Oscillation-center theory at resonance

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Petrosky has shown that it is possible to continue analytically the nonresonant generating function in Lie transform theory and thereby treat resonant interactions in Hamiltonian systems. I will show that this formalism can be derived from the subdynamic theory proposed by the Brussels school. In addition to discussing the relationship between these two formalisms, some of the consequences of the resonant theory in wave-particle interactions are explored. In particular, I derive a complex $K-\chi$ theorem, which relates the resonant ponderomotive Hamiltonian to the imaginary part of the susceptibility of the plasma. This provides a conceptually appealing generalization of the nonresonant results. Finally I discuss other possible applications of the resonant Lie transform theory in plasma physics.

I. INTRODUCTION

The oscillation-center concept[1,2] is a useful technique in nonresonant wave-particle interactions, since it is possible to separate the coherent, high-frequency, response from the total wave-particle system. The residual slow-time-scale motion, is governed by the ponderomotive Hamiltonian, which is related to the linear response of the system.

There have been several attempts to understand the nature of the oscillation-center transformation at wave-particle resonance, and these approaches have been reviewed by Dewar.[3] The major difficulty, however, is that in the Lie-transform approach a suitable generating function is required, but this is a nontrivial problem for the following reason.

The presence of a resonance in a dynamical system is exhibited by the occurrence of a zero divisor in the solution and occurs when two of the natural frequencies of the system are commensurate. The zero divisor does not correspond to a real singularity, however, but indicates that the assumed form of the solution is inappropriate near the resonance condition. Quite generally, the resonance often precludes the construction of an analytic solution within the domain of resonance (which manifests itself in the secular behavior of certain oscillatory terms that appear in the solution), and modifies or destroys adiabatic invariances. From a global viewpoint it can be shown[4] that the motion near every resonance is similar to that of the physical pendulum which can be used to describe chaotic motion near the separatrices associated with resonances.

As expected, an exact analysis of resonant behavior is usually out of the question and therefore the resonant problem is treated by some form of perturbation theory, especially Hamiltonian perturbation methods based on the Hamilton-Jacobi equation. Although most dynamical systems are nonintegrable, one can attempt to expand the generating function, occurring in the Hamilton-Jacobi equation, in powers of a small parameter, at least for systems close to being integrable. Although the appearance of the small divisors prevents the convergence of such series, the expansion may describe the system behavior over some regions of phase space. For nonresonant systems the Poincaré—von Zeipel procedure enables the elimination of all periodic terms in the transformed Hamiltonian. More recently Dewar[2] has shown that canonical transformations, based on Lie series, provide a very powerful tool for solving Hamiltonian problems. In this case, the Lie generators depend upon the new (or old) variables and lead to explicit relations between the new and old variables. These properties give the Lie-transform methods great advantage over the Poincaré—von Zeipel method. One way to include restricted resonant behavior is that for systems admitting a simple deep resonance the appearance of a small divisor may be eliminated by modifying the Poincare—von Zeipel method in accordance with the technique developed by Bohlin.[5] There are two important features of this method: (i) the generating function is no longer purely periodic, and (ii) a solution is sought in powers of the square root of a small parameter.

Dewar[1] has also looked at developing a Hamiltonian theory of resonance wave-particle interactions, although by the von Zeipel method. In this formulation the nonresonant and resonant wave-particle interactions are treated from the outset. This is achieved by making a canonical transformation to the oscillation-center variables before attempting to solve the Vlasov equation. As a result, momentum and energy split into wave and particle components. When resonant interaction is introduced into the problem, a secular breakdown of the generating function will occur, but may be modified by introducing an artificial resonant width and a local-time-averaging operator.

The Lie-transform method promotes a canonical transformation from a function on phase space to a unitary operator on a function space. More generally it may be viewed as a particular measure preserving flow on the Hilbert space of square integrable functions defined on the phase space.[6] Looking at a canonical transformation from this viewpoint it is possible to derive canonical transformations in terms of unitary operators. This is just the framework proposed[7,8] by the Brussels school in their subdynamic theory of classical mechanics and is in fact related to the Lie-transform method.[9]

The subdynamic concept was originally introduced by the Brussels school in an attempt to understand the origin of irreversibility at the microscopic level. The transition between microscopic reversibility and macroscopic irreversibility introduces a time asymmetry in addition to reducing a many-degree-of-freedom dynamical system to a smaller one, which is adequate for the construction of a kinetic or transport theory. Usually the way to reconcile the two levels of description is to resort to coarse graining which reduces the number of degrees of freedom, and to neglect memory terms that break the time symmetry. In the Brussels approach, however, both coarse graining, invoking the random-phase approximation or erasing the memory can be avoided directly by using the subdynamic transformation theory.

In the subdynamic transformation theory, two operators Π and $\hat{\Pi}$ are introduced which project the full distribution function ρ into kinetic and nonkinetic parts, respectively. The $\Pi\rho$ subspace is known as the thermodynamic subspace, since the limit as $t \to \infty$ implies $\Pi\rho(t)$ approaches the equilibrium distribution function. The nonkinetic part is identically zero, which occurs for an equilibrium situation, or it asymptotically vanishes as $t \to \infty$. The term subdynamics arises because it reflects the fact that the evolution of $\Pi\rho$ and $\hat{\Pi}\rho$ become completely decoupled, each one obeying its own evolution law, that is, its own subdynamics. This is quite a remarkable result and is related to the fact that the Π operator commutes with the Liouville operator, even though both operators refer to interacting systems.

It is also possible to use subdynamics as a transformation theory in classical mechanics in its own right,[8,10] without any need to introduce any statistical considerations. Here two projection operators P and Π are introduced which independently govern the motion of the system in the null space of the unperturbed Hamiltonian and the null space of the full Hamiltonian, respectively. This subdivision of the dynamical system can be accomplished by two means. One construction of the transformation operator effecting this decomposition relies on the use of another operator χ which is a solution of an operator equation called the Mandel-Turner equation. Alternatively it is possible to introduce a linear eigenvalue problem where the transformed Hamiltonian appears as a family of eigenvalues while the corresponding eigenfunctions determine χ, and are physically interpreted as the probability distribution for the particle dynamics. These eigenfunctions link classical dynamics and the statistical-mechanics viewpoint.

We should also mention that subdynamics is capable of yielding very useful information concerning nonintegrable systems in general, in particular the construction of dynamical invariants,[11] determining criteria for ergodicity and irreversibility,[10] and finally the construction of a nonequilibrium entropy.[7] These results are based on the properties of the collision operator ψ which plays a central role in the subdynamic formalism and is directly related to the long-time behavior of the system, and thus to the ergodicity and invariants of the system. The behavior of $\psi(z \to i0^+)$, where z is the Laplace transform variable corresponding to the time, determines the asymptotic behavior of the distribution function and determines the form of the collision term for the so-called master equation for the N-particle distribution function. The existence of irreversibility in a dynamical system is therefore closely related to the nonvanishing of the collision operator.

