Transformation properties of the equation $\nabla \times \mathbf{V} = k\mathbf{V}$

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The vector differential equation $\nabla \times \mathbf{V} = k\mathbf{V}$ with constant k has been the subject of recent controversy. Some of its solutions lead to force-free magnetic fields or to standing electromagnetic waves having **E** parallel to **B** everywhere. Recent criticisms of this equation have been given based primarily on its transformation properties, especially under spatial inversions. Here we reformulate this equation so as to reconcile these criticisms.

I. INTRODUCTION

The first-order vector differential equation

$$\nabla \times \mathbf{V} = k\mathbf{V} , \tag{1}$$

where k is a positive constant and **V** has an implicit harmonic time dependence with radian frequency $\omega = ck$, has been the subject of recent controversy. If one identifies **V** with the magnetic field **B**, then some solutions of Eq. (1) lead to "force-free magnetic fields" as discussed, e.g. by Freire.[1] If one identifies **V** with the vector potential **A**, then some solutions of Eq. (1) lead to a type of standing electromagnetic wave having $\mathbf{E} \| \mathbf{B}$ as discussed by Chu and Ohkawa.[2] The $\mathbf{E} \| \mathbf{B}$ type of waves were criticized on physical grounds by Lee[3] and by Salingaros.[4] Subsequently, Salingaros[5] presented three additional criticisms of Eq. (1) based on mathematical grounds.

The purpose of this paper is to defend the validity of Eq. (1). It is organized as follows. In Sec. II we review and discuss the $\mathbf{E} \| \mathbf{B}$ solutions associated with Eq. (1). In Sec. III we reformulate Eq. (1) so as to endow it with the correct behavior under general spatial coordinate transformations (both proper and improper). Finally, in Sec. IV we discuss the above five criticisms in light of this reformulation.

II. $\mathbf{E} \| \mathbf{B}$ SOLUTIONS

We identify **V** with the vector potential **A** and assume, for concreteness, that **A** has an implicit $\cos(\omega t)$ time dependence ($\omega = ck$). A solution of $\nabla \times \mathbf{A} = k\mathbf{A}$ is

$$\mathbf{A} = a[\mathbf{i}\sin(kz) + \mathbf{j}\cos(kz)]\cos(\omega t) , \tag{2}$$

where a is a constant. The associated electric and magnetic fields are

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = ka[\mathbf{i}\sin(kz) + \mathbf{j}\cos(kz)]\sin(\omega t) , \tag{3a}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = ka[\mathbf{i}\sin(kz) + \mathbf{j}\cos(kz)]\cos(\omega t) . \tag{3b}$$

One immediately sees that **E** and **B** (and also **A**) are everywhere parallel. It is difficult to understand any criticism of this electromagnetic wave since the fields (**E** and **B**) clearly satisfy the free-space Maxwell equations. In fact, this electromagnetic wave is simply the superposition

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4) , \tag{4a}$$

$$\mathbf{B} = \tfrac{1}{2}(\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4) , \tag{4b}$$

of the following four plane-polarized traveling electromagnetic waves:

$$\mathbf{E}_1 = ka\cos(kz - \omega t)\mathbf{i}, \quad \mathbf{B}_1 = ka\cos(kz - \omega t)\mathbf{j} ; \tag{5a}$$

$$\mathbf{E}_2 = -ka\sin(kz - \omega t)\mathbf{j}, \quad \mathbf{B}_2 = ka\sin(kz - \omega t)\mathbf{i} ; \tag{5b}$$

$$\mathbf{E}_3 = -ka\cos(kz + \omega t)\mathbf{i}, \quad \mathbf{B}_3 = ka\cos(kz + \omega t)\mathbf{j} ; \tag{5c}$$

$$\mathbf{E}_4 = ka\sin(kz + \omega t)\mathbf{j}, \quad \mathbf{B}_4 = ka\sin(kz + \omega t)\mathbf{i} . \tag{5d}$$

Thus to question the validity of the solution (3), one must object to either the plane-polarized traveling waves (5) or the superposition concept (4).

The standing wave solution (3) may be called "left-handed" because the field lines (**E**, **B**, or **A**) form the pattern of a left-handed screw about the "propagation axis" (z axis in this case). This left-handedness is due to the original differential equation

$$\nabla \times \mathbf{A} = k\mathbf{A} \tag{6}$$

and has nothing to do with singling out the z axis or with the particular phase, $\cos(\omega t)$, of the time dependence. If, however, one were to use the "companion" differential equation

$$\nabla \times \mathbf{A} = -k\mathbf{A} \tag{7}$$

one would obtain right-handed solutions with $\mathbf{E} \| \mathbf{B}$ as is easily seen by changing $k \to -k$ in Eqs. (3).

