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Spatial correlations in multifractals

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We consider spatial correlations within multifractals or fractal measures $\mu(x)$. Correlation functions such as $\langle \mu(x)^m \mu(x+r)^n \rangle$ are argued to scale as $(R/a)^{y(m,n)}(r/a)^{z(m,n)}$, where R is the overall radius of the object and a a short cutoff. The exponents y and z are given in terms of the scaling dimensions D_q of the multifractal. We note that the existence of this single scaling form over the full range of r ($a \ll r \ll R$) is incompatible with any description of the measure as a superposition of simple fractal sets of localized singularities.

The properties of multifractal sets,[1-3] or fractal measures[4,5] (we use these terms synonymously), are relevant in the description of many physical processes such as turbulence,[1-3] diffusive growth,[6-10] electronic transport in disordered systems,[11,12] and chaos.[4,5] A distinguishing feature of a (normalized) fractal measure $\mu(x)$ is that moments $\langle \mu(x)^n \rangle$ of its probability distribution scale with independent powers d_n of the appropriately defined length parameters:

$$\langle \mu^n \rangle \cong (R/a)^{d_n} , \quad (1a)$$

$$d_n \equiv (1-n)D_n - D . \quad (1b)$$

Here R is the overall linear size of the set supporting the measure—taken to be a fractal set X of dimension D and "mass" $M \simeq (R/a)^D$; a is a short cutoff. Angle brackets denote averaging over all "sites" x (i.e., regions of linear size a) where $\mu(x)$ does not vanish. Equation (1a) still applies if a is replaced by any arbitrary coarse-graining length $r \ll R$; this reflects the scale invariance of the whole structure. Equation (1b) may be taken as the definition of the scaling dimensions D_n studied previously.[4,5,7-9] These are all equal to D only for the case of a simple fractal, which will be called a "trivial" multifractal of fractal measure.

In the present Rapid Communication, we indicate how the exponents D_n (or equivalently, d_n) can be used to predict scaling laws for spatial correlation functions such as

$$C_{mn}(r) = \langle \mu(x)^m \mu(x+r)^n \rangle . \quad (2)$$

Here, as before, angle brackets denote an average only over sites $x \in X$. This form is trivially related to the (more familiar) unrestricted correlation function,

$$\tilde{C}_{mn}(r) \equiv \langle\langle \mu(x)^m \mu(x+r)^n \rangle\rangle$$
$$\cong C_{mn}(r)(R/a)^{D-d}(r/a)^{D-d} ,$$

where d is the dimension of space, and double angle brackets denote an average over all sites (empty and nonempty).

We focus initially on fractal measures $\mu(x)$ arising from multiplicative random processes.[3,5,13,14] Such a measure can be generated recursively by a sequence of partitions governed by a scale-invariant probability distribution,[1-3,5] as illustrated in Fig. 1. At the first step we can identify a set of blobs i, each having a certain share μ_i of the normalized measure ($\sum_i \mu_i = 1$). Different blobs may have very different values of μ_i; there exists a distribution $P_\lambda[\mu_i]$, which is in general correlated with spatial position. The subscript $\lambda > 1$ denotes the length rescaling factor—the ratio between the size of a blob at one level and that at the next iteration. (The choice of λ is somewhat arbitrary; it is often useful to consider a "thermodynamic" limit, obtained by letting $\lambda \rightarrow \infty$.) At the next iteration, the measure μ_i within each blob is divided amongst the sub-blobs ij to give values μ_{ij} with probability distribution $P_\lambda[\mu_{ij}/\mu_i]$; here $P_\lambda[y]$ is the same function as before. After many iterations (down to a scale a, identified as the short cutoff) we obtain the measure $\mu_{ijkl\ldots} \equiv \mu(x)$.