If the resolvent of the Liouville operator (of the dynamical system) has a purely discrete spectra, and hence an isolated pole at the origin, then for Hamiltonian systems which have a periodic perturbation, the subdynamic transformation theory may be shown to be equivalent to Lie-transform theory.[9] This is quite interesting since it is possible to derive (as we will show later) the transformed Hamiltonian without the need of a generating function. Instead we now require the solution to an operator equation, the so-called Mandel-Turner equation. It should be mentioned that there are specific criteria necessary for the development of a mathematical consistency of the formalism, particularly the regularity of certain operators and that certain operators, for cases where continuous spectra arise, must be analytically continuable in the lower-half complex plane.

Having already shown that subdynamics may be used as a Lie-transform theory for nonresonant systems (i.e., a discrete spectra) we suggest here that it is possible to use subdynamics and treat resonant systems and in turn develop a resonant Lie-transform theory.

In a study of nonunitary evolution in nonintegrable Hamiltonian systems, Petrosky[12] has noticed that the Lie-generating function assumes the form of a Cauchy integral at resonance. This is important since it is possible to make this integral well defined by analytic continuation, as long as a physical boundary condition is used to determine the Riemann sheet of the continuation. In a sense it is analogous to scattering theory in the quantum-resonance problem and may be understood from the viewpoint of complex scaling and subdynamics.[13] Petrosky found that in the case of a single trajectory in phase space, the generating function is singly valued, while for distribution functions, the Lie-generating function does not reduce to a single function and demonstrates the irreversibility of the system.

In this paper we wish to look at Petrosky's version of the Lie-transform method and show that it may be derived from subdynamics. Furthermore we will show that this theory is the same as Dewar's[1] when the filter width goes to zero. As an important example, we use this formalism and derive the ponderomotive Hamiltonian at resonance. For nonresonant wave-particle interactions the ponderomotive Hamiltonian governs the behavior of quasiparticles in the oscillation-center representation; however, for resonant interactions it is less clear what form the ponderomotive Hamiltonian takes and what is the nature of the oscillation-center representation at resonance. We will show that we have to include the principal part of the resonant denominator in the ponderomotive Hamiltonian. From this resonant Hamiltonian we are able to derive a complex K-χ theorem, where the resonant ponderomotive Hamiltonian is related to the non-Hermitian part of the susceptibility, and provides a conceptually appealing generalization of the nonresonant results of Cary and Kaufman.[14]

II. SUBDYNAMICS AND OSCILLATION-CENTER THEORY

To construct the subdynamic theory, introduce the notion of a free system and an interacting one, described by the Hamiltonians H_0 and H, respectively, by

$$H = H_0 + \epsilon H_1 \ , \tag{1}$$

where ϵ is the coupling parameter. The time evolution of the state ρ is given by the Liouville—von Neumann equation

$$i\partial_t \rho = L\rho \ . \tag{2}$$

Where L is the Liouville operator,

$$L\rho = i\sum_{j=1}^{n}\left[\frac{\partial H}{\partial x_j}\frac{\partial}{\partial p_j} - \frac{\partial H}{\partial p_j}\frac{\partial}{\partial x_j}\right] , \tag{3a}$$

$$L\rho = H\rho - \rho H \ . \tag{3b}$$

For classical systems ρ may be the phase-space dynamical variables (p_n, x_n) while for quantum systems (3b) ρ is the density matrix.

The formal solution of the Liouville-von Neumann equation (2) may be formally expressed in terms of the resolvent R, defined by $R = (L - z)^{-1}$, where z is the Laplace transform variable. In fact,

$$\rho(t) = \exp(-iLt)\rho(0)$$
$$= \frac{1}{2\pi i}\int_C dz e^{-izt}(L - z)^{-1}\rho(0) \ . \tag{4}$$

The contour C goes from right to left in the upper half-plane and is above the real axis. To proceed with the subdynamic formalism define orthogonal projection operators P and Q,

$$P + Q = 1, \quad PQ = QP = 0, \quad P^2 = P, \quad Q^2 = Q \tag{5}$$

such that $P\rho$ is diagonal in a given representation. The resolvent operator then has the expansion

$$\frac{1}{L - z} = [P + \mathscr{C}(z)]\frac{1}{PLP + \psi(z) - z}[P + \mathscr{D}(z)] + \frac{1}{QLQ - z}Q \ . \tag{6}$$

Here we have introduced the collision operator $\psi(z)$,

$$\psi(z) = -PLQ\frac{1}{QLQ - z}QLP \ , \tag{7}$$

the destruction operator,

$$\mathscr{D}(z) = -PLQ\frac{1}{QLQ - z} \ , \tag{8}$$

and the creation operator

$$\mathscr{C}(z) = -\frac{1}{QLQ - z}QLP \ . \tag{9}$$

For systems with $PH = H_0$ and $PH_1 = 0$, then $PLP = 0$ in Eq. (6). If we close the contour C then quite generally the evaluation of the integral (4) gives[15] residue and continuum contributions denoted by I_R and I_C, respectively. Thus,

$$\frac{1}{2\pi i}\int dz\, e^{-izt}(L - z)^{-1} = I_R + I_C \ . \tag{10}$$

If the reduced resolvent has the spectral resolution

$$PLP + \psi(z) = \sum_k \mu_k(z_k)\,|\,u_k(z_k)\rangle\langle v_k(z_k)\,| \ , \tag{11}$$

then it follows that

$$\rho = \Sigma(t)\rho_0 + \hat{\Sigma}(t)\rho_0 \ , \tag{12}$$

where

$$\Sigma(t) = (P + C)\exp(-i\hat{\Theta}t)A(P + D) \ ,$$
$$\hat{\Theta} = \sum_n z_n\,|\,u_n\rangle\langle\bar{v}_n\,| \ ,$$
$$A = \sum_n \frac{1}{1 - \mu_n'(z_n)}\,|\,u_n\rangle\langle v_n\,| \ , \tag{13}$$
$$D = \sum_n |\,\bar{u}_n\rangle\langle v_n\,|\,\mathscr{D}(z_n) \ ,$$
$$C = \sum_n \mathscr{C}(z_n)\,|\,u_n\rangle\langle\bar{v}_n\,| \ .$$

