PHYSICAL REVIEW A VOLUME 35, NUMBER 11 JUNE 1, 1987

Quantum theory of light propagation: Linear medium

I. Abram
Centre National d'Etudes des Telecommunications, 196 Avenue Henri Ravera, 92220 Bagneux, France
(Received 22 December 1986)

We have developed a quantum-mechanical formalism which permits the treatment of light propagation within the conceptual framework of quantum optics. The formalism rests on the calculation of the momentum operator for the radiation field, and yields directly a description for the spatial progression of the electromagnetic waves. In this paper we give a quantum-mechanical treatment for refraction and reflection by applying our formalism to propagation through a linear dielectric. The fidelity with which this formalism reproduces all results known from classical optics demonstrates its validity.

I. INTRODUCTION

Within the conventional formulation of the quantum theory of light, traveling-wave phenomena are described through the Hamiltonian of the electromagnetic field, and particularly through the temporal evolution of the spatial modes of the field. The modal Hamiltonian formalism is quite well suited for the description of radiative emission into the vacuum field:[1] When the Hamiltonian includes a term coupling an emitter to an infinity of modes, energy can be dissipated into the radiation field. On the other hand, the description of nonlinear wave interactions,[2] in which two (or more) waves interact with each other through the nonlinear polarization of a material, is quite cumbersome. The traveling waves have to be analyzed each into an infinity of modes, and a nonlinear polarization term which couples the individual modes is introduced in the Hamiltonian. Thus, the transfer of energy between the two waves is described as a dissipative process from one infinite manifold of modes into another. This complicated quantum-mechanical procedure contrasts sharply with the simplicity of the classical description of nonlinear optical phenomena: the spatial differential equations can be solved directly; energy is then transferred between the two waves as they advance together in the nonlinear material.

When the procedure used for nonlinear traveling-wave phenomena is applied to linear propagation through matter, the modal Hamiltonian formalism is seen to fail. Inclusion of the linear polarization term in the Hamiltonian

$$\mathscr{H}=\frac{1}{8\pi}\int_V (E^2+H^2+4\pi\chi E^2)dv \qquad (1.1)$$

(in standard notation) couples each mode to itself through a term quadratic in the electric field. This term leads to an increase in the energy of all the modes and thus gives rise to a renormalization of their frequencies. This result is, of course, *incorrect* since the frequency of a light beam traversing a dielectric is not changed. The linear polarization of the dielectric gives rise to refraction which affects only the spatial characteristics of light by renormalizing its wave vector and its phase velocity, as is known from classical optics. However, the modal Hamiltonian addresses only the temporal evolution of the free-space normal modes which constitute a fixed set of solutions of the spatial wave equation (with no polarization), and for this reason it cannot account for spatial changes. This failure of the modal Hamiltonian in describing propagation in a linear dielectric indicates that its applicability to the description of nonlinear propagation must also have many limitations.

The problem of propagation through a linear nondispersive dielectric, one of the most basic problems of classical optics, has not been treated yet in quantum optics, to our knowledge. A theory of phenomenological quantum electrodynamics in refractive media was developed quite early,[3] but has not be applied to quantum optics. In quantum optics, on the other hand, a procedure has been devised to circumvent the problem of a dielectric, by redefining the spatial modes in the cavity of quantization;[4,5] the dielectric function of the medium is introduced in the Helmholtz equation for the field in the cavity, and thus a new spatial periodicity (wave vector) is obtained for the modes, different from its free-space value. It is these new modes that are used in constructing traveling wave packets to describe optical phenomena occurring in the medium, including nonlinear wave interactions. Another procedure, based on the space-time analogy for steady-state propagation, has been devised to describe nonlinear propagation. By heuristically converting spatial progression into temporal evolution (through division by the speed of light) this procedure permits us to address nonlinear optical phenomena in a manner analogous to their corresponding classical treatment.[4] Formal space-time analogies have also been pointed out in the differential equations for the propagation of short light pulses.[6]

More recently, the problem of the quantum-mechanical treatment of light propagation has received renewed interest in connection with the description of nonclassical states of the radiation field, such as the "squeezed" states.[7] Work focused mainly on the description of diffraction by applying the Green's function for the classical diffraction problem directly to the quantum-mechanical field operators.[8]