To compute $C_{mn}(r)$, we imagine iterating from the uppermost level (at scale R) down to the scale r. Up to this point, the two points that enter the calculation of $C_{mn}(r)$ lie in the same blob, whereas at subsequent iterations they do not. Thus we may write

$$C_{mn}(r) \cong \langle \tilde{\mu}^{m+n} \rangle \langle [\mu(x)/\tilde{\mu}]^m \rangle \langle [\mu(x+r)/\tilde{\mu}]^n \rangle . \quad (3)$$

Here $\tilde{\mu}$ denotes the total measure in a blob of radius $\cong r$

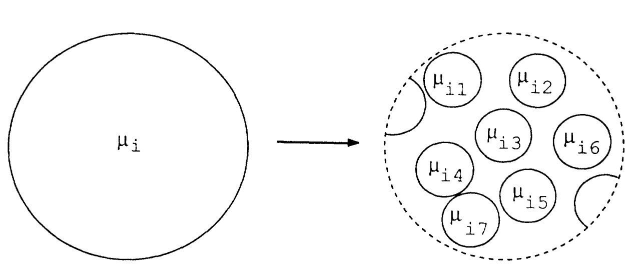

FIG. 1. Schematic representation of the random partitioning process described in the text. At each iteration, the measure μ_i assigned to blob i is shared among its constituent sub-blobs (ij) in a nonuniform way; $\sum_j \mu_{ij} = \mu_i$.

containing the two points. Since the term $\langle\bar{\mu}^{m+n}\rangle$ accounts for all correlations between the points x and $x+r$, the remaining average may be factored, as shown. From Eq. (1a) we then have $\langle\bar{\mu}^{m+n}\rangle \cong (R/r)^{d_{m+n}}$, $\langle[\mu(x)/\bar{\mu}]^m\rangle \cong (r/a)^{d_m}$, and $\langle[\mu(x+r)/\bar{\mu}]^n\rangle \cong (r/a)^{d_n}$. Using these relations in (3) we obtain the very simple result

$$C_{mn}(r) \simeq (R/a)^y (r/a)^z, \ (a \ll r \ll R) \ , \tag{4a}$$

$$y = d_{m+n} \ , \tag{4b}$$

$$z = d_m + d_n - d_{m+n} \ . \tag{4c}$$

Note that $C_{mn}(r)$ is independent of $r(z=0)$ for any "trivial" measure as previously defined. Thus, measurement of spatial correlations can provide an unambiguous quantitative test for multifractal behavior. By extending the blob analysis, it is straightforward to compute scaling forms similar to Eq. (4) for averages involving products of the measure at more than two spatial positions. However, the resulting expressions are cumbersome and will not be given here.

In the above discussion we considered multifractals which can be decomposed into blobs in an unambiguous manner by virtue of their explicitly multiplicative construction. However, we expect the blob picture to be quite general for the purposes of obtaining scaling laws such as Eq. (4). For example, while for many random multifractals (such as the hit probability in diffusion limited aggregation[7,13]) the exact assignment of the boundaries between blobs is ambiguous, we expect $P_\lambda[y]$ to become independent of this assignment in the thermodynamic limit of large λ. This means that our results will still apply for the stated regime $(a \ll r \ll R)$ although there will of course be crossover effects near $r/a \simeq 1$ and $r/R \simeq 1$.

A more interesting complication is if the partitioning of the measure within neighboring blobs is not independent, as was assumed above, but correlated. To show that this does not spoil the argument, we have studied a model in which the measure $\mu(x)$ is generated recursively by a sequence of random multiplications

$$\mu(x) \propto \prod_{j=1}^{N} \eta_j(x/\xi^j) \ , \tag{5}$$

where ξ is a rescaling factor and $\eta_j(y)$ is a random function with long-range correlations:

$$\langle \eta_i(x)^m \eta_j(x+r)^n \rangle = \delta_{ij} g_{mn}(r) + (1-\delta_{ij}) \langle \eta^m \rangle \langle \eta^n \rangle \ ,$$
$$g_{mn}(r) = [A\langle \eta^{n+m}\rangle - \langle \eta^n \rangle\langle \eta^m \rangle] r^{-p} + \langle \eta^n \rangle \langle \eta^m \rangle \ , \quad (p>0, A>0) \ . \tag{6}$$

