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Quantitative test of solvability theory for the Saffman-Taylor problem

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We report on a quantitative test of solvability theory in the case of pattern selection for viscous fingering in a rectangular Hele-Shaw cell. We construct an effective two-dimensional theory that properly takes into account the effect of film draining. In the parameter range where the theory is applicable we find excellent agreement between our predictions and the experimental data. This provides the first precise quantitative assessment of microscopic solvability in a physical system.

Formation of nontrivial patterns in systems displaced far from equilibrium is a subject of considerable current interest.[1] Among such systems, the ones most intensively investigated are viscous fingering in a Hele-Shaw cell,[2] directional solidification of dilute binary mixtures,[3,4] and free dendritic growth in an undercooled melt.[5] One of the central questions in all these investigations is to understand, for a given degree of nonequilibrium which drives the system, the mechanism that governs the selection of an asymptotic state out of various possibilities. The first proposal was the hypothesis of marginal stability,[6] which asserted that the state ultimately selected out of a continuum of possibilities is the one at the boundary between stable and unstable regions. Another scenario proposed more recently is that of "microscopic solvability,"[7] which essentially holds that when all the relevant physical processes are taken into account there is only a discrete spectrum of asymptotic states available, and, of these, only one extremal solution is stable and thus selected by the system.

For theoretical analysis this is an extremely appealing scenario as has been demonstrated recently by a number of authors.[8-11] Of course an issue always at stake is the proper identification and inclusion of the "relevant physics." An important issue open for study is then to test the scenario precisely in experimentally realized situations. Of the three commonly studied examples referred to above, both directional solidification and free dendritic growth are beset by severe problems. Either the model equations describing the dynamics are of necessity oversimplified,[12] or they involve parameters that cannot be measured independently and reliably. Thus, in testing the predictions of one or another scenario of pattern selection, one is faced with the uncertainty that if a discrepancy arises between experiment and theory, one does not know whether to attribute it to the mechanism itself or to the difficulties noted.

In view of this we turn to the example of viscous fingering in a Hele-Shaw cell. In this case, for driving forces not too high, all the equations of motion can be written down with a great deal of confidence, and all the physical parameters can be measured *independently* and *accurately*. We believe that at this time this problem is the prime candidate for a precise quantitative test of ideas relating to mechanisms of pattern selection.

In Fig. 1 the experimental arrangement to study viscous fingering in rectangular geometry is shown schematically. The type of situation which can be handled best theoretically is one in which a gas displaces a liquid between a pair of uniformly spaced plates (of spacing b). The width of the channel is $w \gg b$. For a given driving force and thus a given velocity U of the tip of the finger generated, a specific fraction $\lambda(U)$ of the channel width is swept out. In this example the problem of pattern selection reduces essentially to a prediction of $\lambda(U)$ for given U, with an understanding of the mechanisms at work. If one were to ignore the surface tension between the gas and the liquid, one would find for a given value of U there are steady-state finger solutions for *any arbitrary* λ between 0 and 1.[2] However, if the surface tension is incorporated, this continuum breaks down into a discrete infinity,[13,14] and the solution with the smallest fractional width turns out to be the only linearly stable one.[11] Hence it is reasonable to expect that it is the selected one of the family. This sort of selection has indeed been observed in simulations of model equations.[15-17] To this extent the scenario of microscopic solvability (MS) seems to be obeyed. However, when it comes to quantitative correspondence, there is a serious disagreement (by a factor of 2 or greater) between experimental values of $U(\lambda)$ and the theoretical ones based on previously studied models. This can be seen in Fig. 2. Our objective in this paper is to demonstrate that if all relevant physics is properly included (in the parameter range in which it is possible to do so) theory agrees with experiments within the known error range of about 5%; hence, we have a precise quantitative verification of MS as the mechanism of pattern selection.

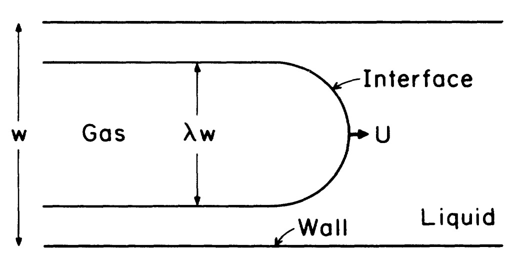

FIG. 1. Experimental arrangement for viscous fingering in a Hele-Shaw cell.