III. THE EQUATION REFORMULATED

One problem associated with Eq. (1) is its behavior under spatial inversions. If k is a scalar quantity then this equation is inconsistent regardless of whether **V** is a vector or a pseudovector quantity. The analogous problem does not occur in Maxwell's equations,

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\ , \tag{8a}$$

$$\nabla\times\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\ , \tag{8b}$$

because **E** is a vector and **B** is a pseudovector (or vice versa if one prefers). To avoid the above problem, we propose to modify Eq. (1) in either of two, essentially equivalent, ways.

One obvious way out of this dilemma is to allow k to take on negative as well as positive values and to assume that it transforms as a pseudoscalar,

$$\nabla\times\mathbf{V}=k\mathbf{V}\ , \tag{9}$$

where k is a pseudoscalar. One can regard this as an eigenvalue problem with eigenvalue $-\infty<k<\infty$. This is analogous to the quantum-mechanical eigenvalue problem for the component of angular momentum along some unit vector **n**,

$$\mathbf{n}\cdot(\mathbf{r}\times\mathbf{p})\psi=\hbar m\psi\ ,$$

in which the quantity m transforms as a pseudoscalar under coordinate inversion.

A second way out of the dilemma is to explicity insert the unit pseudoscalar ξ,

$$\xi=\begin{cases}+1, & \text{for right-handed systems}\\ -1, & \text{for left-handed systems}\end{cases} \tag{10}$$

and retain k as a positive scalar. That is,

$$\nabla\times\mathbf{V}=\xi k\mathbf{V}\ , \tag{11}$$

where k is a positive scalar. This has the advantage of retaining the significance of $k=\omega/c$ as the magnitude of the wave number. We shall adopt this second point of view, i.e., Eq. (11); however, one need only (i) change $\xi k\rightarrow k$ in equation (11) and (ii) change the transformation properties of k from a scalar to a pseudoscalar in order to revert to the interpretation of Eq. (9).

Of course, neither of the above two suggested modifications affects the practical use of Eq. (1) provided that one does not invert the coordinate system; thus the **E**||**B** solutions [Eq. (2) and (3)] of Chu and Ohkawa remain valid.

In terms of components, Eq. (11) is written

$$\frac{1}{\sqrt{g}}\epsilon^{lmn}V_{n;m}=\xi kV^{l}\ , \tag{12}$$

where k is a positive scalar. Here ϵ^{lmn} is the Levi-Civitta symbol and $g=\det(g_{ij})$ is the determinant of the second-rank covariant metric tensor. These transform according to

$$\epsilon'^{lmn}=\epsilon^{ijk}(\partial x'^{l}/\partial x^{i})(\partial x'^{m}/\partial x^{j})(\partial x'^{n}/\partial x^{k})J\ , \tag{13}$$

$$(g')^{1/2}=(gJ^2)^{1/2}=\sqrt{g}\,\sigma J\ , \tag{14}$$

where

$$J=\det(\partial x/\partial x') \tag{15}$$

is the Jacobian of the transformation and

$$\sigma=J/|J|=\begin{cases}+1, & \text{for proper transformations}\\ -1, & \text{for improper transformations}\end{cases} \tag{16}$$

(inversions).

Thus ϵ^{lmn} is a third-rank contravariant tensor density of weight $W=+1$ and $1/\sqrt{g}$ is a pseudoscalar density of weight $W=-1$. The semicolon in Eq. (12) denotes covariant differentiation; because of the antisymmetry of the Levi-Civitta symbol, this may be replaced with ordinary partial differentiation (denoted by a comma),

$$\frac{1}{\sqrt{g}}\epsilon^{lmn}V_{m,n}=\xi kV^{l}\ , \tag{17}$$

where k is a positive scalar. Each side of this equation transforms as a contravariant pseudovector (if V^l is a contravariant vector) or as a contravariant vector (if V^l is a contravariant pseudovector). Equation (17) is presented as the correct version, in component form, of the original Eq. (1).

IV. CRITICISMS OF THE EQUATION

In this section we discuss five recent criticisms of Eq. (1). Criticism (i) was made by Lee,[3] (ii) by Salingaros,[4] and (iii)—(v) by Salingaros.[5]

(i) Lee argued that Eq. (1) is internally inconsistent since it implies that **V** must be both a vector and a pseudovector; hence **V** must vanish. This was rebutted (successfully in our opinion) by Chu.[6] From our point of view, the use of Eq. (11) instead of Eq. (1) resolves any disagreement.

(ii) Salingaros argued that the **E**||**B** waves associated with Eq. (1) lead to an invariance inconsistency. Namely, Salingaros showed that for these waves $(\kappa_1^2+\kappa_2^2)^{1/2}=2\epsilon$ where κ_1 and κ_2 are Lorentz invariants and ϵ is the energy density which is not a Lorentz invariant. However, this equation holds only in one particular Lorentz frame; in other frames the right-hand side would not be simply 2ϵ. Thus, in our opinion, there is no invariance inconsistency.