Thus the random function η_j at each iteration j is chosen independently from an ensemble of such functions; the dilation factor ξ^j in the argument in Eq. (5) provides for the fact that after j iterations the short cutoff is at $a = R/\xi^j$. (The final value of a is R/ξ^N.) However, each function η_j is spatially correlated in a manner described by $g_{mn}(r)$, which, for illustrative purposes, we have taken to have a power-law decay $(\sim r^{-p})$ with position. It may then be shown that

$$\langle \mu(x)^m \mu(x+r)^n \rangle \propto \prod_j g_{mn}(r/\xi^j) \propto \exp\left[\sum_{j=1}^{N} h_{mn}(\ln r - j \ln\xi)\right] \ ,$$

where $h_{mn}(z) \equiv \ln[g_{mn}(e^z)]$. The asymptotic properties of the sum in the exponential are readily found. In particular, since $h_{mn}(z)$ has a much sharper crossover behavior than $g_{mn}(r)$, long-range correlations of the form (6) are harmless, and the results for d_q do not depend on p. [In fact, we obtain $d_q = \ln(\langle\eta^q\rangle)/\ln(\xi)$.] Moreover, Eq. (4) can be explicitly confirmed for this entire class of models. For a breakdown of that equation, the correlation function $g_{mn}(r)$ would have to approach its asymptotic value at large r more slowly than with any negative power of r.

In addition to cases covered by the above arguments, one is often interested in multifractals (such as some strange attractors[4,5]) that arise by chaotic as opposed to random processes. Insofar as such sets are indeed multifractals, it seems likely that an interpretation in terms of blobs can still be made, although now $P_\lambda[y]$ is presumably to be viewed as some kind of frequency distribution[3] rather than a probability. This view is supported by a further argument in favor of Eq. (4), which can be given by considering the effects of a coarse-graining operation. We presume that if $\mu(x)$ is a fractal measure, then so is $\theta_n(x) \equiv \mu(x)^n/\langle\mu^n\rangle$. [The denominator ensures that $\theta_n(x)$ is normalized.] We then define a coarse-grained quantity $\overline{\mu(x)}$ by

$$\overline{\mu(x)} \simeq l^{-D} \int_0^l \mu(x+r) d^D r \ ,$$

with a similar equation for $\overline{\theta_n(x)}$. We now demand that the operations of (i) coarse graining, and (ii) forming $\theta_n(x)$ from $\mu(x)$, can be performed in either order with the same effect. Then

$$\overline{\theta_n(x)} = \overline{\mu(x)^n}/\langle\mu^n\rangle \cong \overline{\mu(x)}^n/\langle\bar{\mu}^n\rangle \ . \tag{7}$$

We now consider the product

$$\langle \overline{\mu(x)^m}\,\overline{\mu(x)^n} \rangle \cong l^{-2D} \int_0^l \int_0^l C_{mn}(x'-x'') d^D x' d^D x'' \ . \tag{8}$$

The left side may be rewritten [using the second equality in (7)] as

$$\langle \overline{\mu^m}\,\overline{\mu^m} \rangle \cong \langle \bar{\mu}^{m+n} \rangle \frac{\langle \mu^m \rangle \langle \mu^n \rangle}{\langle \bar{\mu}^m \rangle \langle \bar{\mu}^n \rangle} \cong (R/a)^y (l/a)^z \ , \tag{9}$$

where to obtain the final expression, Eq. (1a) has been used; y and z are as given in Eqs. (4b) and (4c). Clearly, for Eqs. (8) and (9) to be compatible, $C_{mn}(r)$ must obey Eq. (4).

