0 - d_0^2 \left[\frac{4}{n'} \right] \right]$$

$$= \frac{\lambda_0}{2} [(1-4\lambda_0) + (1-4\lambda_0)^{1/2}] . \tag{5.24}$$

In other words, the minimization of $\varepsilon_2^{(n)}$ has been converted into the one of $\tilde{\varepsilon}_2^{(n')}$, now with respect to the attractive $n'<2$ potential $V_{n'}'$. Then the energy levels of Eq. (5.21) come from Eq. (5.17) via $\tilde{\varepsilon}_2^{(n')} = \varepsilon_2^{(n')}(l,n_r)$. This leads to the extrapolation

$$\varepsilon_2^{(n)}(n>2) = \frac{2-n}{n} F(n) G_n(\lambda_0) (d_0^2)^{4/(2-n)} , \tag{5.25}$$

where

$$d_0^2 = (1+2n_r) \left[d_0 \left[\frac{4}{n} \right] - \frac{1}{2} \right] + [(l+\tfrac{1}{2})^2 - \lambda_0]^{1/2} \tag{5.26}$$

and

$$G_n(\lambda_0) = [g_0(\lambda_0)]^{n/(n-2)} [\tfrac{1}{2} - \lambda_0 + (\tfrac{1}{4} - \lambda_0)^{1/2}]^{-1} . \tag{5.27}$$

We would like to remark that a possible choice for a direct generalization of Eq. (5.20) towards $d_0(n) \neq 1$ implies some questionable points, so that it should be ignored.

Finally, we have to mention that Eqs. (5.11)–(5.13) reproduce identically the energy-levels established recently to first $1/N$ order,[33] insofar as $d_0(n) = d_0^{(N)}(n)$. Then the d_0 evaluation which is responsible for such energy levels reads

$$d_0 = d_0^{(N)}(n;l,n_r) = l + \tfrac{1}{2} + \tfrac{1}{2}(1+2n_r)(2-n)^{1/2} , \tag{5.28}$$

which comes from Eqs. (5.7), (5.11), and (5.12).

VI. CONCLUSIONS

In this paper further steps towards a general quasiclassical description of the energy levels of Eqs. (1.3)–(1.4) have been formulated and discussed. The present approach is based upon the idea of combining Eq. (1.1) with symmetry transformations, as well as with the scaling properties of the underlying PSQ. This makes more efficient the quasiclassical minimization proposed before.[1] Like the usual virial method, one considers the problem of obtaining energies without using wave functions. In this respect Eqs. (5.13) and (5.17) reproduce exactly the well-known results for $n=1$ and -2, and lead to useful and closed approximations for the rest of the n's. The accuracy conditions read $n_r=0$ and/or $l \gg n_r$. This latter condition exhibits certain similarities with the validity attributes of the WKB method, which is a manifestation of the common quasiclassical background. However, we would like to recall that, in general, the WKB integrals are hardly tractable. Accordingly, the present approach is

quickly tractable and quite general, thereby opening the way to perform efficient studies concerning qualitative properties of several classes of potentials. We might notice, however, that in order to make suitable choices among admissible analytic candidates, confrontations with numerical data have to be used. Iterative solutions can also be done. This time we can take as starting points the algebraic equations for the GSE, such as Eqs. (4.3) and (4.4) mentioned above. In practice, we can then consider the power-series expansions $\bar{\varepsilon}_1=\sum_k b_k u^k$ and $d_1^2=\sum_k c_k u^k$, in which $u=\mu^3/\lambda'^2$ for Eq. (4.3), while $u=\alpha^{3/2}/\kappa^{1/2}$ for Eq. (4.4).

A particularly important interconnection concerns the similarity between the quasiclassical minimization of $\delta H(r)$ and the usual minimization of the effective potential considered within the $1/N$ method.[34] In addition, both approaches are subject to mutual completions. Indeed, the $1/N$ method enables us to establish useful d_0 estimates concerning the GSE and the energy levels, such as Eqs. (5.7) and (5.28). In turn, such results are able to be checked in terms of general symmetry transformations and of other qualitative properties characterizing the quasiclassical description of d_0. One realizes that Eq. (5.7) fulfills the symmetry condition (3.17), whereas typical analytic structures and scaling properties are described by Eqs. (1.6) and (2.9). Under such circumstances, d_0 plays the role of the main parameter of the theory. Concrete manifestations of such interconnections for more complex potentials are of further interest.

