PHYSICAL REVIEW A VOLUME 35, NUMBER 11 JUNE 1, 1987

Finite-size scaling for a relativistic Bose gas with pair production

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(Received 6 February 1987)

Scaling hypothesis on the "singular" part of the free-energy density of a finite system is examined in the context of a Bose gas confined to an enclosure of size $L^{d-d'} \times \infty^{d'}$, with $2<d<4$ and $d' \leq 2$, under periodic boundary conditions. Finite-size effects in the various thermodynamic properties of the system, such as the specific heat, the isothermal compressibility, and the condensate density, are predicted in the regions of *both* first-order ($T<T_c$) and second-order ($T \simeq T_c$) phase transition. To test these predictions, a detailed analytical study is carried out in the case of an ideal relativistic Bose gas, which includes the possibility of particle-antiparticle pair production in the system. The various predictions of the scaling hypothesis are fully borne out and the scaling functions governing the critical behavior of the system are found to be universal—irrespective of the severity of the relativistic effects. The influence of the latter enters only through nonuniversal parameters, $\tilde{C}_1 \tilde{t}$ and $\tilde{C}_2$, which depend on the particle mass m and density ρ as well.

I. INTRODUCTION

In a recent paper[1] (hereafter referred to as I) we carried out an analytical study of the various thermodynamic properties of an ideal relativistic Bose gas confined to a cuboidal enclosure, of size $L_1 \times L_2 \times L_3$, under periodic boundary conditions. Taking into account the possibility of particle-antiparticle pair production in the system, we derived explicit expressions for the free energy, the specific heat, and the condensate density of the system at temperatures close to the bulk critical temperature T_c, and examined the special cases of a cube, a square channel, and a film at some length. The most important aspect of that investigation was the comparison of the "singular" parts of the various quantities pertaining to the system as obtained analytically with the predictions following from the finite-size scaling hypothesis of Privman and Fisher[2] near $T=T_c$. In each case the predictions of the hypothesis were fully borne out and, irrespective of the severity of the relativistic effects, the scaling functions governing the critical behavior of the system turned out to be universal.

In the present paper we wish to report the results of a more detailed investigation which extends our previous analysis in several directions. First of all, we employ a generalized geometry, viz., $L^{d-d'} \times \infty^{d'}$ (with $2<d<4$ and $d' \leq 2$), which covers all those cases in which (i) hyperscaling ($d\nu = 2-\alpha$) holds and (ii) the physical properties of the system do not possess a mathematical singularity at any *finite* temperature T. We do, of course, encounter a singularity at $T=0$ K which is characteristic of a d'-dimensional bulk system; at the same time, as $L \to \infty$, a singularity does indeed appear at $T=T_c$ which, quite expectedly, is characteristic of a d-dimensional bulk system. Effecting a crossover between these two situations, we are able to investigate the region of first-order phase transition ($T<T_c$) as well as the region of second-order phase transition ($T \simeq T_c$). Accordingly, the analytical results obtained here are compared with a generalized version[3,4] of the Privman-Fisher hypothesis which covers a broad range of temperatures—from $T \gtrsim T_c$ down to $T=0$ K. This generalized version applies equally well to all O(n) models[5] with $n \geq 2$, with the result that our predictions are of relevance to an interacting Bose gas ($n=2$), with obvious ramification toward superfluid He^4, as well as to an ideal Bose gas ($n=\infty$) which bears a close resemblance to the spherical model of ferromagnetism. Finally, we have included here a study of the isothermal compressibility, as well as the free energy, the specific heat, and the condensate density, of the system; not surprisingly, the critical behavior of the isothermal compressibility shows strong parallels with the corresponding behavior of the zero-field susceptibility of a magnetic system, though there are significant differences in regard to their amplitudes.

In Sec. II we introduce the finite-size scaling hypothesis for the system under study and determine the nonuniversal parameters of the problem from a knowledge of the bulk behavior of the system. The stage is then set for making predictions about the asymptotic behavior, and the associated amplitudes, of the scaling functions governing the various properties of the given finite-sized system; this is done in Sec. III, covering the region of first-order, as well as second-order, phase transition. In Sec. IV we carry out an explicit, analytical evaluation of the free-energy density of the ideal relativistic Bose gas, confined to geometry $L^{d-d'} \times \infty^{d'}$ and subjected to periodic boundary conditions, and derive appropriate expressions for other quantities as well. In Sec. V we compare the behavior of the singular parts of the various quantities of interest with the predictions made in Sec. III and find complete vindication of the scaling hypothesis of Sec. II on which those predictions were based. Section VI of the paper is devoted to a detailed study of the condensate density in the system which, too, is seen to conform to the dictates of the scaling hypothesis. Our analysis thus confirms once again that, with pair production included, the ideal *relativistic* Bose gas belongs to the same universality class as the *nonrelativistic* one. This point has indeed been appreciated previously, but only for *bulk* systems;[6] the present study shows that this is true for *finite* systems as well. Finally, in Sec. VII, we make some concluding remarks on this work and indicate certain directions along which further investigation of this problem may be carried out.

II. FORMULATION OF THE PROBLEM

In accordance with our previous work[3-5] on the lattice models of ferromagnetism with O(n) symmetry ($n \geq 2$), we propose that the singular part of the free-energy density of a system of bosons confined to geometry $L^{d-d'} \times \infty^{d'}$ ($2<d<4, d' \leq 2$) and subjected to periodic boundary conditions may be expressed in the form

$$f^{(s)}(T,\nu_0;L) \approx TL^{-d} Y(x_1,x_2) , \tag{1}$$

where

$$x_1 = \tilde{C}_1 L^{1/\nu}\tilde{t}, \quad x_2 = \tilde{C}_2 L^{\Delta/\nu}\nu_0/T ; \tag{2}$$

here, ν_0 stands for the "Bose field"[7] conjugate to the "order parameter" M_0 ($\equiv \rho_0^{1/2}$, ρ_0 being the condensate density in the system), $\tilde{t}$ is the generalized temperature variable, while other quantities have their usual meanings. In particular, x_1 and x_2 are the *scaled variables* of the finite system, while $\tilde{C}_1$ and $\tilde{C}_2$ are certain *nonuniversal*, system-dependent scale factors whose precise form can be determined from a knowledge of the thermodynamic behavior of the corresponding bulk system. The function $Y(x_1,x_2)$ is then a *universal* function, common to all systems in the same universality class as the given system.

According to (1), the singular part of the zero-field specific heat per unit volume of the system will be given by

$$\begin{aligned} c^{(s)}(T,0;L) &= -T(\partial^2 f^{(s)}/\partial T^2)_{\rho,\,\nu_0=0} \\ &\simeq -TL^{2/\nu}[\partial(\tilde{C}_1\tilde{t})/\partial T]_\rho^2 (\partial^2 f^{(s)}/\partial x_1^2)_{x_2=0} \\ &\approx [T\partial(\tilde{C}_1\tilde{t})/\partial T]_\rho^2 L^{\alpha/\nu} Y_{(1)}(x_1) , \end{aligned} \tag{3}$$

where $Y_{(1)}(x_1) = -(\partial^2 Y/\partial x_1^2)_{x_2=0}$; the isothermal compressibility, on the other hand, will be given by, see Appendix A,

$$\begin{aligned} \kappa(T,0;L) &= [\rho^2(\partial^2 f^{(s)}/\partial\rho^2)_T]_{\nu_0=0}^{-1} \\ &\approx T^{-1}[\rho\partial(\tilde{C}_1\tilde{t})/\partial\rho]_T^{-2} L^{-\alpha/\nu} Y_{(2)}(x_1) , \end{aligned} \tag{4}$$

where $Y_{(2)}(x_1) = -1/Y_{(1)}(x_1)$. In view of the straightforward relationship between the scaling functions $Y_{(1)}$ and $Y_{(2)}$, we conclude that, in *all* temperature regimes, the L dependence of the isothermal compressibility of the system will be simply the reciprocal of the L dependence of the specific-heat density. In fact, quite generally, the product

$$c^{(s)}\kappa = -T[\rho^{-1}(\partial\rho/\partial T)_{\tilde{C}_1\tilde{t}}]^2 , \tag{5}$$

which, in view of its independence of L, may even be obtained directly from the bulk. For simplicity, therefore, we may in the sequel concentrate on only one of these quantities, say $c^{(s)}$, and refer to the other only when need arises. As regards the condensate density in the system, one can argue that

$$\rho_0(T,0;L) \approx \tilde{C}_2^2 L^{-2\beta/\nu} P(x_1) , \tag{6}$$

where $P(x_1)$ is the corresponding scaling function.