The form of Eq. (11) suggests that the non-self-adjoint operator $\psi(z)$ has a discrete spectrum with semisimple eigenvalues $\{\mu_k(z)\}$ and a complete (in the P subspace) biorthogonal set of right $\{\,|\,u_r(z)\rangle\}$ and left $\{\,|\,v_k\rangle\}$ eigenvectors. Suppose $\psi(z)$, defined for $\operatorname{Im} z > 0$, can be analytically continued for $\operatorname{Im} z \le 0$ in some region below the real axis, including the origin. Then

$$\mu_{(+)r}(z) - z = 0 \tag{14}$$

admits one or more solutions with nonpositive imaginary part and that $[PLP + \psi(z) - z]^{-1}$ has no other singularities. The solution to Eq. (14) denoted by z_k for each k insures the semigroup property of the residue contribution to $\rho(t)$, i.e., $\Sigma(t)$, and is the reason why $\psi(z)$ is non-Hermitian. At the initial time $t = 0$, we can also define

$$\Sigma(t = 0) \equiv \Sigma(0) \equiv \Pi, \quad \hat{\Sigma}(0) \equiv \hat{\Pi} \ , \tag{15}$$

where

$$\Pi = (P + C)A(P + D) \ . \tag{16}$$

The superoperator Π is an idempotent projection operator, satisfying

$$\Pi^2 = \Pi, \quad \hat{\Pi} + \Pi = 1, \quad \hat{\Pi}\Pi = \Pi\hat{\Pi} = 0 \ . \tag{17}$$

Note that for systems with a discrete spectrum, Θ vanishes and consequently $\Sigma(t)$ reduces to Π. An important property of Π is that it commutes with the Liouville operator L. This property ensures that the projected part of ρ onto Π and $\hat{\Pi}$ obey separate evolution equations. Hence the term subdynamics. It also can be shown that the spectral resolution given by (11) is equivalent to the more customary approach to subdynamics, where it is assumed that the superoperators are regular at $z=0$; however, this must be justified for the specific system. This enables the operators C and D to be related to $\mathscr{C}$ and $\mathscr{D}$ which in turn follow directly from the $1/z$ coefficient of the resolvent.

In fact we have[7]

$$A=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\partial^n}{\partial z^n}\,|\,\psi(z)+PLP\,|^{\,n}\right]_{z\to+i0}, \tag{18}$$

$$C=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\partial^n}{\partial z^n}\mathscr{C}(z)\right]_{z\to+i0}\theta^n, \tag{19}$$

$$D=\sum_{n=0}^{\infty}(\bar\theta)^n\frac{1}{n!}\left[\frac{\partial^n}{\partial z^n}\mathscr{D}(z)\right]_{z\to+i0}, \tag{20}$$

where $\mathscr{C}$ and $\mathscr{D}$ are given by Eqs. (8) and (9). We also have

$$\theta=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\partial^n}{\partial z^n}\,|\,\psi(z)+PLP\,|\,\right]_{z\to+i0}\theta^n \tag{21}$$

and

$$\bar\theta=\sum_{n=0}^{\infty}(\bar\theta)^n\frac{1}{n!}\left[\frac{\partial^n}{\partial z^n}\,|\,\psi(z)+PLP\,|\,\right]_{z\to+i0}. \tag{22}$$

If Π is decomposed via projection such that

$$P\Pi P=A,\quad P\Pi Q=A\mathscr{D},\quad Q\Pi P=\mathscr{C}A,\quad Q\Pi Q=\mathscr{C}A\mathscr{D}, \tag{23}$$

then it may be shown that

$$\Pi=\oint dz\,(z-L)^{-1}, \tag{24}$$

where the contour encloses the singularity at $z=0$.

Equation (24) actually forms the basis for the comparison of subdynamics and Lie-transform theory.[9] The Lie transform operator C_w is defined by

$$\frac{\partial C_w}{\partial\epsilon}=L_wC_w,\quad C_w(\epsilon=0)=1,$$

where the Lie derivative L_w, generated by w is,

$$L_w\phi=\{\phi,w\}$$

for some arbitrary function ϕ. It is easy to show from Eq. (24) that

$$\Pi=C_w^{-1}PC_w \tag{25}$$

by using the results

$$C_wL_H=L_KC_w$$

and

$$PL_K=L_KP=0$$

in Eq. (24).

The fact that Π may be used as a Hamiltonian-transformation theory follows by defining an operator χ by (not to be confused with the susceptibility)

$$\chi=PC_w^{-1}P\;.$$

This operator is useful, since we defined the new or transformed Hamiltonian K by $K=C_w^{-1}H$, then

$$\chi K=H_0$$

so

$$K=\chi^{-1}H_0\;. \tag{26}$$

More importantly, the equation for χ^{-1} can be derived and is

$$\frac{\partial\chi^{-1}}{\partial\epsilon}=\chi^{-1}AD\frac{\partial C}{\partial\epsilon}\;. \tag{27}$$

For nondissipative systems D and C go to $\mathscr{D}$ and $\mathscr{C}$, respectively. Equation (26) shows us that χ^{-1} operating on H_0 gives the transformed Hamiltonian.

It is also possible to show that

$$C_w^{-1}P=(1+C)\chi=\gamma^+\;. \tag{28}$$

If this equation is operated on the right by P, and using $\chi P=\chi$, we have an expression for C_w^{-1} in terms of C and χ,

$$C_w^{-1}=(1+C)\chi\;. \tag{29}$$

III. PETROSKY'S RESONANT LIE-TRANSFORM THEORY

In this section we will present Lie-transform theory as proposed by Deprit and Dewar[2] and show how to include resonances by analytic continuation and imposing certain boundary conditions. Nonresonant Lie-transform theory uses an infinitesimal generator to perform a canonical transformation from oscillator-center (OC) variables $(\mathbf{J},\mathbf{\Theta})$ to physical (or exact) variables $(\mathbf{I},\boldsymbol{\theta})$. As distinct from (pre-Lie) Hamiltonian perturbation theory, the generator w depends only on the OC variables and thus avoids cumbersome algebra in the mixed generating form, such as von Zeipel theory.

