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Generalized dimensions and entropies from a measured time series

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The correlation-integral method of Grassberger and Procaccia is generalized to yield the whole spectrum of dimensions D_q and entropies K_q from a measured time series with a numerical effort which is only insignificantly larger than that needed to determine the original correlation integral. It is shown that our method yields reliable numerical results for the tent map and for the Mackey-Glass equation.

Recently much progress has been made in the characterization of dissipative nonlinear dynamical systems which display chaotic behavior.[1] The trajectory of these systems in phase space is often attracted to a bounded fractal object called strange attractor for which a whole set of dimensions[2] D_q and entropies[3] K_q has been introduced which generalize the concept of the Hausdorff dimension and the Kolmogorov entropy. Furthermore, it has been shown that the Legendre transformation of these quantities yields information about the distribution $f(\alpha)$ of the singularities in the natural invariant measure on the attractor[4] and about the spectrum of dynamical fluctuations $g(\lambda)$ around the Kolmogorov entropy.[5] However, up until now most experimental data of chaotic systems have only been analyzed by computing the correlation integral introduced by Grassberger and Procaccia[6,7] which yields only D_2 and K_2, i.e., just two points in an infinite spectrum of characteristic variables D_q and K_q. In this article we generalize the method of the correlation integral in such a way that all dimensions D_q and K_q and—via Legendre transformation—also the spectra $f(\alpha)$ and $g(\lambda)$ can be computed from a measured time series with an effort which is only slightly larger[8] than that needed to determine the original correlation integral.

To explain our method we divide the d-dimensional phase space of our system into cubes of size l^d and recall the definition of the generalized dimensions D_q:

$$D_q = \frac{1}{q-1} \lim_{l \to 0} \frac{1}{\ln l} \ln \sum_i^{M(l)} P_i^q \ . \tag{1}$$

Here $P_i(l)$ is the probability that the trajectory $\mathbf{X}_1, \ldots, \mathbf{X}_N$ on the strange attractor[9] visits box i, and $M(l)$ is the number of nonempty boxes. Since $\sum_i P_i^q$ can be written in terms of the natural probability measure $\mu(x)$ on the attractor as[2]

$$\sum_i P_i^q = \int d\mu(x)[\mu(B_l(x))]^{q-1} \ , \tag{2}$$

where $B_l(x)$ denotes a ball of radius l around x, we obtain by ergodicity

$$\sum_i P_i^q = \frac{1}{N} \sum_{j=1}^{N} \tilde{P}_j^{q-1}(l) \ , \tag{3}$$

where $\tilde{P}_j(l)$ is the probability to find a point of the trajectory within a ball of radius l around a point $\mathbf{X}_j$ of the trajectory. The change from q to $q-1$ in the exponents in