For making predictions on the basis of the scaling hypothesis, we must know the form of the nonuniversal scale factors $\tilde{C}_1$ and $\tilde{C}_2$. For this we follow the procedure laid down in Ref. 4 which makes use of the *bulk* correlation function of the corresponding d-dimensional system, namely[8]

$$G(\mathbf{R},T;\infty) = \rho_0(T) + A(T)/R^{d-2} \quad (T<T_c) , \tag{7}$$

where $A(T)$ is another system-dependent coefficient. We thus obtain the following relationships:

$$\begin{aligned} \tilde{C}_1 |\tilde{t}| &= a_1(\rho_0/A)^{1/(d-2)\nu} , \\ \tilde{C}_2 &= a_2\rho_0^{1/2}(A/\rho_0)^{\beta/(d-2)\nu} , \end{aligned} \tag{8}$$

where a_1 and a_2 are universal. Now, in view of the fact that the quantity $A(T)\Upsilon(T)/T\rho_0(T)$, where $\Upsilon(T)$ is the "helicity modulus" of the system (and in the case of a Bose system is directly proportional to the "superfluid density" ρ_s), is also universal,[8] say,

$$A(T)\Upsilon(T)/T\rho_0(T) = a_0 , \tag{9}$$

relations (8) may be written in an alternative form, viz.,

$$\begin{aligned} \tilde{C}_1 |\tilde{t}| &= a_1(\Upsilon/a_0T)^{1/(d-2)\nu} , \\ \tilde{C}_2 &= a_2\rho_0^{1/2}(a_0T/\Upsilon)^{\beta/(d-2)\nu} . \end{aligned} \tag{10}$$

It will be noted that, as $T \to T_{c-}$, $\rho_0(T) \sim |t|^{2\beta}$ while $\Upsilon(t) \sim |t|^{2\beta-\nu\eta} \sim |t|^{(d-2)\nu}$, t being the conventional temperature variable $(T-T_c)/T_c$. In that case

$$\tilde{C}_1 |\tilde{t}| \to C_1 |t| , \quad \tilde{C}_2 \to C_2 , \tag{11}$$

where C_1 and C_2 are the *temperature-independent* nonuniversal parameters pertaining to the original hypothesis[2] for $T \simeq T_c$; our generalized hypothesis thus reduces to the Privman-Fisher form as the region of second-order phase transition is approached. On the other hand, as $T \to 0$, the quantities $\rho_0(T)$ and $\Upsilon(T)$ tend to become constant, with the result that

$$\tilde{C}_1 |\tilde{t}| \sim T^{-1/(d-2)\nu}, \quad \tilde{C}_2 \sim T^{\beta/(d-2)\nu} , \tag{12}$$

in perfect agreement with the recent proposal of Shapiro;[9] see also Ref. 5. Equations (12) are a clear signal of the singularity lurking at $T=0$ K.

The results stated so far are quite general. In the special case of an *ideal* Bose gas, the relevant *bulk* results are known to be[6,10,11]

$$\rho_0(T)=\rho_s(T)=m\Upsilon(T)=\rho\left[1-\frac{W_d(\beta,m)}{W_d(\beta_c,m)}\right],$$
$$A(T)=\frac{\Gamma[(d-2)/2]}{2\pi^{d/2}}\frac{m}{\beta}, \tag{13}$$

so that

$$a_0=\Gamma[(d-2)/2]/(2\pi^{d/2}); \tag{14}$$

here,

$$W_d(\beta,m)=2^{(d+1)/2}\pi^{-1/2}\Gamma\left[\frac{d}{2}\right]\times\sum_{j=1}^{\infty}\frac{\sinh(j\beta m)}{(j\beta m)^{(d-1)/2}}K_{(d+1)/2}(j\beta m), \tag{15}$$

$K_\nu(z)$ being the modified Bessel functions, while

$$W_d(\beta_c,m)=2^{d-1}\pi^{d/2}\Gamma(d/2)(\rho/m^d), \tag{16}$$

which determines the *bulk critical point* $\beta_c(m,\rho)$. Here, ρ denotes the particle (or "charge") density in the system while other symbols have their usual meanings—in particular, the symbol β in Eqs. (13) and (15) stands for the familiar $1/T$ (and should not be confused with the critical exponent β); further note that the units employed here are such that $\hbar=c=k_B=1$. It follows that, for the ideal Bose gas [for which the critical exponents β and ν are $\frac{1}{2}$ and $1/(d-2)$, respectively], the desired scale factors may be written as

$$\tilde{C}_1\tilde{t}=m^{d-1}\beta[W_d(\beta,m)-W_d(\beta_c,m)],$$
$$\tilde{C}_2=(m/\beta)^{1/2}; \tag{17}$$

referring to Eqs. (8), we notice that the universal numbers a_1 and a_2 have been chosen to be

$$a_1=2^{d-1}\pi^{d/2}\Gamma(d/2)a_0, a_2=a_0^{-1/2}. \tag{18}$$

With this choice of a_1 and a_2, our parameters $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$, in the limit $\beta\to\beta_c$, reduce precisely to the parameters C_1t and C_2 adopted in I, i.e.,

$$\tilde{C}_1\tilde{t}\to m^{d-1}\beta_c\left[\frac{dW_d}{d\beta}\right]_c(\beta-\beta_c)=C_1t,$$
$$\tilde{C}_2\to(m/\beta_c)^{1/2}=C_2; \tag{19}$$

see Eqs. (2) and (28) of I. On the other hand, as $\beta\to\infty$, the function $W_d(\beta,m)$ vanishes as $\beta^{-d/2}$ while $W_d(\beta_c,m)$ stays constant, see (16), with the result that

$$\tilde{C}_1|\tilde{t}|\to2^{d-1}\pi^{d/2}\Gamma(d/2)(\rho\beta/m)\sim T^{-1}, \tag{20}$$

in perfect agreement with (12). In passing, we note that Eq. (5) in the present case assumes the explicit form

$$c^{(s)}\kappa=-\beta[W_d(\beta,m)-W_d(\beta_c,m)+\beta dW_d(\beta,m)/d\beta]^2/W_d^2(\beta_c,m), \tag{21}$$

with the limiting results

$$c^{(s)}\kappa\simeq\begin{cases}-\beta\ \ (\beta\to\infty) & \text{(22a)}\\ -\beta_c^3[dW_d(\beta,m)/d\beta]_c^2/W_d^2(\beta_c,m)\ \ (\beta\simeq\beta_c). & \text{(22b)}\end{cases}$$

We are now in a position to make predictions about the various physical quantities and the various mathematical functions involved.

III. PREDICTION OF FINITE-SIZE EFFECTS

For making predictions we go back to the general system of bosons and examine various regimes of T and L one by one.