The two-dimensional model traditionally used to describe the experimental situation has been the following[18] (refer to Fig. 1): In the liquid the velocity $\mathbf{u}$ of the fluid averaged in the direction perpendicular to the plates is given by

$$\mathbf{u} = -\frac{b^2}{12\mu}\nabla p\ , \tag{1}$$

where p is the pressure and μ is the viscosity of the liquid. The assumption of incompressible fluid flow yields

$$\nabla^2 p = 0\ . \tag{2}$$

On the walls one has

$$\mathbf{u}\cdot\mathbf{n} = 0\ , \tag{3}$$

where $\mathbf{n}$ is the unit normal to the wall. Far to the right the velocity field is uniform, parallel to the walls and equal (by mass conservation) to λU. Finally, the pressure drop across the interface is given by the Laplace condition

$$\Delta p = \tau/R\ , \tag{4}$$

where τ is the surface tension and R is the radius of curvature in the plane of the plates. One can write Eqs. (1)–(4) in appropriate units to show that there is only one independent dimensionless parameter B, defined here by $B = 12\mu U w^2/\tau b^2$, which should determine λ. Experimental data in Fig. 2 clearly demonstrate that this is not the case. In fact, there is one more independent parameter, which we may choose as the ratio (b/w). Furthermore, there is a serious discrepancy between the prediction of this model and the experiment, the deviation increasing with (b/w). The corrected theory, as presented below explains these observations and thus yields quantitative consistency with the MS scenario appropriate to this problem.

The essential problem with the traditional theory is that it does not properly take into account the three-dimensional nature of the fluid flow at the interface. Park and Homsy[19] demonstrated that, at the interface, the liquid leaves behind a film of thickness t on each of the two plates (we may ignore gravity for the experiments to which we shall refer[20]) so that

$$u_n(\text{interface}) = U\cos\theta(1 - 2t/b)\ , \tag{5}$$

where θ is the angle between the local normal and the direction parallel to the walls. The thickness t depends on $U\cos\theta$, and this variation in the meniscus shape along the interface leads to an error in the boundary condition (4) which ignores this effect. When this crucial element of physics is incorporated, Eq. (4) becomes[19]

$$\Delta p = \frac{\pi\tau}{4R} + \frac{\tau}{b} f\left(\frac{\mu U\cos\theta}{\tau}\right)\ , \tag{6}$$

where $f(x)$ is a power series in powers of $x^{1/3}$, the first term being proportional to $x^{2/3}$. The film thickness is predicted[19] to be of the form (for sufficiently small $\mu U/\tau$)

$$t = 0.67\left(\frac{\mu U\cos\theta}{\tau}\right)^{2/3} b\ , \tag{7}$$

and this has been experimentally verified by Tabeling and Libchaber.[2] This is the basis for our belief that we have a complete theory represented by the modification in Eq. (6) in at least some parameter range.

The effective two-dimensional (2D) theory of Ref. 13 (which we shall extend) has the inherent assumption that $u_n = U\cos\theta$, and thus from Eq. (5) the very basis of this description breaks down unless $t \ll b$. This already constrains [from Eq. (7)] the velocity U to be sufficiently small, and in our calculations this violation never exceeds 1.5%. Hence we may retain the 2D description, with, however, the important modification represented by Eq. (6).

The prefactor of only the first term in the series expansion of $f(x)$ is independent of the interfacial coordinate in the transverse direction (and thus is consistent with the 2D description) and has been calculated by Park and Homsy. We shall retain only this term, which sets obvious restrictions on the magnitude of $(\mu U/\tau)$. The new boundary condition in place of Eq. (4) is

$$\Delta p = \frac{\pi\tau}{4R} + 7.6\frac{\tau}{b}\left(\frac{\mu U\cos\theta}{\tau}\right)^{2/3}\ . \tag{8}$$

Note that the modification of the boundary condition cannot be thought of as a simple addition to Eq. (4), since even the curvature term has a new prefactor. In essence, both pieces of Eq. (8) involve the same physics. Since the series expansion of $f(x)$ is only in powers of $x^{1/3}$, convergence is rather weak in x. Thus truncation at the first term will restrict the domain of $(\mu U/\tau)$ in the present calculation. There is clearly some scope for expanding the domain of validity by extending the Park-Homsy analysis and calculating higher order prefactors in Eq. (6), doing an appropriate averaging over the transverse direction to eliminate space dependence of Δp in that direction.