The unitary operator C_w is defined by[12]

$$-i\frac{\partial C_w}{\partial\epsilon}=L_wC_w,\quad C_w(\epsilon=0)=1, \tag{30}$$

where the Lie derivative L_w, generated by the function $w(\mathbf{J},\mathbf{\Theta},t)$ is

$$L_w\phi=i\{w,\phi\}, \tag{31}$$

ϵ is some parameter, and ϕ is an arbitrary function. The Poisson brackets are defined by

$$iL_w=\sum_{i=1}^{N}\left[\frac{\partial w}{\partial\mathbf{J}_i}\frac{\partial}{\partial\mathbf{\Theta}_i}-\frac{\partial w}{\partial\mathbf{\Theta}_i}\frac{\partial}{\partial\mathbf{J}_i}\right].$$

C_w^{-1} is the inverse operator defined by

$$-i\frac{\partial C_w^{-1}}{\partial\epsilon}=-C_w^{-1}L_w,\quad C_w^{-1}(\epsilon=0)=1\;. \tag{32}$$

Unitarity follows from the anti-Hermiticity of iL_w. Equation (30) has the formal solution

$$C_w=\mathscr{E}\exp\left[\int_0^{\epsilon}L_w d\epsilon\right], \tag{33}$$

where $\mathscr{E}$ is the ϵ ordering operator. It can be shown that the transformation

$$\theta(\mathbf{J},\mathbf{\Theta},t)=C_w\mathbf{\Theta},\quad \mathbf{I}(\mathbf{J},\mathbf{\Theta},t)=C_w\mathbf{J}\ , \tag{34}$$

is canonical. Physically, $\mathbf{\Theta}$ and $\mathbf{I}$ are the exact phase-space coordinates of a particle with Hamiltonian $H(\boldsymbol{\theta},\mathbf{I},t)$. The coordinates $(\mathbf{J},\mathbf{\Theta})$ are the coordinates of the OC, which behaves like a particle with Hamiltonian $K(\mathbf{J},\mathbf{\Theta},t)$. The nonresonant OC represents the averaged position of the particle so we require w to be uniformly bounded in time. We further make the "gauge" choice

$$\langle w\rangle=0\ , \tag{35}$$

where the brackets denote averaging over the short time scale. For resonant systems, we can relax this condition. It can be shown[2] that w, K, and H are related by the Dewar-Deprit equation

$$\frac{\partial w}{\partial t}+iL_K w=C_w\frac{\partial H}{\partial\epsilon}-\frac{\partial K}{\partial \epsilon}\ , \tag{36}$$

where here and henceforth all implicit dependences are on the same dummy variables $\mathbf{J}$ and $\mathbf{\Theta}$. If a function is to be evaluated at $(\mathbf{I},\mathbf{\Theta})$ this is affected by acting on it with C_w.

In this paper we confine our attention to Hamiltonians H and K, and generators w which are independent of t. Since time dependence can always be formally removed by going to an extended phase space, this presents no real restriction on the theory. As is already implicit in our choice of notation we also assume H, K, and w to be 2π-periodic functions of $\boldsymbol{\theta}$ and $\mathbf{\Theta}$, which can often be achieved by appropriate scaling. Generalization to more than one dimension is straightforward.

Assuming

$$H(\mathbf{J},\mathbf{\Theta})=H_0(\mathbf{J})+\epsilon H_1(\mathbf{J},\mathbf{\Theta})\ , \tag{37}$$

the generator w can be constructed as a power series in ϵ by solving Eq. (36) order by order, with the terms in the power-series expansion of $K(\mathbf{J},\epsilon)$ being determined from the solubility condition at each order. In fact, denoting L_{w_1} by L_1, etc., we have

$$\begin{aligned}C_w&=1+i\epsilon L_1+\tfrac{1}{2}\epsilon^2[(iL_1)^2+iL_2]\cdots\ ,\\ C_w^{-1}&=1-i\epsilon L_1+\tfrac{1}{2}\epsilon^2[(-iL_1)^2-iL_2]-\cdots\ ,\end{aligned} \tag{38}$$

while Eq. (36) becomes

$$-iL_1H_0+H_1=K\ , \tag{39a}$$

$$-iL_2H_0+[(-L_1)^2H_0-2iL_1H_1]=2K_2\ . \tag{39b}$$

K_2 is known as the second-order ponderomotive Hamiltonian. Note we have removed the factor of $\frac{1}{2}$ appearing in Petrosky's version of Eq. (39b) since it is a misprint.

Now let us introduce a "Dirac bra-ket" notation whereby any periodic function $\mathbf{\Theta}$ is represented as an inner product of abstract bra and ket vectors in a Hilbert space

$$f(\mathbf{\Theta})=\langle\mathbf{\Theta}\,|\,f\rangle\ . \tag{40}$$

The momentum $\mathbf{J}$ will be regarded as a parameter, so the complete orthonormal basis of the Fourier expansion is represented by

$$\langle\mathbf{\Theta}\,|\,\mathbf{k}\rangle=\left[\frac{\Delta k}{(2\pi)^N}\right]^{1/2}e^{i\mathbf{k}\cdot\mathbf{\Theta}}\ . \tag{41}$$

We also assume the Fourier expansion for H_1,

$$H_1(\mathbf{J},\mathbf{\Theta})=\Delta k\sum_k H_{1k}(\mathbf{J})e^{i\mathbf{k}\cdot\mathbf{\Theta}}\ , \tag{42}$$

with $\Delta k=\Delta k_1,\Delta k_2,\ldots,\Delta k_N$, $k_i=n_i\Delta k_i$ with integer n_i and $\mathbf{k}\cdot\mathbf{\Theta}=k_1\Theta_1+k_2\Theta_2+\ldots+k_N\Theta_N$. Here N is the number of degrees of freedom and we have assumed that the perturbation has period $2\pi/\Delta k_i$ for $i=1,\ldots,N$.

The complete orthonormality is expressed in terms of a projection operator $P_k\equiv|\,\mathbf{k}\rangle\langle\mathbf{k}\,|$ as

$$\begin{aligned}&\sum_k P_k=1\ ,\\ &P_kP_{k'}=P_k\delta_{k,k'}\ ,\end{aligned} \tag{43}$$

where

$$\delta_{k,k'}=\delta_{k_1,k_1'}\delta_{k_2,k_2'}\cdots\delta_{k_N,k_N'}\ .$$

Thus the Lie derivative is

$$\langle\mathbf{k}\,|\,L_w\,|\,\mathbf{k}'\rangle=\frac{\Delta k}{(2\pi)^N}\int_{-\pi/\Delta k_1}^{\pi/\Delta k_1}d\mathbf{\Theta}_1\cdots\int_{-\pi/\Delta k_N}^{\pi/\Delta k_N}d\mathbf{\Theta}_N e^{-i\mathbf{k}\cdot\mathbf{\Theta}}\times L_w e^{i\mathbf{k}'\cdot\mathbf{\Theta}}\ . \tag{44}$$