(a) For $T\gtrsim T_c$ and $L\to\infty$, we expect our hypothesis to reproduce the standard bulk result for $c^{(s)}$, viz.,

$$c^{(s)}\approx-E_+t^{-\alpha}\ \ (T\gtrsim T_c, L\to\infty), \tag{23}$$

where E_+ is nonuniversal. To recover (23) from (3), we require that, as $x_1\to+\infty$, the scaling function $Y_{(1)}(x_1)$ assumes the asymptotic form

$$Y_{(1)}(x_1)\approx-Y_+x_1^{-\alpha}\ \ (x_1\to+\infty), \tag{24}$$

with Y_+ universal and such that

$$E_+=Y_+C_1^{2-\alpha}. \tag{25}$$

As regards condensate density, we observe that in this regime the total number of particles in the ground state will be $O(1)$, with the result that, for a hypercube of volume L^d, the quantity $\rho_0(T;L)$ will be $O(L^{-d})$. This requires that

$$P(x_1)\approx P_+x_1^{-\gamma}\ \ (x_1\to+\infty), \tag{26}$$

with P_+ universal; accordingly,

$$\rho_0(T;L)\approx P_+C_1^{-\gamma}C_2^2t^{-\gamma}L^{-d}\ \ (T\gtrsim T_c,\ L\to\infty). \tag{27}$$

(b) For $T<T_c$ and $L\to\infty$, our scale factor $x_1\to-\infty$. Since this is equivalent to keeping L finite and letting $T\to0$, see (12), we now require that

$$Y_{(1)}(x_1)\approx-Y_-|x_1|^{\dot{\alpha}(d-2)\nu-2}\ \ (x_1\to-\infty), \tag{28}$$

so that, for $T\to0$, $c^{(s)}\sim T^{-\dot{\alpha}}$ where $\dot{\alpha}$ is the exponent governing the critical behavior of the specific heat of a d'-dimensional bulk system as $T\to T_c(d')=0$. It follows that

$$c^{(s)}\approx-\frac{Y_-}{[(d-2)\nu]^2}\left[\frac{T^2}{\Upsilon}\frac{\partial}{\partial T}\left[\frac{\Upsilon}{T}\right]\right]^2\times\left[\frac{a_1^{(d-2)\nu}\Upsilon L^{d-2}}{a_0T}\right]^{\dot{\alpha}}L^{-d}\quad (T<T_c,\ L\to\infty). \tag{29}$$

It seems worthwhile to point out here that the *approach exponent,* $(d-2)\dot{\alpha}-d$, appearing here, though dependent on d, is totally independent of the critical exponents pertaining to the d-dimensional bulk system. Substituting $\dot{\alpha}=-d'/(2-d')$, see Appendix B, we obtain (for $d'<2$)

$$c^{(s)}\sim-\left[\frac{T^2}{\Upsilon}\frac{\partial}{\partial T}\left[\frac{\Upsilon}{T}\right]\right]^2\left[\frac{T}{\Upsilon}\right]^{d'/(2-d')}\times L^{-2(d-d')/(2-d')}(T<T_c,\ L\to\infty). \tag{30}$$

At this point we note that for $T \lesssim T_c$, where the quantity (Υ/T) behaves as $[C_1|t|]^{(d-2)\nu}$, Eq. (30) takes the form

$$c^{(s)} \sim -|t|^{-2}[(C_1|t|)^{-d'(d-2)\nu}L^{-2(d-d')}]^{1/(2-d')} \tag{31}$$

which may be rewritten as

$$c^{(s)} \sim -|t|^{-\alpha}[C_1^{-d'(d-2)\nu}(|t|^{\nu}L)^{-2(d-d')}]^{1/(2-d')} . \tag{32}$$

It is remarkable that, as the region of second-order phase transition is approached, the critical exponents pertaining to the d-dimensional bulk system do indeed show up.

In passing, we observe that, in the special case $d'=2$, the approach of the system toward standard bulk behavior is expected to be exponential rather than through a power law; cf. Eq. (15) of Appendix B.

As regards condensate density, we first of all expect to recover the bulk result, viz.,

$$\rho_0 \approx B^2|t|^{2\beta} \quad (T \lesssim T_c,\ L \to \infty) , \tag{33}$$

with B^2 nonuniversal. This requires that, for $x_1 \to -\infty$, the scaling function $P(x_1)$ of Eq. (6) be of the form

$$P(x_1) \approx P_-|x_1|^{2\beta} \quad (x_1 \to -\infty) , \tag{34}$$

with P_- universal. It follows that, for *all* $T<T_c$,

$$\rho_0(T;\infty) = P_-\tilde{C}_1^{2\beta}\tilde{C}_2^2|\tilde{t}|^{2\beta} \quad (T<T_c) . \tag{35}$$

Making use of Eqs. (8), we find that Eq. (35) is identically satisfied provided that

$$P_- = 1/(a_1^{2\beta}a_2^2) . \tag{36}$$

As $T \to T_{c-}$, Eq. (35) indeed goes over to (33), with

$$B^2 = P_-C_1^{2\beta}C_2^2 ; \tag{37}$$

accordingly, the quantity

$$\frac{B^2}{C_1^{2\beta}C_2^2} = P_- = \frac{1}{a_1^{2\beta}a_2^2} \tag{38}$$

is universal.

For studying finite-size effects in ρ_0, we must supplement expression (34) with the next leading term in the asymptotic expansion of the function $P(x_1)$ as $x_1 \to -\infty$. Assuming that, to the next approximation,

$$P(x_1) \approx P_-|x_1|^{2\beta} + Q_-|x_1|^{\psi} \quad (x_1 \to -\infty) , \tag{39}$$

we readily obtain

$$\rho_0(T;L) \approx \rho_0(T;\infty) + Q_-[\tilde{C}_1|\tilde{t}|]^{\psi}\tilde{C}_2^2L^{-(2\beta-\psi)/\nu} . \tag{40}$$

Substituting from Eqs. (10), we obtain for the *fractional* finite-size effect in ρ_0

$$\frac{\rho_0(T;L)-\rho_0(T;\infty)}{\rho_0(T;\infty)} \approx Q_-a_1^{\psi}a_2^2\left[\frac{\Upsilon L^{d-2}}{a_0T}\right]^{-(2\beta-\psi)/(d-2)\nu} . \tag{41}$$

As $T \to T_{c-}$, this effect becomes $\sim(|t|L^{1/\nu})^{-(2\beta-\psi)}$; on the other hand, as $T \to 0$, it becomes $\sim(T/L^{d-2})^{(2\beta-\psi)/(d-2)\nu}$. In the latter case, however, we do not expect our *approach exponents* to be dependent on the critical exponents pertaining to the d-dimensional bulk system;[5,9] we, therefore, conjecture that[12]

$$\psi = \nu\eta , \tag{42}$$

so that $(2\beta-\psi)/(d-2)\nu=1$ for *all* n. Equation (41) then takes the form

$$\frac{\rho_0(T;L)-\rho_0(T;\infty)}{\rho_0(T;\infty)} \approx Q_-a_1^{\nu\eta}a_2^2\left[\frac{a_0T}{\Upsilon L^{d-2}}\right] \quad (T<T_c,\ L \to \infty) . \tag{43}$$

The fact that the parameter d' does not appear in this result is not surprising because, so long as $d' \le 2$, this calculation is relevant only for the case $d'=0$; see Sec. VI.

(c) Finally, in the "core" region, where $|x_1|=O(1)$ and hence $|t|=O(L^{-1/\nu})$, the functions $f^{(s)}$, $c^{(s)}$, and ρ_0, for a fixed value of x_1, are proportional to L^{-d}, $L^{\alpha/\nu}$, and $L^{-2\beta/\nu}$, respectively; see Eqs. (1), (3), and (6). Accordingly, the quantities

$$\begin{aligned} &f^{(s)}(T_c;L)L^dT_c^{-1} , \\ &c^{(s)}(T_c;L)L^{-\alpha/\nu}C_1^{-2} , \\ &\rho_0(T_c;L)L^{2\beta/\nu}C_2^{-2} , \end{aligned} \tag{44}$$

evaluated at the erstwhile critical point $T=T_c$, must be universal. This completes the set of predictions, following from hypothesis (1), which will now be tested for the special case of an ideal Bose gas.

IV. THERMODYNAMICS OF AN IDEAL RELATIVISTIC BOSE GAS WITH PAIR PRODUCTION

We consider an ideal Bose gas composed of N_1 particles and N_2 antiparticles, each of mass m, confined to a d-dimensional enclosure of sides L_i $(i=1,\ldots,d)$. Since particles and antiparticles are supposed to be created in pairs, the system is governed by the conservation of the number Q $(=N_1-N_2)$, rather than of the numbers N_1 and N_2 separately; the conserved quantity Q may be looked upon as a kind of generalized "charge." In equilibrium the chemical potentials of the two species will be equal and opposite: $\mu_1=-\mu_2=\mu$, say, with the result that[13]

$$N_1=\sum_{\varepsilon}(e^{\beta(\varepsilon-\mu)}-1)^{-1},\quad N_2=\sum_{\varepsilon}(e^{\beta(\varepsilon+\mu)}-1)^{-1} , \tag{45}$$

where $\varepsilon=(k^2+m^2)^{1/2}$. Both ε and μ here include the rest energy m of the particle (or the antiparticle) and, for the mean occupation numbers in the various states to be positive definite, we must have $|\mu| \le m$. Assuming that, to begin with, $\mu>0$, it readily follows that $N_1>N_2$ and hence $Q>0$. In view of the conservation of Q, μ then stays positive under all circumstances. Without loss of generality, we shall assume this to be the case.