One immediate consequence of Eq. (8) is that now, when the equations are written in dimensionless form, there are *two* independent control parameters determined by B and (b/w) as noted above. This at least has the po-

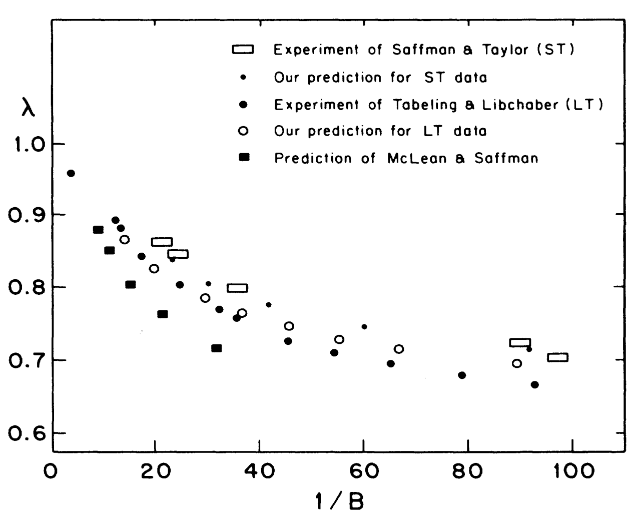

FIG. 2. Comparison of experimental data and predictions of various theories.

tential to explain one aspect of the experiments. This also has interesting implications for the number of control parameters for the radial Hele-Shaw flow.[21,22] To summarize, the new model retains Eqs. (2) and (3) but replaces Eq. (4) with Eq. (8). Now we turn our attention to the problem of finding $\lambda(U)$.

To solve systematically for all the steady-state fingers we extend the formalism developed by McLean and Saffman[13] and Vanden Broeck[14] and write down a system of integro-differential equations incorporating the modified boundary condition. (Details of derivations will be presented elsewhere.[23]) The system is defined in terms of two functions $q(s)$ and $\theta(s)$ with $0 \le s \le 1$. The equations and boundary conditions are

$$q-\cos\theta=k_1 qs\frac{d}{ds}\left[qs\frac{d\theta}{ds}\right]+k_2\frac{\cos\theta}{|\sin\theta|^{1/3}}qs\frac{d\theta}{ds}, \tag{9}$$

$$\ln q(s)=-\frac{s}{\pi}\mathrm{P}\int_0^1\frac{\theta(s')ds'}{s'(s'-s)}, \tag{10}$$

$$q(0)=1,\ \theta(0)=0,\ q(1)=0, \tag{11}$$

where "P" indicates principal part and

$$k_1=\frac{\pi^3 b^2\tau}{12\mu(1-\lambda)^2 U w^2}, \tag{12}$$

$$k_2=1.53\left[\frac{2bk_1}{w(1-\lambda)}\right]^{1/3}. \tag{13}$$

The new variable s locates position along the finger, with $s=1$ corresponding to the "nose" and $s=0$ corresponding to the "tail." In the above system, $\theta(1)$ is left as a free parameter. Physical solutions are those for which $\theta(1)=-\pi/2$, since the interface is required to be smooth at the nose.

The first step toward numerically solving the system (9)-(13) accurately is to have an analytic understanding of $\theta(s)$ and $q(s)$ in the neighborhood of the limits $s=0$, $s=1$. This information is used to make appropriate transformations on the independent variable s so that, with respect to this new variable, all necessary derivatives are bounded. We find that, with the new boundary condition, the convergence of θ and q at the tail end is algebraic in distance from the tip rather than exponential (which is the case in the old theory). In fact, as $s \to 0$,