Petrosky[12] has shown, using the definition of L_w, that

$$\langle\mathbf{k}\,|\,L_w\,|\,\mathbf{k}'\rangle=\sqrt{\Delta k}\left[-w_{k-k'}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial}{\partial\mathbf{J}}+\left[\frac{\partial}{\partial\mathbf{J}}\cdot\mathbf{k}'w_{k-k'}\right]\right]\sqrt{\Delta k}\ . \tag{45}$$

This may be used to solve Eqs. (39). Introduce

$$\langle 0\,|\,H_0\rangle=(2\pi)^{N/2}(\Delta k)^{-1/2}H_0(\mathbf{J})$$

and

$$\langle\mathbf{k}\,|\,H_1\rangle=(2\pi)^{N/2}(\Delta k)^{1/2}H_{1k}(\mathbf{J})\ .$$

Then Eq. (39a) is

$$-i\langle\mathbf{k}\,|\,L_1\,|\,0\rangle\langle 0\,|\,H_0\rangle+\langle\mathbf{k}\,|\,H_1\rangle=\langle\mathbf{k}\,|\,K_1\rangle\delta_{\mathbf{k},0}\ . \tag{46}$$

For $\mathbf{k}=\mathbf{0}$, Eq. (46) has the solution $K_1(\mathbf{J})=\Delta kH_0(\mathbf{J})$ which is the secular term of the transformed Hamiltonian. For $k\neq 0$,

$$w_{k1}(\mathbf{J})=\frac{iH_{1k}(\mathbf{J})}{\mathbf{k}\cdot\boldsymbol{\omega}}\ , \tag{47}$$

where $\omega_i=\partial H_0/\partial\mathbf{J}_i$. Let us first look at the form of w_k due to the results of Petrosky but then we will show how it arises more directly from dissipative subdynamics. As is well known, $w_{k1}(\mathbf{J})$ has a small denominator and is not well defined if $\mathbf{k}\cdot\boldsymbol{\omega}=0$. This is a major restriction since this implies that resonances must be excluded. Petrosky has shown, however, under suitable conditions, that the generating function $w_{k1}(\mathbf{J})$ becomes well defined at resonance [at least to $O(\epsilon)$] by a suitable analytic continuation of $\mathbf{k}$ to the complex plane. In this paper we will use this new expression into Eq. (39b) and derive K_2, the second-order ponderomotive Hamiltonian. The basic idea in generalizing Eq. (47) to include resonances is the following. From Eq. (34) we know that to $O(\epsilon)$,

$$\mathbf{J}=C_w^{-1}\mathbf{I}=\mathbf{I}-\epsilon\Delta k\sum_k \frac{k_i H_{1k}}{\mathbf{k}\cdot\boldsymbol{\omega}}e^{i\mathbf{k}\cdot\boldsymbol{\Theta}} . \tag{48}$$

In the continuous limit $\Delta k_1 \to 0$, the second term of Eq. (48) reduces to a Cauchy integral

$$\Phi(z)=\int_{-\infty}^{\infty} dx \frac{f(x)}{x-z} , \tag{49}$$

which is evaluated on the real z axis. Let us denote the upper half-plane by S_+ and the lower half-plane by S_-. At present the integral Eq. (49) has no meaning as it stands for real z, but when z tends towards the real value y by taking only values in S_+, then $\Phi(z)$ tends towards a definite limit $\Phi^{(+)}(y)$. Similarly for values in S_-, $\Phi(z)$ tends towards a limit $\Phi^{(-)}(y)$ which is generally different from $\Phi^{(+)}(y)$. In fact, $\Phi^{(+)}(z)$ and $\Phi^{(-)}(z)$ are not elements of the same analytical function. This implies that $\Phi(z)$ is discontinuous on the real axis; however, it is continuous from above and from below. For $z\in S_+$, and by defining a contour for the analytic continuation of $\Phi^{(+)}(z)$, it follows that

$$\Phi^{(+)}(y-i\epsilon)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dx \frac{f(x)}{x-y+i\epsilon}+f(y-i\epsilon) , \quad \epsilon>0 \tag{50}$$

which defines the analytic continuation into S_- of the function $\Phi(z)$ for $z\in S_+$. It is different from the integral of Eq. (49) evaluated at $y-i\epsilon$. A similar argument also holds for z initially in S_- and continued into S_+ A useful picture of what is happening is to discuss the analytical behavior of $\Phi(z)$ from the point of view of Riemann surfaces. Since $\Phi(z)$ is a two-valued function (by means of analytic continuation), its Riemann surface has a cut along the real axis when the sheets interact. However, a point on one sheet will remain on that sheet when it goes from S_+ to S_-.

To make w_{k1} well defined we need a physical boundary condition to determine the Riemann sheet of the analytic continuation of k_1. Petrosky imposes the boundary condition that the perturbed solution $J_i(t)$ in Eq. (48) reduces to the unperturbed solution in the limit of $t\to-\infty$. Since the Hamiltonian is cyclic the solutions of $J_i(t)$ and $\Theta_i(t)$ are $J_i(t)=\text{const}$ and $\Theta_i(t)=\omega_i t+\Theta_{0i}$, and using Eq. (50), we have

$$\mathbf{J}_{1+}(t)=\mathbf{I}_1-\epsilon\Delta k_2\sum_{k_2}\int_{-\infty}^{\infty} dk_1 \frac{k_1 H_{1k}(\mathbf{J})}{\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon}e^{ik(\omega t+\Theta_{01})} , \tag{51}$$

where ϵ is a positive infinitesimal. The corresponding generating function is

$$w_{1+}=\Delta k_2\sum_{k_2}\int_{-\infty}^{\infty} dk_1 \frac{iH_{1k}(\mathbf{J})}{\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon}e^{i\mathbf{k}\cdot\boldsymbol{\Theta}} . \tag{52}$$

If we impose the boundary condition that $\mathbf{J}\to\mathbf{I}$ in the limit of $t\to+\infty$, then we have similar equations for $\mathbf{J}_{1-}$ and w_{1-} except that ϵ is replaced by $-\epsilon$. The above formalism has some quite remarkable properties. Firstly we notice that the generating function of C_w does not reduce to a single function w_{1+} and secondly C_w is no longer unitary. These results follow because the analytic continuation of the matrix elements of C_w, which is put between the states, $|\mathbf{k}_2\rangle$ and $|\mathbf{k}'\rangle$ to $O(\epsilon)$ as

$$\begin{aligned}\langle \mathbf{k}_2 | C_w | \mathbf{k}'\rangle &\cong i\epsilon\langle \mathbf{k}_2 | L_{1-} | \mathbf{k}'\rangle , \\ \langle \mathbf{k}' | C_w | \mathbf{k}_2\rangle &\cong i\epsilon\langle \mathbf{k}' | L_{1+} | \mathbf{k}_2\rangle .\end{aligned} \tag{53}$$

Here $L_{1\pm}$ is the Lie derivative generated by $w_{1\pm}$, respectively. It is also possible to show that C_w is now star unitary,

$$C_w^{\dagger}(L_H)=C_w^{-1}(-L_H) , \tag{54}$$

where $C_w^{\dagger}$ is the Hermitian conjugate of C_w.