Under *periodic* boundary conditions, the eigenvalues k_i of the wave vector $\mathbf{k}$ are given by

$$k_i=(2\pi/L_i)n_i \quad (n_i=0,\pm1,\pm2,\dots)\ ; \tag{46}$$

the pressure $\mathscr{P}$ in the grand canonical ensemble may then be written as

$$\begin{aligned}\mathscr{P}&=-\frac{1}{\beta V}\sum_{\varepsilon_n}[\ln(1-e^{-\beta(\varepsilon_n-\mu)})+\ln(1-e^{-\beta(\varepsilon_n+\mu)})]\\&=\frac{2}{\beta\prod_i L_i}\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{j}\sum_{\{n_i\}}\exp\left[-j\beta m\left[1+\frac{4\pi^2}{m^2}\sum_i\frac{n_i^2}{L_i^2}\right]^{1/2}\right].\end{aligned} \tag{47}$$

Employing techniques developed in earlier papers,[1,14,15] we obtain (correct to *all* powers of the parameters λ/L_i where λ denotes the mean thermal wavelength $\sqrt{2\pi\beta/m}$, or the Compton wavelength $1/m$, of the particles)

$$\mathscr{P}(\beta,\mu;L)=\mathscr{P}_B(\beta,\mu)+\frac{2}{\pi^{d/2}\beta}\left[\frac{y}{L}\right]^d\mathscr{K}\left[\frac{d}{2}\middle|d^*;y\right], \tag{48}$$

where $\mathscr{P}_B(\beta,\mu)$ is the standard bulk expression[6] for $\mathscr{P}$,

$$\mathscr{P}_B(\beta,\mu)=\frac{2^{(3-d)/2}m^{d+1}}{\pi^{(d+1)/2}}\times\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{(j\beta m)^{(d+1)/2}}K_{(d+1)/2}(j\beta m)\ , \tag{49}$$

y is the *thermogeometric parameter* of the system,

$$y=\tfrac{1}{2}(m^2-\mu^2)^{1/2}L\ , \tag{50}$$

while

$$\mathscr{K}(\nu\,|\,d^*;y)=\sum_{\mathbf{q}(d^*)}{}'\frac{K_\nu(2yq)}{(yq)^\nu}\qquad[q=(q_1^2+\cdots+q_{d^*}^2)^{1/2}>0]\ ; \tag{51}$$

it will be noted that, for simplicity, we have adopted the geometry $L^{d^*}\times\infty^{d'}$, with $d^*+d'=d$. Using the thermodynamic relation $\rho=(\partial\mathscr{P}/\partial\mu)_T$, see Eq. (6) of Appendix A, we obtain for the charge density in the system

$$\rho\equiv\frac{Q}{V}=\rho_B(\beta,\mu)+\frac{\mu}{\pi^{d/2}\beta}\left[\frac{y}{L}\right]^{d-2}\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y\right], \tag{52}$$

where $\rho_B(\beta,\mu)$ is the corresponding bulk expression,[6] viz.,

$$\rho_B(\beta,\mu)=\frac{2^{(3-d)/2}m^d}{\pi^{(d+1)/2}}\times\sum_{j=1}^{\infty}\frac{\sinh(j\beta\mu)}{(j\beta m)^{(d-1)/2}}K_{(d+1)/2}(j\beta m)\ . \tag{53}$$

For a given value of ρ, Eq. (52) determines μ as a function of β and L; Eq. (48) then gives $\mathscr{P}(\beta,L)$.

In the region of phase transition ($\mu\simeq m$), Eq. (52), for $2<d<4$, takes the form

$$\rho=\rho_B(\beta,m)-\frac{m}{2^{d-1}\pi^{d/2}\beta}\left|\Gamma\left[\frac{2-d}{2}\right]\right|(m^2-\mu^2)^{(d-2)/2}+\frac{m}{\pi^{d/2}\beta}\left[\frac{y}{L}\right]^{d-2}\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y\right]+O(m^2-\mu^2)^1\ , \tag{54}$$

the *bulk* critical point β_c being given by the conditions $L\to\infty$ and $\mu\to m$, i.e., by

$$\rho=\rho_B(\beta_c,m)\ . \tag{55}$$

At the same time, Eq. (48) takes the form

$$\begin{aligned}\mathscr{P}(\beta,\mu;L)=\mathscr{P}_B(\beta,m)&+\rho_B(\beta,m)(\mu-m)\\&+\frac{1}{2^{d-1}\pi^{d/2}d\beta}\left|\Gamma\left[\frac{2-d}{2}\right]\right|(m^2-\mu^2)^{d/2}\\&+\frac{2}{\pi^{d/2}\beta}\left[\frac{y}{L}\right]^d\mathscr{K}\left[\frac{d}{2}\middle|d^*;y\right]\\&+O(m^2-\mu^2)^2\ .\end{aligned} \tag{56}$$

The *thermal* free energy density of the system is then given by

$$f\equiv\frac{F-mQ}{V}=\frac{(\mu Q-\mathscr{P}V)-mQ}{V}=(\mu-m)\rho-\mathscr{P} \tag{57}$$

from which the singular part $f^{(s)}$ can be readily extracted. Using Eqs. (54), (56), and (57), we find that

$$f^{(s)}(\beta,\mu;L)=\frac{2}{\pi^{d/2}\beta}\left[\frac{y}{L}\right]^d\left[\frac{1}{d}\Gamma\left[\frac{4-d}{2}\right]-\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y\right]-\mathscr{K}\left[\frac{d}{2}\middle|d^*;y\right]\right]. \tag{58}$$

The corresponding expressions for the specific heat density and the isothermal compressibility of the system turn out to be

$$c^{(s)}(\beta,\mu;L)=-\beta^2[\partial^2(\beta f^{(s)})/\partial\beta^2]_\rho$$
$$=-\frac{4\pi^{d/2}\beta^2[\rho_B(\beta,m)-\rho+\beta\partial\rho_B(\beta,m)/\partial\beta]^2(y/L)^{4-d}}{m^2\left[\Gamma\left[\frac{4-d}{2}\right]+2\mathcal{K}\left[\frac{d-4}{2}\middle|d^*;y\right]\right]} \tag{59}$$

and

$$\kappa(\beta,\mu;L)=\rho^{-2}(\partial\rho/\partial\mu)_\beta$$
$$=\frac{m^2\left[\Gamma\left[\frac{4-d}{2}\right]+2\mathcal{K}\left[\frac{d-4}{2}\middle|d^*;y\right]\right]}{4\pi^{d/2}\beta\rho^2(y/L)^{4-d}}\,. \tag{60}$$

We note that, as observed in Sec. II, the product

$$c^{(s)}\kappa=-\beta[\rho_B(\beta,m)-\rho+\beta\partial\rho_B(\beta,m)/\partial\beta]^2/\rho^2 \tag{61}$$

is quite independent of L. Furthermore, it agrees with expression (21), for

$$\frac{\rho_B(\beta,m)}{W_d(\beta,m)}=\frac{\rho}{W_d(\beta_c,m)}=\frac{m^d}{2^{d-1}\pi^{d/2}\Gamma(d/2)}\;; \tag{62}$$

see Eqs. (15), (16), (53), and (55).

We are now in a position to compare our analytical results for the various quantities of interest with the predictions made in Sec. III.