The basic assumptions about the concavity or convexity attributes of d_0^2 have interesting consequences referring to the negative regions of the underlying couplings. Such regions refer, e.g., to the $\lambda'<0$ ($\mu>0$) (Ref. 35) and the $\mu<0$ ($\lambda'>0$) (Ref. 36) phases of the anharmonic oscillator.[37] One should then have a critical point $\lambda'=\lambda_c'<0$ (Ref. 38) at which $d_0^2(\lambda')$ vanishes, whereas $d_0^2(\mu)$ tends to infinity as $\mu\to-\infty$. Accordingly, $d_0^2(\lambda')$ becomes negative for $\lambda'<\lambda_c'$, which signals the onset of resonances. In contradistinction, the limit $d_0^2(\mu)\to\infty$ may be understood in terms of the scaling $d_0^2\sim(\mu_0^3/\lambda'^2)^{\delta_c}$, where δ_c is a critical exponent having the magnitude order of unity.[39] These few examples show that the limits $d_0^2\to 0$ and $d_0^2\to\infty$ involve nontrivial manifestations requiring further attention. Of course, for more complicated potentials, superpositions of convex and concave d_0^2 contributions should also be considered, which is related to the existence of several scaling variables.

ACKNOWLEDGMENT

I would like to thank Professor H. D. Doebner and Professor R. Weiner for useful discussions.

APPENDIX: GENERAL PARAMETRIZATIONS FOR d_0 AND d_0^2

The λ dependence of d_0 has been discussed in Ref. 2 with the help of the condition

$$d_0^2=d_0+\lambda+l(l+1)\ , \tag{A1}$$

where $\lambda\neq 0$. This equation has the typical solutions

$$d_0=d^{(\pm)}(l,\lambda)=\tfrac{1}{2}\pm[\lambda+(l+\tfrac{1}{2})^2]^{1/2}\ , \tag{A2}$$

which serve as inputs for further generalizations. Alternatively, one would also have the proposal

$$d_0^4=d_0^2+\lambda+l(l+1)\ , \tag{A3}$$

which competes with (A1). This is a concrete manifestation of Eq. (4.9) of Ref. 1. Next we observe that $d^{(\pm)}(l,\lambda)$ is subject to further generalizations. For this purpose, we shall consider the more general parametrizations

$$\begin{aligned} d_0&=d^{(+)}(l,n_r,\lambda)\\ &=f_0(n_r)[d_0(n)-\tfrac{1}{2}]+[\lambda+(l+\tfrac{1}{2})^2]^{1/2}\ , \end{aligned} \tag{A4}$$

and the same for d_0^2,

$$d_0^2=d^{(+)}(l,n_r,\lambda)\ , \tag{A5}$$

which rely on Eqs. (A1) and (A3), respectively. Equations (A4) and (A5) have also been constructed by choosing the positive sign in the front of the square root. This is, in fact, the only representation which works irrespective of the sign of λ, insofar as $n<2$ and $n\gamma(n)<0$. In general, $f_0(n_r)$ is a positive function of the radial quantum number n_r. Whether more general proposals are useful is an open problem. As a matter of fact $d^{(+)}(l,n_r,0)$ turns out to be fairly approximated by a bilinear function in n_r and l, where $\lambda=0$. We might notice that this choice seems not to be unique, as biquadratic functions in n_r and l can also be invoked.[40] However, we need to specify that such a sensitivity has been checked for short-range Yukawa-like potentials. Next, let us remember that $d_0(n)$ expresses, in agreement with Eq. (2.14), the PSQ which is responsible for the GSE of Eq. (1.3). Then the energy levels of H_n come from the combination of Eqs. (2.14) and (A4), now for $\lambda=0$. Of course, for the GSE one has $f_0(0)=1$. On the other hand, Eq. (1.6) can be rewritten as