We wish to solve Eq. (39b) for the ponderomotive Hamiltonian, but first, however, we have to evaluate explicitly the generating function, and this requires taking the limit as $\epsilon\to 0^+$ of Eq. (52). Thus

$$w_{1+}=\Delta k_2\sum_{k_2}\int_{-\infty}^{\infty} dk_1 iH_{1k}(\mathbf{J}) \lim_{\epsilon\to 0^+}\frac{1}{\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon}e^{ik\cdot\Theta} . \tag{55}$$

Using the Plemelj formulas

$$\lim_{\epsilon\to 0^+}\frac{1}{x+i\epsilon}=P\frac{1}{x}\pm i\pi\delta(x) , \tag{56}$$

in Eq. (55), we find

$$w_{1+}=-\pi\Delta k\sum_k H_{1k}\delta(\mathbf{k}\cdot\boldsymbol{\omega})e^{ik\cdot\Theta} . \tag{57}$$

Here we have used the fact that $k_1\omega_1=-k_2\omega_2$. We also have used the result that the principal part of Eq. (56) does not contribute to w_{1+} due to the symmetry of the function of $\mathbf{k}$. To second order, Eq. (36) is

$$\begin{aligned}&-i\sum_{\mathbf{k}'}\langle \mathbf{k} | L_2 | \mathbf{k}'\rangle\langle \mathbf{k}' | H_0\rangle \\ &\quad -\sum_{k'}\sum_{k''}\langle \mathbf{k} | L_1 | \mathbf{k}'\rangle\langle \mathbf{k}' | L_1 | \mathbf{k}''\rangle\langle \mathbf{k}'' | H_0\rangle \\ &\quad -2i\sum_{k'}\langle \mathbf{k} | L_1 | \mathbf{k}'\rangle\langle \mathbf{k}' | H_1\rangle=2\langle \mathbf{k} | K_2\rangle ,\end{aligned} \tag{58}$$

which follows from Eq. (39b). We can use, in view of Eq. (37), that $PH_0=H_0$ and $QH_1=H_1$ and $PH_1=0$, where from Eq. (43), $P\equiv|0\rangle\langle 0|$ and $Q\equiv|\mathbf{k}\rangle\langle\mathbf{k}|$. Thus $\langle \mathbf{k} | H_0\rangle=\langle 0 | H_0\rangle\delta_{\mathbf{k},0}$. The secular component of Eq. (58) allows us to determine K_2 via

$$\langle 0|L_2|0\rangle\langle 0|H_0\rangle-\sum_{k'}\langle 0|L_1|\mathbf{k}'\rangle\langle \mathbf{k}'|L_1|0\rangle\langle 0|H_0\rangle -2i\sum_{k'}\langle 0|L_1|\mathbf{k}'\rangle\langle \mathbf{k}'|H_1\rangle=2\langle 0|K_2\rangle\ . \quad (59)$$

From Eq. (45) we have (we will drop the one subscript on w_{1k} for convenience)

$$\langle 0|L_1|\mathbf{k}'\rangle=\sqrt{\Delta k}\left[w_{-k'}\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}}+\frac{\partial}{\partial \mathbf{J}}\cdot\mathbf{k}'w_{-k'}\right]\sqrt{\Delta k}\ ,$$
$$\langle \mathbf{k}'|L_1|0\rangle=\sqrt{\Delta k}\left[-w_{k'}\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}}\right]\sqrt{\Delta k}\ . \quad (60)$$

Equation (59) then becomes

$$\tfrac{1}{2}\sum_k(\Delta k)\left[w_{-k}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}w_k(\mathbf{k}\cdot\boldsymbol{\omega})+w_k(\mathbf{k}\cdot\boldsymbol{\omega})\frac{\partial}{\partial \mathbf{J}}\cdot\mathbf{k}w_{-k}\right]\Delta k-i\sum_k(\Delta k)\left[w_{-k}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}H_{1k}+H_{1k}\frac{\partial}{\partial \mathbf{J}}\cdot\mathbf{k}w_{-k}\right]\Delta k=K_2(\mathbf{J})\ . \quad (61)$$

To evaluate this expression, we need an expression for w_k. If w has the Fourier representation

$$w=\int d\mathbf{k}\,w_k(\mathbf{J})e^{i\mathbf{k}\cdot\Theta}\ ;$$

it follows from Eq. (57) that

$$w_k=-\lim_{\epsilon\to 0}\frac{H_{1k}}{\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon}\ . \quad (62)$$

In the nonresonant case, Eq. (47) can be used for w_k, and then Eq. (61) gives

$$K_2=-\frac{\Delta k}{2}\sum_k\frac{\partial}{\partial \mathbf{J}}\cdot\mathbf{k}\frac{\Delta k\,|H_{1k}|^2}{\mathbf{k}\cdot\boldsymbol{\omega}}\ . \quad (63)$$

For the resonant case, it is clear that both terms of Eq. (62) are important since w_k is nonzero when $\mathbf{k}\cdot\boldsymbol{\omega}=0$. By including resonant terms, Eq. (61) is

$$\mathscr{K}=-\frac{\Delta k}{2}\sum_k{}'\frac{\partial}{\partial \mathbf{J}}\cdot\mathbf{k}\Delta k w_{-k}w_k(\mathbf{k}\cdot\boldsymbol{\omega})$$
$$=\frac{i\Delta k}{2}\sum_k\frac{\partial}{\partial \mathbf{J}}\cdot\mathbf{k}\Delta k w_k H_{-1k}\ . \quad (64)$$
$$\equiv K_2+iK_2^{\text{res}}$$

IV. SUBDYNAMICS OF RESONANCE

Although the form of w_k given by Eq. (62) arises from the physically motivated form of w_{1+}, we would like to derive it directly from dissipative subdynamics, since this is in line with the earlier results of deriving the nonresonant generating function from nondissipative subdynamics.