V. VERIFICATION OF THE SCALING PREDICTIONS

To bring the results of Sec. IV into a scaled form, we start with the observation that the scaled variable x_1 of this problem may be written as, see Eqs. (17) and (62),

$$x_1\equiv\tilde{C}_1L^{1/\nu}\tilde{t}$$
$$=m^{d-1}\beta[W_d(\beta,m)-W_d(\beta_c,m)]L^{d-2}$$
$$=2^{d-1}\pi^{d/2}\Gamma(d/2)(\beta/m)[\rho_B(\beta,m)-\rho]L^{d-2}\,. \tag{63}$$

Equation (54) then takes the form

$$x_1\approx2^{d-2}\Gamma(d/2)y^{d-2}\times\left[\left|\Gamma\left[\frac{2-d}{2}\right]\right|-2\mathcal{K}\left[\frac{d-2}{2}\middle|d^*;y\right]\right], \tag{64}$$

which determines y as a function of x_1. It is now straightforward to see that Eqs. (58)—(60) indeed conform to the scaled forms (1), (3), and (4), respectively, with $\alpha=(d-4)/(d-2)$,

$$Y(x_1,0)=\frac{2}{\pi^{d/2}}y^d\left[\frac{1}{d}\Gamma\left[\frac{4-d}{2}\right]-\mathcal{K}\left[\frac{d-2}{2}\middle|d^*;y\right]-\mathcal{K}\left[\frac{d}{2}\middle|d^*;y\right]\right] \tag{65}$$

and

$$Y_{(1)}(x_1)=-\frac{y^{4-d}}{4^{d-2}\pi^{d/2}[\Gamma(d/2)]^2\left[\Gamma\left[\frac{4-d}{2}\right]+2\mathcal{K}\left[\frac{d-4}{2}\middle|d^*;y\right]\right]}\,. \tag{66}$$

We may now examine the behavior of the scaling functions Y and $Y_{(1)}$ in the various regimes of interest; however, since these two function are very closely related to one another, recall that $Y_{(1)}(x_1)=-\partial^2Y(x_1)/\partial x_1^2$, it seems sufficient to discuss here only one of them, say $Y_{(1)}$, in full detail.

(a) $T\gtrsim T_c$ and $L\rightarrow\infty$. In this regime $x_1\rightarrow+\infty$, with the result that y diverges while the functions $\mathcal{K}(\nu\,|\,d^*;y)$ vanish exponentially. The constraint equation (64) then gives

$$y\approx\left[\frac{x_1}{2^{d-2}\Gamma\left[\frac{d}{2}\right]\left|\Gamma\left[\frac{2-d}{2}\right]\right|}\right]^{1/(d-2)}\left[1+\frac{d^*\pi^{1/2}}{\Gamma\left[\frac{4-d}{2}\right]}\left[\frac{2\xi(\infty)}{L}\right]^{(d-1)/2}e^{-L/\xi(\infty)}\right], \tag{67}$$

where $\xi(\infty)$ denotes the correlation length of the bulk system.[16] At the same time, Eq. (66) takes the form

$$Y_{(1)}(x_1)\approx-Y_+x_1^{(4-d)/(d-2)}\left[1-\frac{2d^*\pi^{1/2}}{\Gamma\left[\frac{4-d}{2}\right]}\left[\frac{2\xi(\infty)}{L}\right]^{(d-3)/2}e^{-L/\xi(\infty)}\right], \tag{68}$$

with

$$Y_+ = \left\{ 2^{d-1}(d-2)\pi^{d/2} \left[\Gamma\left(\frac{d}{2}\right)\right]^{d/(d-2)} \left| \Gamma\left(\frac{2-d}{2}\right)\right|^{2/(d-2)} \right\}^{-1} . \tag{69}$$

Indeed, as $L \to \infty$, Eq. (68) reduces to the expected result (24). The specific-heat function $c^{(s)}$ may now be determined explicitly from Eqs. (3), (19), and (68),

$$c^{(s)} \approx -Y_+ C_1^{2-\alpha} t^{-\alpha} \left[1 - \frac{2d^* \pi^{1/2}}{\Gamma\left(\frac{4-d}{2}\right)} \left(\frac{2\xi(\infty)}{L}\right)^{(d-3)/2} e^{-L/\xi(\infty)} \right] . \tag{70}$$

Not only does this result agree with the bulk expression (23) and the scaling relation (25) but it also gives us the leading finite-size correction to the bulk expression. As expected under *periodic* boundary conditions, this correction is indeed exponentially small; see also Refs. 3 and 4.

(b) $T < T_c$ and $L \to \infty$. In this regime $x_1 \to -\infty$, with the result that y tends to zero while the functions $\mathscr{K}(\nu \mid d^*;y)$ diverge.[17] Equation (64) now gives

$$y \approx \begin{cases} \left[2^{d-2}\pi^{(d-d')/2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{2-d'}{2}\right) \Big/ |x_1| \right]^{1/(2-d')} & (d'<2) \\ \text{const} \times \exp[-|x_1|/2^{d-1}\pi^{(d-2)/2}\Gamma(d/2)] & (d'=2) , \end{cases} \tag{71}$$

whereupon (66) takes the form

$$Y_{(1)}(x_1) \approx \begin{cases} -Y_- |x_1|^{-(4-d')/(2-d')} \\ -\text{const} \times \exp[-|x_1|/2^{d-2}\pi^{(d-2)/2}\Gamma(d/2)] & (d'=2) , \end{cases} \tag{72}$$

with[18,19]

$$Y_- = \frac{2}{(2-d')} \left[2^{d-2}\pi^{(d-2)/2}\Gamma\left(\frac{d}{2}\right)\right]^{d'/(2-d')} \left[\Gamma\left(\frac{2-d'}{2}\right)\right]^{2/(2-d')} . \tag{73}$$

Equation (72) for $d'<2$ indeed agrees with our scaling prediction (28), with $\dot{\alpha} = -d'/(2-d')$. By implication, Eqs. (29)–(32) are also verified, with the helicity modulus Υ given by Eqs. (13), (15), and (16),

$$\frac{\Upsilon}{T} = \frac{m^{d-1}\beta}{2^{d-1}\pi^{d/2}\Gamma(d/2)}[W_d(\beta_c,m) - W_d(\beta,m)] = \frac{a_0}{a_1}\tilde{C}_1 |\tilde{t}| , \tag{74}$$

while C_1 is given by Eq. (19),

$$C_1 = m^{d-1}\beta_c^2 |dW_d/d\beta|_c . \tag{75}$$

The specific-heat function $c^{(s)}$ may now be written explicitly as

$$c^{(s)} \approx \begin{cases} -Y_- \left(\beta \frac{\partial(\tilde{C}_1|\tilde{t}|)}{\partial\beta}\right)^2 [\tilde{C}_1|\tilde{t}|]^{-(4-d')/(2-d')} L^{-2(d-d')/(2-d')} & (d'<2) \\ -\text{const} \times \left(\beta \frac{\partial(\tilde{C}_1|\tilde{t}|)}{\partial\beta}\right)^2 L^{-(4-d)} \exp\left[-\frac{\tilde{C}_1|\tilde{t}|L^{d-2}}{2^{d-2}\pi^{(d-2)/2}\Gamma(d/2)}\right] & (d'=2) . \end{cases} \tag{76}$$

Equations (76) provide complete dependence of $c^{(s)}$ on *both* L and T for *all* $T < T_c$. For $T \lesssim T_c$, they fully agree with our previous results[1,14] for the special case $d=3$.

