Quantum behavior of a four-wave mixer operated in a nonlinear regime

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A four-wave mixer model is presented for which exact analytic results can be obtained in a regime where the usual perturbation procedures no longer apply. It is shown that under suitable conditions the output of such a four-wave mixer consists of a superposition of macroscopically distinguishable states. Such states can be identified with a homodyne detector which is able to observe the interference between the macroscopically distinguishable components. Further, it is shown that the device can act as an even-odd filter which will switch incoming pump light between two output ports depending on whether there is an even or odd number of pump photons.

Four-wave mixers have been extensively discussed[1–6] as sources of nonclassical light referred to as squeezed light. In fact, three of the recently successful experimental efforts to generate squeezed light employed four-wave mixers.[7–9] In order to generate squeezed light, four-wave mixers are operated under conditions of small pump depletion. In this regime, where the coupling between the signal and pump modes is weak, an adequate theoretical description of four-wave mixing is obtained via a perturbation expansion to low orders in the signal amplitude and the pump noise amplitude. Here we explore some quantum phenomena arising when a four-wave mixer is operated in a regime where the coupling between the pump and signal modes is strong. A model is presented for which exact analytic results can be obtained. It is shown that for a suitable choice of coupling between the signal and pump mode an initial state consisting of vacuum in the signal mode and coherent state light in the pump mode will evolve into a state in which the light is in a coherent superposition of being entirely in the signal mode or entirely in the pump mode. This superposition is generated because the device acts as an even-odd filter in which incoming pump light is transferred to the signal mode if the number of pump quanta is even. The two parts of this coherent superposition will be macroscopically distinguishable if the initial pump state has a large amplitude. By combining the light in the signal and pump mode via a beam splitter, a homodyne detector can be used to exhibit the interference that arises between the two macroscopically distinguishable parts of the coherent superposition.

The possibility of generating coherent superpositions of macroscopically distinguishable states via an amplitude dispersive process which could be regarded as self-four-wave mixing has already been discussed.[10,11] Here a four-wave mixer is modeled more realistically in that there is a separate signal and pump mode.

Let a and b denote the boson annihilation operators for the pump and signal mode, respectively. Before describing the Hamiltonian it is useful to introduce the operators[12]

$$\begin{aligned} J_x &= \tfrac{1}{2}(a^\dagger b + b^\dagger a)\ , \\ J_y &= -\frac{i}{2}(a^\dagger b - b^\dagger a)\ , \\ J_z &= \tfrac{1}{2}(a^\dagger a - b^\dagger b)\ , \end{aligned} \tag{1}$$

which satisfy the usual angular-momentum commutation relations. The Casmir invariant can be put into the form

$$J^2 = \frac{N}{2}\left[\frac{N}{2}+1\right]\ , \tag{2}$$

where N is the operator for the total number of photons,

$$N = a^\dagger a + b^\dagger b\ . \tag{3}$$

It is useful to note that

$$\begin{aligned} J_z \,|\, n\rangle_a \,|\, 0\rangle_b &= \frac{n}{2}\,|\, n\rangle_a \,|\, 0\rangle_b\ , \\ J^2 \,|\, n\rangle_a \,|\, 0\rangle_b &= \frac{n}{2}\left[\frac{n}{2}+1\right]|\, n\rangle_a \,|\, 0\rangle_b\ , \end{aligned} \tag{4}$$

that is, $|\, n\rangle_a \,|\, 0\rangle_b$ is the $|\, j,m\rangle$ state given by $j=n/2$ and $m=n/2$.