$$\theta(s)\sim(-\ln s)^{-3},\qquad 1-q(s)\sim(-\ln s)^{-3}. \tag{14}$$

For $s \to 1$

$$\theta(1)-\theta(s)\sim\sqrt{(1-s)},\qquad q(s)\sim(1-s), \tag{15}$$

when $\theta(1)=-\pi/2$. For other (unphysical) values of $\theta(1)$ (in which there is a cusp at the nose), there are actually divergences in the derivatives (a point not mentioned in Vanden Broeck's work, but one equally applicable). Since the full singularity structure at the tip is not known [for $\theta(1)\neq-\pi/2$], we do not know if these divergences are removed by the transformations. To obtain accurate numerical solutions, more care and more grid points are needed. We have checked that the numerically obtained value of $\theta(1)$ converges rather quickly as the number of grid points is increased. However, we also notice that the spectrum of λ for physical solutions is rather insensitive to the number of grid points (typically 50-100). As a check on the numerical accuracy of our scheme, we reproduced many of the results of Saffman and McLean and Vanden Broeck.

Numerical calculations were carried out for parameter values corresponding to the Saffman-Taylor and the more accurate Tabeling-Libchaber experiments. Notice from Fig. 2 that apart from changing the singularity structure at the tail, the major effect of the modified boundary condition is to shift the physical solutions to higher values of λ compared to those for the old version of the theory. The importance of this will be discussed further below. Essentially the shift is a reflection of the extra pressure drop across the interface, caused by the velocity-dependent term, which causes a slowdown at the tip and enhanced flow at the sides. The finger widens in response.

A full stability analysis for these solutions has not yet been carried out. However, physical arguments strongly suggest that the velocity-dependent term has a stabilizing effect. For the moment, given the structural similarities between the solutions with and without the velocity-dependent boundary condition, we shall assume that only the solution with the smallest width is linearly stable. Figure 2 shows the experimental data and our predictions for the experiments of Tabeling and Libchaber as well as those of Saffman and Taylor. We are not certain about the error bars associated with the latter experiment, and the boxes shown are taken from the paper of Tabeling and Libchaber.

The important predictions of the present theoretical analysis to be noted in Fig. 2 are the following. (i) For a given λ the value of $1/B$ depends significantly on the value of the second independent parameter (b/w), as is seen in the experiments. (ii) For the experiments of Libchaber and Tabeling, as $(1/B)\to 0$, our predictions agree with the nominal experimental data to within 5%, which is also the error range quoted for the experiment (arising mostly from uncertainty in the measured value of the surface tension[24]). This should be compared with the traditional theory which shows about a factor of 2 disagreement in the same range of λ. Note that Tabeling and Libchaber[2] attempted to account phenomenologically for the discrepancy by constructing an effective surface tension that averaged the velocity-dependent correction over the interface. Unlike our calculation this procedure introduces a new parameter which has to be adjusted to fit the experimental data and which cannot be computed from first principles. As $(1/B)$ increases, a gap between the prediction and the experiment arises and is expected for the reasons noted above. Theoretical values are missing for λ very close to 1, since here k_1 rises rapidly necessitating the use of a very large number of grid points. Hence the computation time goes up very rapidly as $\lambda\to 1$.

To summarize, we have produced what we feel is the first demonstration that the MS scenario is indeed valid in a precise quantitative way for the case of viscous fingering in a Hele-Shaw cell (although demonstrations of stability remain to be made). At this time it is the only case in which the equations of motion can be systematically derived from first principles (the Navier-Stokes equation in this case) and some aspects can also be tested *independently* (film thickness, for example). We have stretched the two-dimensional theory almost to the limit of its validity, and in the domain where our description is internally consistent and accurate, there is quantitative agreement with experiment. We should mention that there is one more crucial implication of the boundary condition (6). We pointed out that the extra pressure drop due to the velocity-dependent correction leads to a more space-filling structure. This is exactly what is seen in the case of *radial* Hele-Shaw fingering with high rate of fluid displacement.[25,26] We conjecture that this is what leads to dense branching morphology as opposed to fractal structure of the interface. Further work is in progress to verify whether this will indeed explain the crossover between the two apparently different growth morphologies as the velocity-dependent correction is tuned.

After this paper was prepared we came across work of Schwartz and DeGregoria[27] which also reports some of our results, although the point of view and the techniques employed are considerably different.

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