While for $d'<2$ our result is complete in every respect, for $d'=2$ an unknown universal factor still remains to be determined. In the case of a three-dimensional film, this factor can be obtained exactly. This is made possible by the fact that, for $d=3$ and $d'=2$, the functions $\mathscr{K}(y)$ appearing in Eqs. (64) and (66) can be expressed in a closed form, viz.,

$$\mathscr{K}\left(\tfrac{1}{2} \mid 1;y\right) = -\frac{\sqrt{\pi}}{y}\ln(1-e^{-2y}) \tag{77}$$

and

$$\mathscr{K}\left(-\tfrac{1}{2} \mid 1;y\right) = \sqrt{\pi}/(e^{2y}-1) ; \tag{78}$$

the scaling function $Y_{(1)}$ is then given by

$$Y_{(1)}(x_1) = -\frac{1}{\pi^3} y \tanh y , \tag{79}$$

where

$$y(x_1) = \sinh^{-1}[\tfrac{1}{2}\exp(x_1/2\pi)] . \tag{80}$$

For $x_1 \to -\infty$,

$$y \approx \tfrac{1}{2}\exp[-|x_1|/2\pi]\ , \tag{81}$$

so that

$$Y_{(1)}(x_1) \approx -\frac{1}{4\pi^3}\exp[-|x_1|/\pi]\ . \tag{82}$$

Comparing this result with Eq. (72), we find that the unknown factor in this case is $1/(4\pi^3)$. Accordingly, for $d=3$ and $d'=2$,

$$c^{(s)} \approx -\frac{1}{4\pi^3}\left[\beta\frac{\partial(\tilde{C}_1|\tilde{t}|)}{\partial\beta}\right]^2 L^{-1}\exp\left[-\frac{1}{\pi}\tilde{C}_1|\tilde{t}|L\right]. \tag{83}$$

As $T\to 0$, our parameter $C_1\tilde{t}$ assumes the limiting form (20); Eqs. (76) then become

$$c^{(s)} \approx -L^{-d}\begin{cases}\dfrac{2}{(2-d')}\left[\Gamma\left[\dfrac{2-d'}{2}\right]\right]^{2/(2-d')}\times\left[\dfrac{mT}{2\pi\rho L^{d-2}}\right]^{d'/(2-d')} & (d'<2)\\[2ex] \text{const}\times\left[\dfrac{\rho L^{d-2}}{mT}\right]^2\exp\left[-\dfrac{2\pi\rho L^{d-2}}{mT}\right] & (d'=2)\ .\end{cases} \tag{84}$$

It is gratifying to note that, in keeping with the argument of Sec. II, the foregoing expressions display the same kind of singularity at $T=0$ as encountered in the case of a d'-dimensional bulk system; cf. Eqs. (9) and (15) of Appendix B. It may be added here that, in the special case $d=3$ and $d'=2$, the unknown factor in (84) is simply π.

(c) Finally, in the core region, where $|x_1|=O(1)$ and hence $|t|=O(L^{-(d-2)})$, the thermogeometric parameter y will be $O(1)$. Accordingly, the functions $f^{(s)}$ and $c^{(s)}$ will be $O(L^{-d})$ and $O(L^{-(4-d)})$, respectively; this agrees with the results predicted by Eqs. (1) and (3). Moreover, at the erstwhile critical point $T=T_c$, the precise value of the parameter y will be a universal number, y_0, as determined by the constraint equation (64), with $x_1=0$, i.e., by

$$\mathscr{K}\left[\frac{d-2}{2}\middle|d^*;y_0\right]=\tfrac{1}{2}\left|\Gamma\left[\frac{2-d}{2}\right]\right|\ ; \tag{85}$$

for $d=3$, y_0 turns out to be 0.970, 0.756, or 0.481, as $d'=0$, 1, or 2, respectively.[20] Consequently, the quantities $f^{(s)}(T_c;L)L^dT_c^{-1}$ and $c^{(s)}(T_c;L)L^{4-d}C_1^{-2}$ will be given by the respective scaling functions Y and $Y_{(1)}$ evaluated at $y=y_0$—clearly, universal numbers.[21]

This completes the verification of the various predictions made for the functions $f^{(s)}$ and $c^{(s)}$ in Sec. III.

VI. SCALING BEHAVIOR OF THE CONDENSATE DENSITY

The condensate Q_0 in the system is given by the expression, see Eqs. (45),

$$Q_0=(N_1)_0-(N_2)_0 = (e^{\beta(m-\mu)}-1)^{-1}-(e^{\beta(m+\mu)}-1)^{-1}\ . \tag{86}$$

In the region of phase transition ($\mu\simeq m$), we may write

$$Q_0 \simeq \frac{1}{\beta(m-\mu)} \simeq \frac{2m}{\beta(m^2-\mu^2)}\ . \tag{87}$$

For the total charge Q, we multiply Eq. (52) with volume $V\,(=L^{d-d'}\times L_{\|}^{d'}$ where $L_{\|}$ will ultimately go to infinity) and, for simplicity of argument, integrate over $\mathbf{q}(d^*)$. Remembering that $\mu\simeq m$, we obtain

$$Q=Q_B(\beta,\mu)+\frac{2^{1-d'}}{\pi^{d'/2}}\Gamma\left[\frac{2-d'}{2}\right]\times\frac{m}{\beta(m^2-\mu^2)^{(2-d')/2}}L_{\|}^{d'}+\cdots\ , \tag{88a}$$

with corrections arising from the difference between the summation over $\mathbf{q}(d^*)$, as required in Eq. (52), and the integration over $\mathbf{q}(d^*)$, as carried out here. It is important to note that, for $d'=0$, the second term in (88a) is precisely Q_0, so that

$$Q=Q_B(\beta,\mu)+Q_0+\cdots\quad(d'=0)\ . \tag{88b}$$

This means that, in the special case of the "block" geometry, the condensation phenomenon is similar to the one in the bulk system—except for finite-size effects which will be examined in the sequel. For $d'>0$, however, we note two things: (i) the second term in (88a) is now precisely $Q_B^{(d')}(\beta,\mu)$ for a d'-dimensional bulk system, see Eq. (6) of Appendix B, and (ii) the term Q_0 is now missing [for it normally arises from an integration over full $\mathbf{q}(d)$, but that provision has already been curtailed by the fact that at an earlier stage of the calculation we found it expedient to set $\mathbf{q}(d')=0$]. We may, therefore, include this component at the present stage of the calculation on purely physical grounds and write

$$Q=Q_B^{(d)}(\beta,\mu)+Q_B^{(d')}(\beta,\mu)+Q_0+\cdots\quad(0<d'<2)\ . \tag{88c}$$

Now, in the case of a block geometry ($d'=0$), as $\mu\to m$, $Q_B^{(d)}(\beta,\mu)$ tends to a *finite* limit, see Eqs. (15) and (53),

$$Q_B^{(d)}(\beta,m)=\frac{m^dL^d}{2^{d-1}\pi^{d/2}\Gamma(d/2)}W_d(\beta,m)\quad(d>2)\ , \tag{89}$$

which, at $\beta=\beta_c$, equals Q. For $\beta>\beta_c$, $Q_B^{(d)}(\beta,m)<Q$, with the result that a macroscopic fraction of the particles falls into the ground state, leading to the condensate

$$Q_0=[Q-Q_B^{(d)}(\beta,m)]+\cdots\quad(d'=0,\ \beta>\beta_c)\ . \tag{90}$$

For $d'>0$, on the other hand, the term $Q_B^{(d')}(\beta,\mu)$, in the same limit, is *unbounded* and does not let Q_0 become $O(Q)$ unless $Q_B^{(d')}(\beta,m)$ itself is also $O(Q)$. The first condition requires that

$$(m^2-\mu^2)=O(m/\beta Q)\ll 1\ , \tag{91}$$

see Eq. (87); the second condition then requires that

$$\frac{m}{\beta(m/\beta Q)^{(2-d')/2}} L_{\parallel}^{d'} = O(Q) , \tag{92}$$

see Eq. (88), which means that

$$(m/\beta Q) = O(L_{\parallel}^{-2}) . \tag{93}$$

In terms of the charge density ρ ($=Q/L^{d-d'}L_{\parallel}^{d'}$), condition (93) takes the form

$$\beta = O\left[\frac{m}{\rho}\frac{L_{\parallel}^{2-d'}}{L^{d-d'}}\right] \quad (0<d'<2) . \tag{94}$$

For $d'=2$, we obtain instead

$$\beta = O\left[\frac{m}{\rho L^{d-2}} \ln\left[\frac{L_{\parallel}}{L}\right]\right] \quad (d'=2) . \tag{95}$$

It follows that, in a system with $0<d'\leq 2$, condensation on a *macroscopic* scale does not develop unless the temperature of the system is low enough to satisfy condition (94) or (95), as the case may be. And, as $L_{\parallel}\to\infty$, this essentially requires that $T\to 0$. Looking back at Eq. (88c), with $Q^{(d')}$ dominating over $Q^{(d)}$, we infer that, insofar as condensation is concerned, our system behaves very much like a d'-dimensional bulk system (for which T_c is indeed at absolute zero). Accordingly, we may henceforth consider only the block geometry for which $d^*=d$. Furthermore, since d^*, being the number of dimensions in which the system is finite, *ought* to be integral and $2<d<4$, the only case meriting discussion here is the one with $d=d^*=3$.