If we work to $O(\epsilon)$, then it follows from Eqs. (19)—(22) that

$$C=\mathscr{C}_1(+i0) \quad (65)$$

where the one subscript of $\mathscr{C}$ denotes that Eq. (9) is to be evaluated to $O(\epsilon)$ only. That is,

$$\mathscr{C}_1=\epsilon\frac{1}{z-QL_0Q}Q\delta LP\ , \quad (66)$$

where δL is the Liouville operator for the perturbed part of the Hamiltonian. In the matrix notation defined earlier, we have

$$\langle \mathbf{k}|\mathscr{C}_1(+i0)|0\rangle=\lim_{\epsilon\to 0^+}\left[\epsilon\sqrt{\Delta k_1}\frac{H_{1k}}{\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\sqrt{\Delta k_1}\right]\ . \quad (67)$$

But we know from Eq. (38) that to $O(\epsilon)$,

$$C_w^{-1}=1-i\epsilon L_1\ .$$

Using Eq. (45) it is clear that we can identify

$$w_k=-\lim_{\epsilon\to 0^+}\frac{H_{1k}}{\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon}\ . \quad (68)$$

To derive $\mathscr{K}_2$, we have to solve Eq. (27) which necessitates the construction of the destruction operator D. Again it follows that to $O(\epsilon)$, $\mathscr{D}_1=D_1$ where

$$\mathscr{D}_1=\epsilon P\delta LQ\frac{1}{z-QL_0Q}\ .$$

Using the trivial result $\langle \mathbf{k}|QL_0|\mathbf{k}'\rangle=(\mathbf{k}\cdot\boldsymbol{\omega})$, it follows that

$$\langle 0|\mathscr{D}_1(+i0)|\mathbf{k}\rangle=-\lim_{\epsilon\to 0^+}\left[\epsilon\sqrt{\Delta k_1}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\frac{H_{-1k}\sqrt{\Delta k_1}}{(\mathbf{k}\cdot\boldsymbol{\omega}-i\epsilon)}\right]\ . \quad (69)$$

We know[8] that to $O(\epsilon^2)$, the Mandel-Turner equation has the series solution

$$\chi^{-1}=1+\frac{DC}{2}\ . \quad (70)$$

So from Eq. (26)

$$\mathscr{K}=H_0+\frac{DC}{2}H_0\ . \quad (71)$$

It is clear then that

$$\mathscr{C}_1=\Delta k_1 w_k\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\ ,$$
$$\mathscr{D}_1=-\Delta k_1\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}w_k\ . \quad (72)$$

Denoting $\mathscr{K}_2$ as the ϵ^2 term of Eq. (71), we have

$$\mathscr{K}_2=-\frac{\Delta k}{2}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}[w_{-k}w_k(\mathbf{k}\cdot\boldsymbol{\omega})\Delta k]\ ,$$

which agrees with Eq. (64) if we use the first-order Dewar-Deprit equation,

$$-iw_k(\mathbf{k}\cdot\boldsymbol{\omega})=H_{1k}\ . \quad (73)$$

We have thus shown that it is possible to derive the generating function at resonance by using the subdynamic theory. Furthermore the derivation of the ponderomotive Hamiltonian from the Lie-transform approach or from the superoperator theory give the same results. In Sec. V we will consider the consequences of the ponderomotive Hamiltonian given by Eq. (64).

V. THE COMPLEX K-χ THEOREM

In this section we would like to show that the resonant ponderomotive Hamiltonian may be used to construct a K-χ theorem valid at resonance. This theorem relates the ponderomotive Hamiltonian and the linear response and for nonresonant cases, is given by[14]

$$\int d\Theta\, d\mathbf{J}\, f_0(\Theta,\mathbf{J},t)K_2=\tfrac{1}{2}\left\langle \int d\Theta(\rho\Phi-\mathbf{j}\cdot\mathbf{A})\right\rangle_t , \quad (74)$$

where the angled brackets denote temporal averaging, while Φ and $\mathbf{A}$ are the first-order scalar and vector potentials. ρ and $\mathbf{j}$ are the first-order density and current density. Cary and Kaufman have shown, for nonresonant wave-particle interactions, that Eq. (74) reduces to

$$\int d\Theta\, d\mathbf{J}\, f_0(\Theta,\mathbf{J},t)K_2=-\frac{1}{16\pi}\int d\Theta\, \mathbf{E}^*\cdot\hat{\chi}_H\cdot\mathbf{E} , \quad (75)$$

where $\hat{\chi}_H$ is the Hermitian part of the susceptibility tensor

$$\hat{\chi}=-\frac{4\pi e^2}{m\omega^2}\int d^3\mathbf{p}\, f_0\left[1+\frac{\mathbf{k}\mathbf{v}_0+\mathbf{v}_0\mathbf{k}}{\omega-\mathbf{k}\cdot\mathbf{v}_0}+\frac{\mathbf{v}_0\mathbf{v}_0 k^2}{(\omega-\mathbf{k}\cdot\mathbf{v}_0)^2}\right] . \quad (76)$$

Here $\mathbf{p}=m\mathbf{v}_0$ is the canonical momentum. Let us look at these relationships using the formalism of Sec. IV. Let the Hamiltonian of the wave-particle interaction be given by

$$H=\frac{p^2}{2m}-\frac{\epsilon e}{m}A\cos(kx-\omega t) . \quad (77)$$

We can eliminate the time dependence in H by going to the wave frame. This is achieved by introducing the generating function

$$F_2=(kx-\omega t)J_\theta$$

giving

$$p=\frac{\partial F_2}{\partial x}=kJ_\theta , \quad (78a)$$

$$\theta=\frac{\partial F_2}{\partial J_\theta}=kx-\omega t . \quad (78b)$$

The new Hamiltonian is

$$H'=H+\frac{\partial F_2}{\partial t}$$

$$=H_0+\epsilon H_1 , \quad (79)$$

where

$$H_0=\frac{k^2J_\theta^2}{2m}-\omega J_\theta , \quad (80)$$

and

$$H_1=-ekJ_\theta A\cos\theta . \quad (81)$$

Note also that

$$\omega_\theta\equiv\frac{\partial H_0}{\partial J_\theta}=\omega+\frac{k^2}{m}J_\theta=\omega-kv_x , \quad (82)$$

so when $\Delta k_\theta\to 0$, $\omega-kv_x\to 0$, so we can use the Petrosky formalism.