This last argument, though perhaps less transparent physically than the one based on blobs, suggests the conjecture that Eq. (4) applies whenever the quantities involved are well defined, i.e., for any object to which the multifractal formalism embodied in Eqs. (1a) and (1b) is itself applicable. This conclusion will not surprise anyone familiar with the success of blob arguments in describing the spatial correlations of ordinary fractals.[14]

It is noteworthy, however, that the scaling law Eq. (4) is *incompatible* with the idea[5] that a fractal measure or multifractal can be represented as a superposition of simple fractal sets, having fractal dimension $f(\alpha)$, of local singularities of strength α. (Singularities of "strength" α are defined as points around which the measure $\delta\mu$ within a small blob of linear size l behaves as $\delta\mu \sim l^{\alpha}$.) In view of the extent to which this "overlaid singularity" description has been adopted in the recent literature,[5-8,12,15] it is remarkable that whenever Eq. (4) holds (and we have argued above that it holds generally), such a decomposition is not possible.

To demonstrate this, we first presume that the decomposition is possible, and then obtain a contradiction with Eq. (4). In the overlaid singularity picture, it is well known[5] that the spectral function $f(\alpha)$ and the scaling dimensions D_q are related by the Legendre transform $(1-q)D_q = \max_\alpha[f(\alpha)-\alpha q]$. This follows directly from applying the steepest descents method to the quantity

$$(R/a)^{(1-q)D_q} \cong \sum_x \mu(x)^q \cong \int d\alpha (R/a)^{f(\alpha)} (a/R)^{\alpha q} \ .$$

[In the integral over α, $(R/a)^{f(\alpha)}$ is the number of sites x where $\mu(x) \cong (a/R)^{\alpha}$.] The same formalism can be extended to predict correlation functions such as $C_{mn}(r)$. Denoting the density of the set of singularities corresponding to a given α by $\rho_\alpha(x)$, we have [using Eq. (2), and paying attention to normalization factors]

$$C_{mn}(r) \cong \int d\alpha d\alpha' \Lambda_{\alpha\alpha'}(r) (a/R)^{\alpha m + \alpha' n} \ , \tag{10a}$$

$$\Lambda_{\alpha\alpha'}(r) = \frac{Q_{\alpha\alpha'}(r)}{\int d\alpha d\alpha' Q_{\alpha\alpha'}(r)} \ , \tag{10b}$$

$$Q_{\alpha\alpha'}(r) = \langle\langle \rho_\alpha(x) \rho_{\alpha'}(x+r) \rangle\rangle \ . \tag{10c}$$

Now, the fractal sets ρ_α are, presumably, correlated with one another in a complicated way. Nonetheless, since each ρ_α is itself a scale invariant set, the density-product $Q_{\alpha\alpha'}(r)$ must scale homogeneously under variation in the short cutoff a, so long as this always remains much less than r:

$$Q_{\alpha\alpha'}(r) \sim a^{\tau(\alpha,\alpha')}, \ (a \ll r \ll R) \ .$$

(We use "$\sim$" to indicate that there is also a dependence on r and R, which is not relevant to our present discussion.) From this we obtain, using (10b),

$$\Lambda_{\alpha\alpha'}(r) \sim a^{\sigma(\alpha,\alpha')}, \ (a \ll r \ll R) \ , \tag{11}$$

with a suitable choice of $\sigma(\alpha,\alpha')$.

To show that Eq. (11) is incompatible with Eq. (4), we invert the double Legendre transform, Eq. (10a), and insert Eq. (4) to obtain

$$\Lambda_{\alpha\alpha'}(r) \cong (R/a)^{\psi(\alpha,\alpha',t)} \ , \tag{12a}$$

$$\psi(\alpha,\alpha',t) = \max_{m,n}[y(m,n)+z(m,n)+\alpha m + \alpha' n + z(m,n)t] \ , \tag{12b}$$

$$t = \frac{\ln(r/R)}{\ln(R/a)} \ , \tag{12c}$$

where y and z are as defined in Eqs. (4b) and (4c). Now, if ψ were to vary linearly with t, say as

$$\psi(\alpha,\alpha',t) = \psi_0(\alpha,\alpha') + \psi_1(\alpha,\alpha')t \ , \tag{13}$$