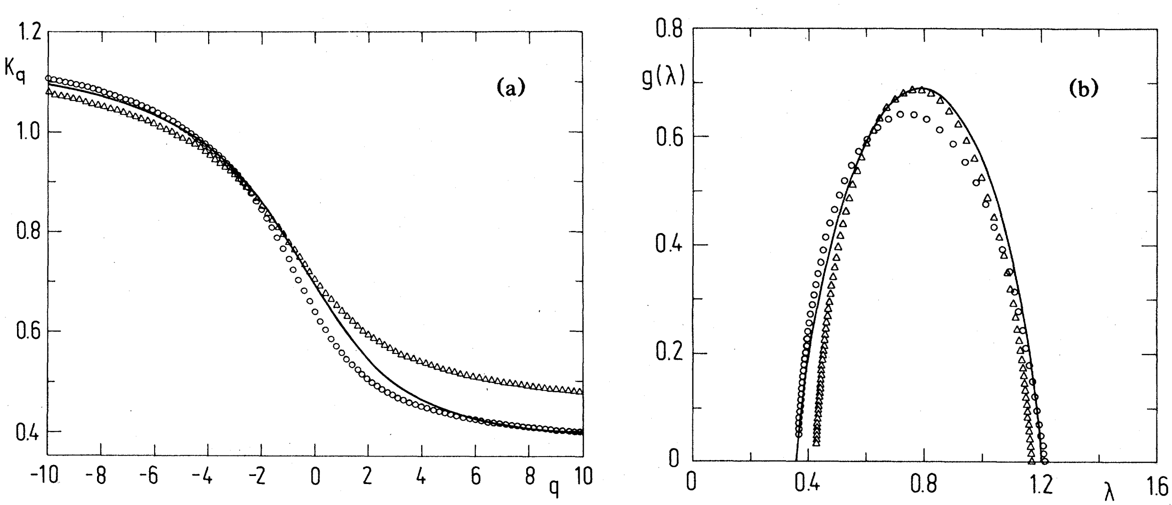

FIG. 1. The generalized entropies K_q (a) and the corresponding spectrum $g(\lambda)$, (b) for the tent map at $\eta = 0.3$. The points Δ,0 were computed from a time series of 1000 and 2000 points, the line is the theoretical result obtained by Legendre transformation of K_q in Ref. 15.