$$\varepsilon_2^{(n)}=\frac{n-2}{n}F(n)(d_0^2)^{2n/(n-2)}\ , \tag{A6}$$

if $n<2$, which is tied to the symmetrical $d_0(n)=1$ input. Accordingly, $d_0^2+\lambda$ ($l=0$) can be rewritten as d_0^4, where, this time, Eq. (A6) has been used. One sees that Eq. (A6) reproduces identically Eq. (2.14) if $d^{(+)}(0,0,0)=1$, i.e., if $d_0=1$. It is obvious that this symmetry ceases to be preserved if $d_0(n)\neq 1$. The main point is that the combination of Eqs. (A5) and (A6) leads to the extrapolation of Eq. (1.6) towards energy levels, now for $d_0(n)\neq 1$. Next we would like to anlayze the influence of the symmetry transformation (3.8) on Eqs. (A4) and (A5). Accordingly, Eq. (A5) becomes

$$\tilde{d}_0^2=\tilde{f}_0(n_r)[d_0(n^*)-\tfrac{1}{2}]+\rho_1(l+\tfrac{1}{2})\ , \tag{A7}$$

in which $\lambda=\Lambda$, where Eqs. (3.7) and (3.15) have been used. For the sake of generality $f_0(n_r)$ has also been re-

quoted as $\tilde{f}_0(n_r)$. One realizes that Eqs. (A6) and (A7) enable us to establish the explicit form of $\tilde{\varepsilon}_{n^*}$. Under such conditions Eq. (3.15) can be written equivalently as

$$\tilde{d}_0^{\,2}=\rho_0 d_0 \ , \tag{A8}$$

where $\rho_0=|\rho_1|$. Then

$$d_0(n^*)=\tfrac{1}{2}+\frac{2}{2-n}[d_0(n)-\tfrac{1}{2}] \ , \tag{A9}$$

provided that $\tilde{f}_0(n_r)=f_0(n_r)$. In other words, the emergence of the symmetry behavior is basically determined by Eq. (A9). This also means that Eq. (A5) has been chosen to be consistent with Eq. (3.8).

[1]E. Papp, Phys. Rep. **136**, 103 (1986).

[2]E. Papp, Phys. Rev. A **34**, 47 (1986).

[3]Invoking the notations of Ref. 2, one would have $\gamma(n)=\gamma_n d_0^n$ and $d_0^2=d_2^2(n)$, so that $\lambda_0=\gamma_2^0 d_0^2$.

[4]J. N. Silverman and J. C. van Leuven, Chem. Phys. Lett. **7**, 37 (1970); **7**, 640 (1970). See also Appendix A of Ref. 2.

[5]See Appendix B of Ref. 2.

[6]H. Orland, Phys. Rev. Lett. **42**, 285 (1979); G. Rosen, Phys. Rev. A **20**, 1287 (1979); F. M. Fernández and E. A. Castro, *ibid.* **27**, 2735 (1983).

[7]G. A. Arteca *et al.*, J. Math. Phys. **25**, 932 (1984); F. M. Fernández and E. A. Castro, Phys. Rev. A **27**, 663 (1983); C. Esebbag *et al.*, J. Phys. A **18**, 3505 (1985).

[8]V. Fock, Z. Phys. **63**, 855 (1930); E. Brändas and P. Froelich, Phys. Rev. A **16**, 2207 (1977); B. R. Junker, *ibid.* **27**, 2785 (1983); T. K. Rebane, Theor. Math. Phys. **56**, 432 (1983), etc.

[9]B. Simon, Ann. Phys. **58**, 76 (1970).

[10]Above $\partial_l^2\varepsilon=0$ should be understood as a limiting case.

[11]N. Rowley, J. Phys. A **12**, L7 (1979); M. C. Dumont-Lepage *et al.*, *ibid.* **13**, 1243 (1980).

[12]Equation (4.3) comes up immediately by squaring Eq. (4.2) of Ref. 2 and by rescaling the parameters.

[13]R. E. Langer Phys. Rev. **51**, 669 (1937); E. C. Kemble, *The Fundamental Principles of Quantum Mechanics* (McGraw-Hill, New York, 1937).

[14]F. M. Fernández and E. A. Castro, J. Chem. Phys. **79**, 321 (1983). Here the energies have been given to three digits. However, we shall give our results to six digits, thereby mentioning the inputs.