The charge density, ρ_0, in the case of interest is given by

$$\rho_0 \equiv \frac{Q_0}{V} \simeq \frac{2m}{\beta(m^2-\mu^2)L^3} = \frac{m}{2\beta y^2 L} , \tag{96}$$

which indeed conforms to the scaling relation (6), with scaling function

$$P(x_1) = 1/(2y^2) . \tag{97}$$

(a) For $T \gtrsim T_c$ and $L\to\infty$, $y(x_1)$ is given by Eq. (67), i.e.,

$$y \approx x_1/2\pi \quad (x_1\to +\infty) , \tag{98}$$

whence

$$\rho_0 \approx \frac{2\pi^2 m}{\beta_c C_1^2 t^2 L^3} = \frac{2\pi^2}{m^3\beta_c^5 |dW_3/d\beta|_c^2} t^{-2}L^{-3} . \tag{99}$$

This verifies prediction (27), with $P_+=2\pi^2$ and γ [$=2/(d-2)$]$=2$; it also agrees with our recent result for condensate density in an Einstein universe[22] of radius a (and hence of volume $2\pi^2 a^3$).

(b) For $T<T_c$ and $L\to\infty$, $y(x_1)$ is given by Eq. (71), i.e.,

$$y \approx \pi |x_1|^{-1/2} \quad (x_1\to -\infty) , \tag{100}$$

whence

$$\rho_0 \approx \frac{m\tilde{C}_1|\tilde{t}|}{2\pi^2\beta} = \frac{m^3}{2\pi^2}[W_3(\beta_c,m)-W_3(\beta,m)] . \tag{101}$$

Using (16), this takes the standard bulk form[6]

$$\rho_0 = \rho\left[1-\frac{W_3(\beta,m)}{W_3(\beta_c,m)}\right] . \tag{102}$$

To go beyond this result, we need a better approximation for $y(x_1)$ than the one provided by Eq. (71). For this we go back to the constraint equation (64) which in the present case reads

$$x_1 \approx \pi y\left[2-\sum_{\mathbf{q}(3)}{}' \frac{e^{-2yq}}{yq}\right] . \tag{103}$$

Now, as $y\to 0$, the sum appearing here assumes the asymptotic form[11,23,24]

$$\sum_{\mathbf{q}(3)}{}' \frac{e^{-2yq}}{yq} = \frac{\pi}{y^3} + \frac{C_3}{\pi y} + 2 + O(y) , \tag{104}$$

with the result that

$$P(x_1) = \frac{1}{2y^2} \approx \frac{|x_1|}{2\pi^2} - \frac{C_3}{2\pi^2} \quad (x_1\to -\infty) ; \tag{105}$$

here, C_3 is a universal number with value $-8.913\,633\ldots$. Equation (105) verifies prediction (39), with $P_-=1/(2\pi^2)$, $Q_-=|C_3|/(2\pi^2)$; it also conforms to our conjecture (42), with $\psi=0$. The finite-size effect in ρ_0 is then given by

$$\begin{aligned}\frac{\rho_0(T;L)-\rho_0(T;\infty)}{\rho_0(T;\infty)} &\approx \frac{|C_3|}{|x_1|} = \frac{|C_3|}{\tilde{C}_1|\tilde{t}|L} \\ &= \frac{|C_3|m}{2\pi^2\beta\rho_0(T;\infty)L} = \frac{|C_3|}{2\pi^2}\left[\frac{T}{\Upsilon L}\right] ,\end{aligned} \tag{106}$$

in perfect agreement with prediction (43), with $\eta=0$ and $a_2=a_0^{-1/2}$; see (18).

(c) Finally, in the core region, we encounter the quantity $\rho_0(T_c;L)LC_2^{-2}$, see (44), which turns out to be $1/(2y_0^2)$—clearly, a universal number.

VII. CONCLUDING REMARKS

We have shown analytically that the various predictions of the finite-size scaling hypothesis on the hyperuniversality of finite systems are fully borne out in the case of an ideal relativistic Bose gas confined to geometry $L^{d-d'}\times\infty^{d'}$ ($2<d<4$, $d'\leq 2$) and subjected to periodic boundary conditions. With pair production included, the scaling functions governing the behavior of the system, for $T<T_c$ as well as $T\simeq T_c$, are found to be universal —irrespective of the severity of the relativistic effects. The influence of the latter enters only through the nonuniversal parameters $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$ which depend on the particle mass m and the charge density ρ as well and for all T (from $T\simeq T_c$ down to $T=0$) are determined by the quantities $\rho_0(T)$ and $A(T)$ appearing in the *bulk* correlation function of the system. Once these parameters are known, no more nonuniversal amplitudes are needed to describe the behavior of the system, regardless of whether it is finite or infinite in extent. It is remarkable that the

approach exponents governing the region of first-order phase transition ($T<T_c$), while dependent on the total dimensionality d of the system, bear no relation to the critical exponents pertaining to the d-dimensional bulk system near $T=T_c$; they are, on the other hand, intimately related to the critical exponents pertaining to a d'-dimensional bulk system near $T=0$.

The present investigation suggests several directions in which further work on this problem may be carried out. While one readily thinks of extensions to higher dimensions ($d\geq 4$) and to other boundary conditions (antiperiodic, Dirichlet, Neumann), a study of Bose condensation in curved spaces is also a matter of some interest.[15,25] On the more practical side, our results for the case $n=2$, which pertain to the XY-model and therefore have a direct bearing on the problem of superfluidity in liquid He^4 confined to restricted geometries, indicate an obvious line to pursue. Such a pursuit will indeed require information on the bulk properties of the system, such as the condensate density $\rho_0(T)$ and the superfluid density $\rho_s(T)$, so that one may be able to construct the relevant parameters $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$ and make predictions about the *singular* parts of the various quantities, such as the specific heat, of the finite-sized system. Combining this with information on the *regular* parts of these quantities, which again belongs to the domain of the bulk system, one may then predict, for instance, the behavior of the total specific heat of the system and hopefully compare it with experiment. Work along these lines is currently in progress.

ACKNOWLEDGMENT

Financial support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

APPENDIX A

In this appendix we shall derive a suitable expression for the isothermal compressibility of the *two-component* system of N_1 particles and N_2 antiparticles. We start with the Gibbs free energy of the system, viz.,

$$G=N_1\mu_1+N_2\mu_2=\mu(N_1-N_2)=\mu Q\ , \tag{A1}$$

where Q is total charge in the system. It follows that

$$dG=\mu\, dQ+Q\, d\mu\ . \tag{A2}$$

At the same time the change in the internal energy of the system is given by

$$\begin{aligned} dU&=T\,dS-\mathscr{P}\,dV+\mu_1\,dN_1+\mu_2\,dN_2\\ &=T\,dS-\mathscr{P}\,dV+\mu\,dQ\ , \end{aligned} \tag{A3}$$

so that $U=TS-\mathscr{P}V+\mu Q$ and hence

$$\mu Q=G=U-TS+\mathscr{P}V\ , \tag{A4}$$

as usual. Now, combining Eqs. (A2)–(A4), we obtain the important relationship

$$Q\,d\mu=-S\,dT+V\,d\mathscr{P}\ , \tag{A5}$$

whence

$$\rho\equiv Q/V=(\partial\mathscr{P}/\partial\mu)_T\ . \tag{A6}$$

The isothermal compressibility is then given by

$$\begin{aligned} \kappa_T&=-\frac{1}{V}\left[\frac{\partial V}{\partial\mathscr{P}}\right]_{Q,T}\\ &=-\frac{1}{V}\left[\frac{\partial V}{\partial\rho}\right]_{Q,T}\left[\frac{\partial\rho}{\partial\mu}\right]_{Q,T}\left[\frac{\partial\mu}{\partial\mathscr{P}}\right]_{Q,T}\\ &=-\frac{1}{V}\frac{-Q}{\rho^2}\left[\frac{\partial\rho}{\partial\mu}\right]_T\frac{1}{\rho}=\frac{1}{\rho^2}\left[\frac{\partial\rho}{\partial\mu}\right]_T\ , \end{aligned} \tag{A7}$$