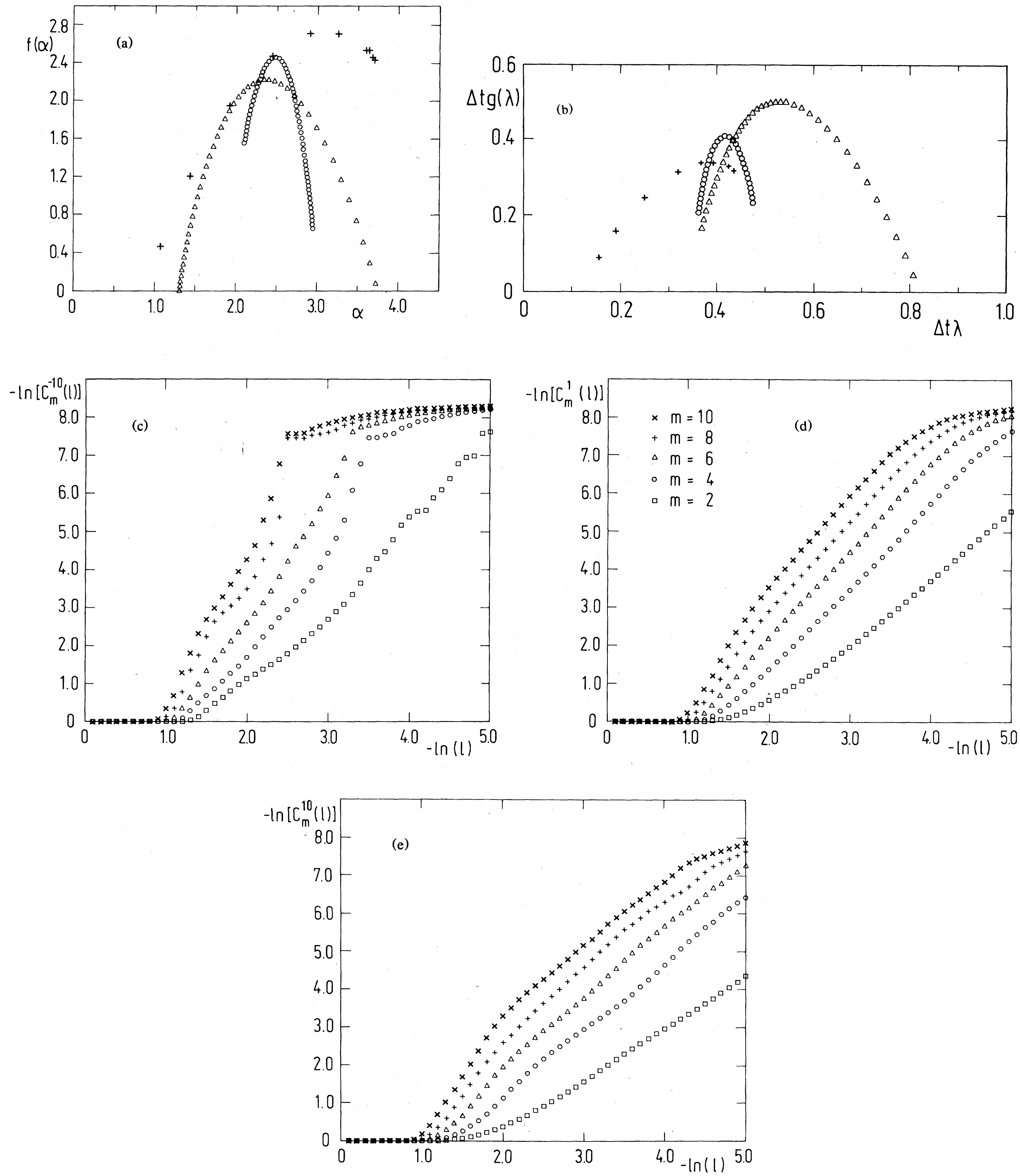

FIG. 2. (a) The spectrum of static scaling indices $f(\alpha)$ and (b) the spectrum of dynamic scaling indices $g(\lambda)$, computed via Eqs. (10a) and (10b) from 2000, 4000 points (Δ,0) which have been obtained from the Mackey-Glass equation (Ref. 14) with delay-constant $\tau = \Delta t = 23$. The crosses (+) denote the results obtained from a series of 10^4 points using the return time method from Ref. 16. (c)–(e) Examples of $-\ln C_m^q(l)$ vs $-\ln l$ from a one-dimensional time series of the Mackey-Glass equation for $q = -10, 1, 10$, respectively. The embedding dimensions m are 2,4, . . . ,10.

Eq. (3) is due to the fact that we switch from $P_i(l)$, i.e., the probability to find the trajectory in one of the homogeneously distributed boxes introduced above, to $\tilde{P}_j(l)$ which denotes the probability to find the trajectory within a ball around one of the inhomogeneously distributed points of the trajectory. The latter can be written as

$$\tilde{P}_j(l)=\frac{1}{N}\sum_i \Theta(l-|\mathbf{X}_j-\mathbf{X}_i|) \ , \tag{4}$$

where $\Theta(x)$ is the Heaviside step function. By combining Eqs. (1)-(4) we obtain

$$D_q=\lim_{l\to 0}\frac{1}{\ln l}\ln C^q(l) \ , \tag{5}$$

where

$$C^q(l)=\left[\frac{1}{N}\sum_i\left(\frac{1}{N}\sum_j\Theta(1-|\mathbf{X}_i-\mathbf{X}_j|)\right)^{q-1}\right]^{1/q-1} \tag{6}$$

is the generalized correlation integral.[10] It reduces for $q=2$ to the well-known result of Grassberger and Procaccia.[6] It is now straightforward to extend our method to the computation of the generalized entropies K_q which are defined as[3]

$$K_q=\lim_{l\to 0}\lim_{n\to\infty}\frac{-1}{q-1}\frac{1}{n\Delta t}\ln\sum_{i_1,\ldots,i_n}P^q_{i_1,\ldots,i_n} \ , \tag{7}$$

where $P_{i_1,\ldots,i_n}$ is the joint probability that the trajectory visits the boxes $i_1,\ldots,i_n$ in the mesh of cubes introduced above. By using joint probabilities instead of simple ones we obtain in analogy to Eqs. (5) and (6) the result

$$K_q=\lim_{l\to 0}\lim_{n\to\infty}\left(-\frac{1}{n\Delta t}\ln C^q_n(l)\right) \ , \tag{8a}$$

where

$$C^q_n(l)=\left\{\frac{1}{N}\sum_i\left[\frac{1}{N}\sum_j\Theta\left(l-\left[\sum_{m=0}^{n-1}(\mathbf{X}_{i+m}-\mathbf{X}_{j+m})^2\right]^{1/2}\right)\right]^{q-1}\right\}^{1/q-1} . \tag{8b}$$

This yields, e.g., for $q=1$ an expression from which the K entropy can be obtained experimentally:

$$K=K_1=\lim_{l\to 0}\lim_{n\to\infty}\frac{1}{n\Delta t}\frac{1}{N}\sum_i\ln\left[\frac{1}{N}\sum_j\Theta\left(l-\left[\sum_{m=0}^{n-1}(\mathbf{X}_{i+m}-\mathbf{X}_{j+m})^2\right]^{1/2}\right)\right] \ , \tag{9}$$

which agrees with previous results.[11]