then (12a) would give

$$\Lambda_{\alpha\alpha'}(r) \cong (R/a)^{\psi_0(\alpha,\alpha')} (r/R)^{\psi_1(\alpha,\alpha')} \sim a^{-\psi_0(\alpha,\alpha')} \ ,$$

which corresponds to an acceptable a dependence [cf. Eq. (11)]. However, it is clear that in general $\psi(\alpha,\alpha',t)$ depends *nonlinearly* on t, once the minimization over m,n in Eq. (12b) is carried out. [An exception is when the measure is trivial, since then $z(m,n)=0$.] Thus a single expression of the form (13) does not exist to describe all values of a and r in the range $a \ll r \ll R$, which is the domain of validity of Eq. (4); correspondingly the dependence of $\Lambda_{\alpha\alpha'}(r)$ on a cannot have the required scaling form (11) over this range.

The above argument may appear somewhat delicate, since one might naively expect that the inequalities $a \ll r \ll R$ would imply $t \to 0$, so that the unacceptable nonlinear terms would vanish. However, for $R \gg a$, values of r arise for which $a \ll r \ll R$ (so that the scaling arguments are valid) while t is of order unity (so that the unacceptable nonlinearities arise). Specifically, this occurs for all r of the form $r = a^s R^{1-s}$ with s a parameter obeying $0 < s < 1$.

We thus argue that Eq. (4), which describes the spatial scaling of a multifractal or fractal measure $\mu(x)$ in the blob picture, and Eq. (11), which describes the spatial scaling of the fractal sets [of dimension $f(\alpha)$] of local singularities (of strength α) into which $\mu(x)$ is purportedly decomposable, cannot both be correct. Since we expect Eq. (4) to hold generally, we conclude that the overlaid singularity picture does not provide an adequate description of the spatial properties of multifractals. Blob concepts[15] seem to provide a much more suitable description of multifractal correlations. Note, however, that the spectral function $f(\alpha)$ remains a useful object to study, since it can be identified with the thermodynamic limit of the partitioning probability $P_\lambda[y]$, when this is expressed in scaling form by a suitable transformation to logarithmic variables.[7,8,10]

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[1]B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974); *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[2]U. Frisch and G. Parisi, in *Turbulence and Predictability of Geophysical Fluid Dynamics and Climate Dynamics,* Proceedings of the International School of Physics "Enrico Fermi" Course LXXXVIII, Varenna, Italy, edited by M. Ghil (North-Holland, New York, 1985).

[3]R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **18**, 3521 (1984).

[4]H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983); P. Grassberger and I. Procaccia, *ibid.* **13**, 34 (1984).

[5]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[6]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[7]P. Meakin, H. E. Stanley, A. Coniglio, and T. A. Witten, Phys. Rev. A **32**, 2364 (1985); P. Meakin, A. Coniglio, H. E. Stanley, and T. A. Witten, *ibid.* **34**, 3325 (1986); P. Meakin, *ibid.* **33**, 1365 (1986); C. Amitrano, A. Coniglio, and F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).

[8]P. Meakin, Phys. Rev. A **34**, 710 (1986).

[9]M. E. Cates and T. A. Witten, Phys. Rev. Lett. **56**, 2497 (1986).

[10]M. E. Cates and T. A. Witten, Phys. Rev. A **35**, 1809 (1987).

[11]R. Rammal, C. Tannous, P. Breton, and A. M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985); L. de Archangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985).

[12]C. Castellani and L. Peliti, J. Phys. A **19**, L429 (1986).

[13]T. A. Witten, in *Chance and Matter,* Les Houches Summer School Proceedings, Vol. 46, edited by J. Souletie, J. Vannimenus, and R. Stora (North-Holland, Amsterdam, in press).

[14]P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell Univ. Press, Ithaca, NY, 1979).

[15]L. Pietronero and A. P. Siebesma, Phys. Rev. Lett. **57**, 1098 (1986).