The Hamiltonian is taken to be

$$H = \omega N + \Omega J_x^2\ . \tag{5}$$

The nonlinear term ΩJ_x^2, besides containing the usual four-wave mixing terms $\frac{1}{4}\Omega(a^\dagger a^\dagger bb + aab^\dagger b^\dagger)$, has the nonlinear term $\frac{1}{2}\Omega a^\dagger a b^\dagger b$. It is reasonable to consider such a Hamiltonian since realizable four-wave mixing media will generally also exhibit a Kerr-effect nonlinearity proportional to $a^\dagger a b^\dagger b$.[13] Under the action of H an initial state $|\text{in}\rangle$ evolves into the state $|\text{out}\rangle$ according to the unitary transformation (we take $\hbar=1$)

$$|\,\text{out},t\,\rangle = e^{-i(\omega N+\Omega J_x^2)t}\,|\,\text{in}\,\rangle\ , \tag{6}$$

where t is the interaction time.

The initial state will consist of the pump in the coherent state $|\,\alpha\rangle_a$ and the signal in the vacuum state $|\,0\rangle_b$:

$$|\,\text{in}\rangle = |\,\alpha\rangle_a \,|\, 0\rangle_b\ . \tag{7}$$

Since N commutes with J_x^2, Eq. (6), using the number basis expansion for $|\alpha\rangle_a$, can readily be put into the form

$$|\text{out},t\rangle = e^{-(|\alpha|^2/2)} \sum_{j=0,1/2,1,3/2,\ldots} \frac{(\alpha e^{-i\omega t})^{2j}}{\sqrt{(2j)!}} e^{-i\Omega J_x^2 t} |j,j\rangle \ . \tag{8}$$

One can now bring to bear the mathematical machinery for the rotation group. Since $e^{-i(\pi/2)J_y}$ performs a rotation by an angle $\pi/2$ about the y axis, Eq. (10) can be put into the form

$$|\text{out},t\rangle = e^{-(|\alpha|^2/2)} \sum_{j=0,1/2,1,3/2,\ldots} \frac{(\alpha e^{i\omega t})^{2j}}{\sqrt{(2j)!}} e^{i(\pi/2)J_y} e^{-i\Omega t J_z^2} e^{-i(\pi/2)J_y} |j,j\rangle \ . \tag{9}$$

Introducing the usual[14] rotation matrix elements

$$d^j_{mn}(\beta) = \langle j,m | \exp(-i\beta J_y) | j,n\rangle \ , \tag{10}$$

the state $e^{-i(\pi/2)J_y}|j,j\rangle$ can be expanded in a $|j,m\rangle$ eigenbasis and Eq. (9) can be further written as

$$|\text{out},t\rangle = e^{-(|\alpha|^2/2)} \sum_{j=0,1/2,1,3/2,\ldots} \frac{(\alpha e^{-i\omega t})^{2j}}{\sqrt{(2j)!}} e^{i(\pi/2)J_y} \sum_m e^{-im^2\Omega t} d^j_{mj}\left[\frac{\pi}{2}\right] |j,m\rangle \ . \tag{11}$$

There are specific values of t for which Eq. (13) is particularly easy to evaluate. When $t = 8\pi k/\Omega$ where k is an integer, then $e^{-im^2\Omega t} = 1$ and

$$\left|\text{out},\frac{8\pi k}{\Omega}\right\rangle = |\alpha e^{-i(8\pi\omega k/\Omega)}\rangle_a |0\rangle_b \ , \tag{12}$$

that is, periodically with a period of $8\pi/\Omega$ the photons will reassemble themselves into a coherent state in the pump mode.

Of interest is the case when $t = 2\pi/\Omega$. Then $e^{-im^2\Omega t} = 1$ if j is an integer and $e^{-im^2\Omega t} = -i$ if j is a half integer. Equation (11) then reduces to

$$|\text{out},2\pi/\Omega\rangle = \frac{1}{\sqrt{2}}[(e^{-i(\pi/4)}|\alpha e^{-i(2\pi/\Omega)\omega}\rangle_a + e^{i(\pi/4)}|-\alpha e^{-i(2\pi/\Omega)\omega}\rangle_a)]|0\rangle_b \ . \tag{13}$$

The pump is thus in a coherent superposition of two coherent states 180° out of phase with each other while the signal is in the vacuum state. As α is made large, $|\text{out},2\pi/\Omega\rangle$ consists of two macroscopically distinguishable pieces. Similar behavior has been noted in a simpler model consisting of an anharmonic oscillator in Ref. 10.