which is exactly of the form we have in the case of a *one-component* system.[19] Finally, since $\mu=(\partial f/\partial\rho)_T$ where f is the Helmholtz free energy density of the system, Eq. (A7) may also be written as

$$\kappa_T=[\rho^2(\partial^2 f/\partial\rho^2)_T]^{-1}\ . \tag{A8}$$

APPENDIX B

In this appendix we shall examine the nature of the singularity encountered by a *bulk* system of dimensionality $d'\leq 2$ as $T\rightarrow 0$. Right away we note that the condensate density ρ_0 in this case does not assume a macroscopic value unless the temperature of the system is reduced to a value *infinitesimally* close to absolute zero. Following the line of argument developed in Sec. VI, one can show that, for a d'-dimensional system of side $L_{\parallel}$, a macroscopic measure of condensate appears only if

$$\beta=\begin{cases} O(mL_{\parallel}^{2-d'}/\rho(d'))\quad (d'<2)\\ O(m\ln[\rho(d')L_{\parallel}^2]/\rho(d'))\quad (d'=2)\ . \end{cases} \tag{B1}$$

Clearly, as $L_{\parallel}\rightarrow\infty$, the required value of $\beta\rightarrow\infty$. Thus, for a *bulk* system of dimensionality $d'\leq 2$,

$$\rho_0(T>0)=0\ . \tag{B2}$$

However, at $T=0$, ρ_0 must be equal to ρ. The situation is clearly *singular* at $T=0$ which may, therefore, be regarded as the *critical* temperature of the system.

As in the case of a d-dimensional *bulk* system, the pressure $\mathscr{P}$ and the charge density ρ in the present case are given by, see Eqs. (49) and (53),

$$\mathscr{P}(\beta,\mu)=\frac{2^{(3-d')/2}m^{d'+1}}{\pi^{(d'+1)/2}}\times\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{(j\beta m)^{(d'+1)/2}}K_{(d'+1)/2}(j\beta m) \tag{B3}$$

and

$$\rho=\frac{2^{(3-d')/2}m^{d'}}{\pi^{(d'+1)/2}}\sum_{j=1}^{\infty}\frac{\sinh(j\beta\mu)}{(j\beta m)^{(d'-1)/2}}K_{(d'+1)/2}(j\beta m)\ . \tag{B4}$$

For $d'\leq 2$, the behavior of the foregoing expressions, as $\mu\rightarrow m$, is markedly different from that of the corresponding expressions for a system of dimensionality greater than 2. For instance, if $d'<2$, Eqs. (B3) and (B4) assume the form

$$\mathscr{P}(\beta,\mu)=\mathscr{P}(\beta,m)-\frac{2^{1-d'}}{\pi^{d'/2}d'}\Gamma\left[\frac{2-d'}{2}\right]\frac{(m^2-\mu^2)^{d'/2}}{\beta}+O(m^2-\mu^2)^1 \tag{B5}$$

and

$$\rho=\frac{2^{1-d'}}{\pi^{d'/2}}\Gamma\left[\frac{2-d'}{2}\right]\frac{m}{\beta(m^2-\mu^2)^{(2-d')/2}}+O(m^2-\mu^2)^0 \tag{B6}$$

which may be compared with the bulk terms of Eqs. (54) and (56), respectively. It follows that the chemical potential of the system at low temperatures is given by the expression

$$(m^2-\mu^2)\simeq\left[\frac{2^{1-d'}}{\pi^{d'/2}}\Gamma\left[\frac{2-d'}{2}\right]\left[\frac{m}{\rho\beta}\right]\right]^{2/(2-d')}\sim T^{2/(2-d')} . \tag{B7}$$

The singular part of the free energy density of the system is given by

$$f^{(s)}(\beta,\mu)\simeq\frac{2^{1-d'}}{\pi^{d'/2}d'}\Gamma\left[\frac{4-d'}{2}\right]\frac{(m^2-\mu^2)^{d'/2}}{\beta}\sim T^{2/(2-d')} . \tag{B8}$$

The singularity in the specific-heat density is, therefore, of the form

$$c^{(s)}\equiv -T\left[\frac{\partial}{\partial T}\left[\frac{\partial}{\partial T}f^{(s)}\right]\right]_\rho \sim T^{d'/(2-d')} \tag{B9}$$

and the corresponding critical exponent is

$$\dot{\alpha}=-d'/(2-d')\quad (d'<2) . \tag{B10}$$

Exponents for other quantities can be found in a similar manner.[26] For possible reference in the future, we note that the exponent $\dot{\lambda}$ for the isothermal compressibility of the system turns out to be $2/(2-d')$—the same as for the zero-field susceptibility of a corresponding magnetic system.[3,5]

For $d'=2$, on the other hand, we find that

$$\mathscr{P}(\beta,\mu)\simeq\mathscr{P}(\beta,m)-\frac{(m^2-\mu^2)}{4\pi\beta}\left[\ln\left[\frac{2m}{\beta(m^2-\mu^2)}\right]+C\right] , \tag{B11}$$

where C is a constant of order unity, and

$$\rho\simeq\frac{m}{2\pi\beta}\left[\ln\left[\frac{2m}{\beta(m^2-\mu^2)}\right]+C-1\right] . \tag{B12}$$

It follows that

$$(m^2-\mu^2)\sim(m/\beta)\exp(-2\pi\rho\beta/m) , \tag{B13}$$

with the result that

$$f^{(s)}\simeq\frac{(m^2-\mu^2)}{4\pi\beta}\sim\frac{m}{\beta^2}\exp(-2\pi\rho\beta/m) \tag{B14}$$

and

$$c^{(s)}\equiv -\beta^2\left[\frac{\partial^2}{\partial\beta^2}(\beta f^{(s)})\right]_\rho \sim \frac{\rho^2\beta}{m}\exp(-2\pi\rho\beta/m) . \tag{B15}$$

Thus, for $d'=2$, the power-law behavior is replaced by an exponential one.

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[12]This conjecture draws strength from the fact that for the case $n=\infty$, which has been analyzed in detail in the subsequent sections, the exponent ψ, like η, turns out to be zero; see Eq. (105).
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[16]It may be noted that, under *periodic* boundary conditions, a straightforward relationship exists between y and $\xi(L)$, viz., $y=L/2\xi(L)$; for details, see Ref. 4. For $T\gtrsim T_c$ and $L\to\infty$, one may, however, write $y\simeq L/2\xi(\infty)$. This explains the origin of the quantity $L/\xi(\infty)$ in Eq. (67).
[17]For asymptotic behavior of the functions $\mathscr{K}(\nu\,|\,d^*;y)$, as $y\to 0$, see S. Singh and R. K. Pathria, Phys. Rev. B **31**, 4483 (1985).
[18]It will be noted that the universal number Y_- for an ideal Bose gas confined to block geometry is exactly equal to 1. This could have been anticipated on the basis of the fact that, in the limit $T\to 0$, our analysis of Sec. III would in this case give $\kappa_T\approx L^d/Y_-T$ whereas considerations based on the rela-

tionship between the isothermal compressibility of a fluid and the integral of its correlation function over the space occupied by the system, see Ref. 19, would, in the same limit, give $\kappa_T \approx L^d/T$.

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[21]A comparison with Ref. 17 shows that the universal number $f^{(s)}(T_c;L)L^dT_c^{-1}$ for the ideal Bose gas is exactly *twice* the corresponding number for the spherical model of ferromagnetism. Since these two systems are supposed to be in the same universality class, this factor of 2 seems to be a consequence of the fact that the symmetry properties of the real Bose gas pertain to the case $n=2$ and, even when interactions are switched off (and the universality class *effectively* reduced to $n=\infty$), the "duplicity" arising from the complex nature of the order parameter still persists. We would like to pursue this question in greater detail on some other occasion.

[22]See Ref. 15, Eq. (36).

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[26]In view of the fact that the singularity studied here occurs at absolute zero, the resulting values of the various critical exponents are expected to be valid for *all* $n \geq 2$.