[15]The accuracy of such relationships could then be better for large l values, owing to the suppression of the Langer-Kemble terms.

[16]M. Seetharaman *et al.*, J. Phys. A **16**, 455 (1983). See Table I.

[17]R. A. Bertlmann, Phys. Rep. **134**, 279 (1986). Here the GSE's are given to four digits, see Table I.

[18]C. Quigg and J. L. Rosner, Phys. Rep. **56**, 167 (1979). See Table VI.

[19]Above 0^- (0^+) means that zero is approached from the left (right).

[20]S. A. Maluendes *et al.*, Phys. Rev. D **34**, 1835 (1986).

[21]L. D. Mlodinow and N. Papanicolaou, Ann. Phys. **128**, 314 (1980). See Eq. (3.30) which is responsible for the GSE.

[22]M. Seetharaman and S. S. Vasan, J. Phys. A **17**, 2485 (1984).

[23]F. M. Fernández *et al.*, J. Phys. A **18**, 1389 (1985). See Table VIII.

[24]S. S. Vasan and M. Seetharaman, J. Phys. A **17**, 2493 (1984). See Table I.

[25]The relative error is $r\%=100(\varepsilon_n-\varepsilon_n^{(\text{exact})})/\varepsilon_n^{(\text{exact})}$, as usual.

[26]The numerical estimates of the energy levels involved above will be reproduced by $\Delta B(1)\cong 0.129\,955$, $\Delta B(5)\cong 0.125\,836$, and $\Delta B(25)\cong 0.124\,103$. These estimates are centered reasonably around the average value $\Delta B\cong 0.126\,631$.

[27]F. M. Fernández *et al.*, Phys. Lett. **111A**, 104 (1985). See Table I.

[28]See, e.g., Sec. IV.4 of Ref. 9.

[29]For the sake of simplified notation we have suppressed the λ parameter.

[30]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1959), pp. 128, 129.

[31]I. Benjamin and R. D. Levine, Phys. Rev. A **33**, 2833 (1986).

[32]See also E. Papp. Phys. Rev. A **34**, 4405 (1986).

[33]A. Pagnamenta and U. Sukhatme, Phys. Rev. D **34**, 3528 (1986). Our kinetic energy should also be rescaled: $p^2/m_0\to p^2/2m_0$.

[34]T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Rev. D **29**, 1669 (1984); L. D. Mlodinow and M. P. Shatz, J. Math. Phys. **25**, 943 (1984).

[35]J. Killingbeck, M. N. Jones, and M. J. Thompson, J. Phys. A **18**, 793 (1985).

[36]J. L. Richardson and R. Blankenbecler, Phys. Rev. D **19**, 496 (1979); R. Balsa *et al.*, *ibid.* **28**, 1945 (1983).

[37]Mention that Refs. 35 and 36 refer to the one-dimensional oscillator.

[38]Fitting, e.g., the point $\lambda'=-0.02$ (see Table III of Ref. 35) leads to the extrapolation $d_0^2(\lambda')=(1+A\lambda')^{-1}[d_0^2(-2)+A\lambda' d_0^2(-4)]$, where $A\cong 1.505\,517$. This yields the critical point $\lambda_c'\cong -0.395\,116$. Mention that in the one-dimensional case one has $d_0(-2)=0.5$ and $d_0(-4)\cong 0.648\,283$. A special property of this extrapolation is the vertical asymptote at $\lambda'\cong -0.664\,223$.

[39]So far the large $\mu<0$ data quoted in Table IV of Balsa *et al.* (see Ref. 36) seem to indicate that $\delta_c\cong\frac{1}{2}$ ($N=1$).

[40]A. E. S. Green, J. M. Schwartz, and A. Khosravi, Phys. Rev. A **33**, 2087 (1986). Using the notations of this reference, one easily finds that, at zero energy, $d_0=(2Z_c/\exp 1)^{1/2}$, where Z_c is the critical coupling characterizing the Yukawa potential ($\nu=-1$). This corresponds to a biquadratic dependence of d_0 on l and n_r. However, the coefficients of the quadratic terms take relatively small values.