Even more bizarre behavior occurs when $t = \pi/\Omega$. When j is an integer, $e^{-im^2\Omega t} = (-1)^m$. If j is half integer, $e^{-im^2\Omega t} = e^{-i(\pi/4)}$. The state $|\text{out},\pi/\Omega\rangle$ can thus be written as

$$|\text{out},\pi/\Omega\rangle = e^{-(|\alpha|^2/2)}\left[\sum_{j=0,1,2,\ldots} \frac{\beta^{2j}}{\sqrt{(2j)!}} e^{(i\pi/2)J_y} \sum_m (-1)^m d^j_{mj}\left[\frac{\pi}{2}\right]|j,m\rangle + e^{-i(\pi/4)} \sum_{j=1/2,3/2,5/2,\ldots} \frac{\beta^{2j}}{\sqrt{(2j)!}}|j,j\rangle\right] , \tag{14}$$

where $\beta = \alpha e^{-i\pi\omega/\Omega}$. By noting that

$$\langle j,m|\mathscr{D}|j,j\rangle = (-1)^m d^j_{mj}\left[\frac{\pi}{2}\right] , \tag{15}$$

where the rotation operator $\mathscr{D}$ is given by[13]

$$\mathscr{D} = \exp(-i\pi J_z)\exp\left[-i\frac{\pi}{2}J_y\right] , \tag{16}$$

and that

$$e^{i(\pi/2)J_y} e^{-i\pi J_z} e^{-i(\pi/2)J_y} = e^{i\pi J_x}$$

and using $e^{i\pi J_x}|j,j\rangle = (-1)^j|j,-j\rangle$, Eq. (18) can be simplified to

$$|\text{out},\pi/\Omega\rangle = \frac{1}{\sqrt{2}}(|S\rangle + |P\rangle) \ , \tag{17}$$

where

$$|S\rangle = \frac{1}{\sqrt{2}}|0\rangle_a(|i\beta\rangle_b + |-i\beta\rangle_b) \ ,$$
$$|P\rangle = \frac{e^{-i(\pi/4)}}{\sqrt{2}}(|\beta\rangle_a - |-\beta\rangle_a)|0\rangle_b \ . \tag{18}$$

Equation (17) can be recognized as consisting of a coherent superposition of two macroscopically distinguishable parts: one, $|S\rangle$, in which all the photons exit the four-wave mixer through the signal port, and the other, $|P\rangle$, in which all the photons exit the pump port. Expanded in the number basis, $|S\rangle$ is a superposition of only even number states while $|P\rangle$ is a superposition of only odd number states. Hence, the device acts as an even-odd filter. In order to distinguish such a state from a statistical mixture in which incoming pump light is randomly switched between the signal and pump exit ports, it is necessary to construct an experiment which exhibits the interference between the two components of the coherent superposition. This can be accomplished by combining the light coming out of the signal port and the pump port via a beam splitter and then viewing the light exiting the beam splitter with a homodyne detector.

More rigorously, consider a beam splitter having the mode transformation

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} . \tag{19}$$

Such an overall mode transformation can be realized with any 50-50 beam splitter provided suitable phase shifters are placed in its input and output ports. The state $|\text{out},\pi/\Omega\rangle$ upon passing through the beam splitter becomes[10]

$$|\text{out},\pi/\Omega\rangle = \tfrac{1}{2}(|\gamma\rangle_c - e^{-i(\pi/4)}|-\gamma\rangle_c)|i\gamma\rangle_d + \tfrac{1}{2}(|-\gamma\rangle_c + e^{-i(\pi/4)}|\gamma\rangle_c)|-i\gamma\rangle_d , \tag{20}$$