The nonresonant ponderomotive Hamiltonian given by Eq. (63) is then

$$K_2=-\frac{k^2e^2}{4m^2}|\Phi|^2\frac{\partial}{\partial J_\theta}\frac{J_\theta^2}{\omega_\theta} .$$

Using $h_2=e^2|E|^2/4m\omega^2$, we have

$$K_2=\frac{e^2|E|^2}{4m\omega^2}\left[1+\frac{2kv_x}{(\omega-kv_x)}+\frac{k^2v_x^2}{(\omega-kv_x)^2}\right] , \quad (83)$$

which is the one-dimensional version of the well-known result[14]

$$K_2=\frac{e^2}{4m\omega^2}\mathbf{E}^*\cdot\left[1+\frac{\mathbf{k}\mathbf{v}_0+\mathbf{v}_0\mathbf{k}}{\omega-\mathbf{k}\cdot\mathbf{v}_0}+\frac{\mathbf{v}_0\mathbf{v}_0k^2}{(\omega-\mathbf{k}\cdot\mathbf{v}_0)^2}\right]\cdot\mathbf{E}^* . \quad (84)$$

The usefulness of the K-χ theorem is that this expression can be deduced by Eq. (75), by functional differentiation with respect to f_0. We should mention that we have inserted a factor of four in Eq. (84) so as to make Cary and Kaufman's notation conform with ours. This is due to the definition of the complex conjugate used to define real quantities, that is $(A+A^*)/2$ instead of $A+A^*$ which Cary and Kaufman use.

In view of our expression for the resonant ponderomotive Hamiltonian we postulate a complex generalization of the K-χ theorem, namely,

$$\int d\Theta\, d\mathbf{J}\, f_0(\Theta,\mathbf{J},t)K_2^{\text{res}}=-\frac{1}{16\pi}\int d\Theta\, \mathbf{E}^*\cdot\hat{\chi}_a\cdot\mathbf{E} , \quad (85)$$

where $\hat{\chi}_a$ is the anti-Hermitian part of the susceptibility. By following the previous analysis, Eq. (64) shows

$$\int dJ\, d\Theta\, f_0K_2^{\text{res}}=-\frac{i\pi e^2k^2|E|^2}{m^2\omega^2}\int dJ\, d\Theta\frac{\partial}{\partial J_\theta}J_\theta^2\delta(\omega_\theta)$$

$$=-\frac{i\pi\omega_p^2}{k}\frac{\partial f_0}{\partial v}\bigg|_{v=\omega/k}\int E^2d\theta . \quad (86)$$

We have to relate this to Landau damping and the appropriate form of the susceptibility. It is well known that in Landau damping the contribution of the pole at $v_x=\omega/k$ is needed in the Landau contour. The dielectric function ϵ is

$$\epsilon(\mathbf{k},\omega)=1-\frac{\omega_p^2}{k^2}\int dv_x\frac{\partial f_0(v_x)/\partial v_x}{v_x-\omega/k} . \quad (87)$$

Using the Plemelj formula, this expression is

$$\epsilon(\mathbf{k},\omega)=1-\frac{\omega_p^2}{k^2}P\int_{-\infty}^{\infty}dv_x\left[\frac{\partial f_0(v_x)/\partial v_x}{v_x-\omega/k}\right]-\pi i\frac{\omega_p^2}{k}\frac{\partial f_0}{\partial v_x}\bigg|_{v_x=\omega/k} . \quad (88)$$

The anti-Hermitian part of ϵ corresponds to the anti-Hermitian part of χ (since $\epsilon=1+\chi$) so clearly from Eq. (86) and Eq. (88), the complex K-χ theorem, given by Eq. (85), is satisfied. Finally it should be mentioned that these ideas may readily be extended to the magnetized plasma case.

VI. DISCUSSION

We have shown that Petrosky's resonant Lie-transform theory arises from the subdynamic formalism and that it may be used to derive a complex K-χ theorem. This theorem allows us to calculate the resonant contribution to the ponderomotive Hamiltonian from the susceptibility of the system. Although we discussed Petrosky's ideas from a scattering viewpoint, we would like to mention some of the mathematical assumptions in the subdynamic theory, in particular when dissipation is present.

It is well known that dissipative and nondissipative subdynamics is distinguished by the spectral properties of the resolvent. In fact, the Liouville operator has a continuous spectrum when dissipation occurs. Subdynamics also relies on the $z=0$ contribution and the requirement of regularity of certain operators in the lower-half complex plane. These assumptions allow us to develop a subdynamics with dissipation and is an analytic continuation of the nondissipative theory. We will not discuss this in much detail here, but when dissipation occurs, such as at resonance, subdynamics can be rigorously justified by the method of complex scaling,[15,16] although some aspects are still controversial.[17] In fact, it is possible[13] to use a complex scaled Liouville operator for the Landau resonance in the resolvent and produce a rigorously consistent mathematical theory of resonant wave-particle interactions. We mention this only to point out that the resonant theory proposed in this paper may be justified from complex-scaling arguments and not so much by the conventional approaches of the Brussels school.[7] This follows because under certain specific mathematical assumption complex scaling shows that there exists a meromorphic continuation of some of the matrix elements of the resolvent operator, into the lower-half plane.

It is possible to show that Dewar's 1973 theory follows when the filter width $\Delta\omega\rightarrow 0$. Note the $\Pi(\omega_k-\mathbf{k}\cdot\mathbf{p}/m)$ operator in Dewar's work reduces to the principal part operator. It is easy to show then that the renormalized energy (or the ponderomotive Hamiltonian) is essentially the same as in our paper, although Dewar considers a spectrum of waves. Our theory is, of course, done in the Lie-transform language and not in the von Zeipel formalism.

An interesting consequence of the complex ponderomotive Hamiltonian $\mathscr{K}$ is that if we solve the oscillation-center equation[2]

$$\frac{\partial F}{\partial t}+L_{\mathscr{K}}F=0\ , \tag{89}$$

then the imaginary part of $\mathscr{K}\equiv K+iK^{\mathrm{res}}$ provides a natural resonance broadening. The nonresonant result introduces propagators proportional to $1/(\omega-\mathbf{k}\cdot\partial K/\partial\mathbf{p})$. If we replace K by $\mathscr{K}$, as expected from Eq. (89), we have $1/(\omega-\mathbf{k}\cdot\partial K/\partial\mathbf{p}-i\mathbf{k}\cdot\partial K^{\mathrm{res}}/\partial\mathbf{p})$, so the imaginary part is the resonant width for Landau damping. It is expected that resonant oscillation-center theory will have applications in plasma-turbulence theory.

Other possible applications that are of particular interest are (1) rederiving Johnston's[18] induced scattering results by the resonant theory in this paper; (2) calculation of adiabatic invariants across the separatrix;[19] (3) using $\mathscr{K}$ as the generator in the dissipative bracket formulations;[20,21] (4) deriving diffusion coefficients[4] and (5) developing a resonant-averaged Lagrangian theory[22] since ξ (the displacement vector) is related[23] to the generating function by $\xi=\partial w/\partial\mathbf{p}$.

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