(iii) Salingaros argued that Eq. (1), with **V** chosen as the magnetic field **B**, leads to a gauge inconsistency. He asserts that under the gauge transformation

$$\mathbf{A}'=\mathbf{A}+\nabla\lambda \text{ and } \mathbf{B}'=\mathbf{B}\ , \tag{18a}$$

one has

$$\nabla\times\mathbf{B}=k\mathbf{B}=k\nabla\times\mathbf{A}\Longrightarrow\mathbf{B}=k\,\mathbf{A}+\nabla\phi \tag{18b}$$

and (presumably) similarly

$$\mathbf{B}'=k\,\mathbf{A}'+\nabla\phi\ , \tag{18c}$$

which leads to a contradiction. In our opinion there is no reason why the same function ϕ should appear in both Eqs. (18b) and (18c). Thus Eq. (18c) should read $\mathbf{B}=k\,\mathbf{A}+\nabla\phi'$; in fact, with $\phi'=\phi-k\lambda$ there is no contradiction. (*Note added in proof.* In a recent Letter, Maheswaren[7] reaches a similar conclusion concerning the function ϕ.)

(iv) Salingaros argued that Eq. (1) is internally inconsistent on parity grounds. This is essentially the same argument as in (i) above.

(v) One may obtain a solution $\mathbf{V}$ of the vector differential equation (1) by employing a scalar function ψ such that

$$\psi=\psi(y,z)\ , \tag{19a}$$

$$(\partial^2/\partial y^2+\partial^2/\partial z^2+k^2)\psi=0\ . \tag{19b}$$

It is then readily verified that

$$\mathbf{V}=\mathbf{i}\psi+(1/k)(\mathbf{j}\partial\psi/\partial z-\mathbf{k}\partial\psi/\partial y) \tag{19c}$$

is indeed a solution of Eq. (1). Salingaros showed that Eq. (19c) does not maintain its form under a general coordinate transformation $(y,z)\rightarrow[u(y,z),v(y,z)]$; Salingaros interprets this lack of form invariance as a deficiency of Eq. (1). In our opinion relations such as Eq. (19c) are obviously intended only for Cartesian coordinate systems; one cannot expect them to maintain their form under more general coordinate transformations. We now reformulate Eqs. (19) in a coordinate-free notation; this will lead to equations which have the correct transformation properties under arbitrary coordinate transformations (including curvilinear coordinates and coordinate inversions).

Let $\mathbf{u}$ be any "constant" vector and let $\psi(\mathbf{r})$ be such that $\nabla\psi$ is orthogonal to $\mathbf{u}$, i.e.,

$$\nabla\mathbf{u}=0 \text{ and } \mathbf{u}\cdot\nabla\psi=0\ , \tag{20a}$$

and such that ψ satisfies

$$(\nabla^2+k^2)\psi=0\ . \tag{20b}$$

It is then readily verified that

$$\mathbf{V}=\mathbf{u}\psi-\frac{1}{\xi k}\mathbf{u}\times\nabla\psi \tag{20c}$$

is indeed a solution of Eq. (11). In this manner one may generate a vector solution $\mathbf{V}$ using a scalar function ψ or a pseudovector solution $\mathbf{V}$ using a pseudoscalar function ψ. Equations (20) are the desired coordinate-free analogs (with $\mathbf{u}=\mathbf{i}$) of Eqs. (19). For completeness, we write Eqs. (20) in component form,

$$u_{i;j}=0 \text{ and } u^i\psi_{;i}=0\ , \tag{21a}$$

$$g^{ij}\psi_{;ij}+k^2\psi=0\ , \tag{21b}$$

$$V^l=u^l\psi-\frac{1}{\xi k}\frac{1}{\sqrt{g}}\epsilon^{lmn}u_m\psi_{;n}\ . \tag{21c}$$

Equations (21) have the correct transformation properties under arbitrary coordinate transformations. One can also verify that $\mathbf{V}$, as given by Eq. (21c), does indeed satisfy the differential equation (17).

V. RESULTS

The original vector differential equation

$$\nabla\times\mathbf{A}=k\mathbf{A}\ , \tag{1'}$$

where k is a positive constant has been reformulated to read

$$\nabla\times\mathbf{A}=\xi k\mathbf{A}\ , \tag{11'}$$

where k is a positive scalar and ξ is the unit pseudoscalar. Equation (11) is clearly manifestly covariant for arbitrary spatial coordinate transforations (including inversions). Five recent criticisms of Eq. (1) were discussed; some of these are reconciled by the use of Eq. (11) instead of Eq. (1), others were shown to be incorrect for other reasons.

The above two equations are clearly equivalent for right-handed systems. Therefore previous solutions, in particular the $\mathbf{E}||\mathbf{B}$ standing waves of Chu and Ohkawa, remain valid.

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