where $\gamma = 2^{-1/2}\beta$. A homodyne detector[15–18] placed in the output port c will measure the variable

$$x = \frac{1}{\sqrt{2}}(e^{i\theta}e^{i\omega t}c + e^{-i\theta}e^{-i\omega t}c^{\dagger}) , \tag{21}$$

where the local oscillator phase θ is at the experimenter's control. Similarly, a homodyne detector in the port d will measure the variable

$$y = \frac{1}{\sqrt{2}}(e^{i\psi}e^{i\omega t}d + e^{-i\psi}e^{-i\omega t}d^{\dagger}) . \tag{22}$$

For simplicity we will set ψ equal to θ. By standard techniques,[19] the x,y representation $\psi(x,y) = \langle x,y\,|\,\text{out},\pi/\Omega\rangle$ for $t=\pi/\Omega$ can be constructed

$$\psi(x,y) = \tfrac{1}{2}[\psi_\gamma(x) - e^{-i(\pi/4)}\psi_{-\gamma}(x)]\psi_{i\gamma}(y) + \tfrac{1}{2}[\psi_{-\gamma}(x) + e^{-i(\pi/4)}\psi_\gamma(x)]\psi_{-i\gamma}(y) , \tag{23}$$

where

$$\psi_\gamma(x) = \frac{1}{\pi^{1/4}} \exp\left[-\frac{x^2}{2} + \sqrt{2}x\gamma e^{i\theta'} - \frac{|\gamma|^2}{2} - \tfrac{1}{2}(\gamma e^{i\theta'})^2\right] \tag{24}$$

and $\theta' = \theta + \pi\omega/\Omega$. The probability distribution $P(x)$ for the current delivered by the homodyne detector in port c is given by

$$P(x) = \frac{1}{2\pi^{1/2}}\left(\exp\{-[x - |\alpha|\cos(\theta+\phi)]^2\} + \exp\{-[x + |\alpha|\cos(\theta+\phi)]^2\} + 2^{1/2}\exp[-x^2 - |\alpha|^2\cos^2(\theta+\phi)]\sin[2x|\alpha|\sin(\theta+\phi)] + \text{terms of order } e^{-|\alpha|^2}\right) , \tag{25}$$

where ϕ is defined by $\alpha = |\alpha|e^{i\phi}$. When the pump light is macroscopic ($|\alpha|$ large), the terms of order $e^{-|\alpha|^2}$ can be neglected. The third term on the right-hand side of Eq. (25) gives rise to interference fringes in the probability distribution which can be maximized by choosing the local oscillator phase θ such that $\sin(\theta+\phi)=1$; then

$$P(x) = \frac{1}{2\pi^{1/2}}e^{-x^2}[2 + 2^{1/2}\sin(2x|\alpha|)] . \tag{26}$$

These fringes arise from the interference of $|S\rangle$ with $|P\rangle$ in the coherent superposition Eq. (17). For a statistical mixture of $|S\rangle$ and $|P\rangle$ passing through a beam splitter, the third term of Eq. (25) will not be present and the interference fringes of Eq. (26) will not arise. The combination of beam splitter and homodyne detector thus provides a means by which the macroscopic superposition Eq. (17) can be distinguished from a statistical mixture.

It has been pointed out[10,11,20,21] that such states are extremely fragile. The interference fringes can be washed out with the absorption, on average, of a single photon. This places severe demands on the amount of loss that can be tolerated. However, in view of the fact that four-wave mixers and homodyne detectors are likely to undergo considerable development in an effort to generate and detect light with large amounts of squeezing, the experiments proposed here may become feasible. It has come to our attention that Mecozzi and Tombesi[22] have investigated the generation of macroscopic superpositions of coherent states via nonlinear birefringence modeled by a Hamiltonian similar to Eq. (5).

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