By using the embedding theorem[12] to replace the time series $\{\mathbf{X}_i\}$, by the time series of a single measured signal $\{x_j\}$, we can combine Eqs. (5), (6), and (8) to

$$\lim_{l\to 0}\lim_{n\to\infty}\ln C^q_n(l)=D_q\ln l-n\Delta t K_q \ , \tag{10a}$$

with

$$C^q_n(l)=\left\{\frac{1}{N}\sum_i\left[\frac{1}{N}\sum_j\Theta\left(l-\left[\sum_{m=0}^{n-1}(x_{i+m}-x_{j+m})^2\right]^{1/2}\right)\right]^{q-1}\right\}^{1/q-1} . \tag{10b}$$

Equations (10a) and (10b) are our main result. They show that the generalized correlation integral $C^q_n(l)$ which can be obtained from an experimental time series yields in a plot $\ln C^q_n(l)$ vs $\ln l$ straight lines with slopes D_q whose distances in y direction converge for $n\to\infty$ to $\Delta t K_q$.

In order to test our method we used Eqs. (10a) and (10b) to compute numerically the K_q and $g(\lambda)$ spectra for the tent map[13] and the $f(\alpha)$ and $g(\lambda)$ spectra for the Mackey-Glass equation.[14] Figure 1 shows that for the tent map our results converge rapidly to the rigorous theoretical curves.[15] For the Mackey-Glass equation[14] we obtain with moderate numerical effort reasonable $f(\alpha)$ and $g(\lambda)$ curves which are, by using the same time series, significantly better than those obtained by calculating the probabilities appearing in the definitions of D_q and K_q via return times[16] (see Fig. 2).

To conclude, we have generalized the correlation integral method to yield all generalized dimensions D_q and entropies K_q from a measured time series. From the D_q's and K_q's the spectra $f(\alpha)$ and $g(\lambda)$ of the singularities in the invariant measure and of the dynamical fluctuations around the Kolmogorov entropy, respectively, can be obtained by straightforward Legendre transformation. The computer time which is needed to calculate the generalized correlation integral is only insignificantly larger than that needed for the original Grassberger-Proccacia method. It is hoped that our method stimulates precise classifications of experimental chaotic systems.

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[2]H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).
[3]P. Grassberger and I. Procaccia, Phys. Rev. A **28**, 2591 (1983).
[4]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[5]J. P. Eckmann and I. Procaccia, Phys. Rev. A **34**, 659 (1986).
[6]P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).
[7]*Dimension and Entropies in Chaotic Systems,* Springer Series in Synergetics, Vol. 32, edited by G. Mayer Kress (Springer, New York, 1986).

[8]For a time series of length N the computer time needed to evaluate the correlation integral is of order N^2. The computation of D_q's and K_q's adds to this a time of order $q_{\max}N$, i.e., the relative change in computer time is of order $q_{\max}/N$ which is for typical values $N \sim 10^7 - 10^5$, $q_{\max} \sim 10^2$ of the order of a few percent.

[9]Here the continuous trajectory $\mathbf{X}(t)$ has been chopped into points $\mathbf{X}_i = \mathbf{X}(t_0 + i\Delta t)$, where Δt is an elementary time step in analogy to an experimental time series.

[10]Similar quantities have been considered to characterize inhomogeneous attractors, however, without establishing their connection to the D_q's. G. Paladin and A. Vulpani, Lett. Nuovo Cimento **46**, 82 (1982).

[11]A. Cohen and I. Procaccia, Phys. Rev. A **31**, 1872 (1985).

[12]*Geometry Symposium Utrecht 1980,* Proceedings of a Symposium Held at the University of Utrecht, The Netherlands, 27–29 August, 1980, Springer Lecture Notes in Mathematics, Vol. 894, edited by E. Looijenga, D. Stersma, and F. Takens (Springer, New York, 1981).

[13]The tent map is defined by $x_{n+1} = x_n/\eta$ for $0 < x_n < \eta$ and $x_{n+1} = (1-x_n)/(1-\eta)$ for $\eta < x_n < 1$; $x_n \in [0,1]$.

[14]M. C. Mackey and L. Glass, Science **197**, 287 (1977).

[15]For the tent map one obtains $K_q = (q-1)^{-1}\ln[\eta^q + (1-\eta)^q]$ in analogy to the results in Ref. 5.

[16]M. H. Jensen, L. P. Kadanoff, A. Libchaber, I. Procaccia, and J. Stavans, Phys. Rev. Lett. **55**, 2798 (1985).