Clearly, it is necessary to extend the traditional theory of quantum optics to describe propagative phenomena by developing a quantum-mechanical formalism that can treat traveling-wave phenomena without invoking the modal Hamiltonian, and can treat both linear and nonlinear propagation on the same footing. In developing this formalism, we may be guided by classical linear optics: Introduction of a dielectric in the path of a light beam does not change the total energy; the incident energy is simply redistributed into the transmitted and reflected waves. The quantum-mechanical Hamiltonian must thus remain unchanged. On the other hand, both the local energy density and the momentum of the wave (its wave vector) change because of the polarization induced in the dielectric. Clearly, the momentum is the concept that is appropriate for the description of refraction, and more generally of propagative phenomena within the framework of quantum optics: Since quantum mechanically space and momentum are conjugate variables, the momentum operator of the electromagnetic field permits the calculation of its spatial progression, just like the Hamiltonian permits the calculation of its temporal evolution. Use of the momentum operator to describe the spatial progression of a wave was first proposed by Shen.[4] However, this is the first paper where this proposal is thoroughly examined.

In this series of papers we develop the formalism for the quantum-mechanical treatment of the spatial progression of electromagnetic waves in matter, through the calculation of the momentum operator of the electromagnetic field. In this paper, we examine propagation through a linear dielectric, while in the following paper[9] we treat propagation through a nonlinear medium. When applied to a situation already treated in classical optics (such as refraction or reflection from a dielectric surface), this formalism reproduces all the known classical results. At the same time, however, it permits the treatment of purely quantum-mechanical propagative phenomena, such as propagation of nonclassical states, or spontaneously initiated stimulated emission.[9] The paper is organized as follows. In Sec. II we review some well-known results from classical electrodynamics, recasting them in a way that permits their direct application to the quantum-mechanical description of light propagation. The notation is introduced in Sec. III by examining quantum-mechanical propagation in the trivial case of plane waves propagating in free space. In Sec. IV we establish the quantum-mechanical Maxwell equations. These equations are solved for a linear dielectric in Sec. V, where we present a quantum-mechanical treatment of the refractive index, of propagation through a refractive medium, and of reflection on a dielectric surface. Finally, in Sec. VI we present a discussion of our results.

II. A REMINDER OF CLASSICAL ELECTRODYNAMICS

Propagation of the electromagnetic field is described through the Maxwell equations,

$$\nabla\times\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\ , \tag{2.1a}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\ , \tag{2.1b}$$

$$\nabla\cdot\mathbf{B}=0\ , \tag{2.1c}$$

$$\nabla\cdot\mathbf{D}=0\ , \tag{2.1d}$$

where $\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$ is the electric displacement, $\mathbf{B}$ is the magnetic induction, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths, respectively, $\mathbf{P}$ is the (linear or nonlinear) polarization induced in the medium, and c is the speed of light. We assume that there are no free charges or currents, and that we are dealing with nonmagnetic materials, so that $\mathbf{B}=\mathbf{H}$. We use Gaussian units throughout the paper.

For simplicity, we shall consider only the case of plane waves propagating along the z axis, with the electric field polarized along x, and the magnetic field along y. This reduces the Maxwell equations into scalar differential equations, the directions of all vectors being implicit. We shall further assume that light is propagating in linear dielectric, where the induced polarization is at all times proportional to the incident electric field,

$$P=\chi E\ , \tag{2.2a}$$

where χ is the (linear) susceptibility of the material, which we assume for simplicity to be a scalar (neglecting its tensorial properties), independent of frequency (no dispersion). It is convenient to define also the dielectric function ϵ of the material

$$\epsilon=1+4\pi\chi \tag{2.2b}$$

and the refractive index n

$$n=\sqrt{\epsilon}\ . \tag{2.2c}$$

The Maxwell equations can be rearranged to give the electromagnetic wave equation, which is usually the starting point for the classical treatment of (linear or nonlinear) light propagation. Alternatively, they may be rearranged to give energy and momentum flow equations, which can also be used as the starting points for treating propagation. This latter approach to the propagation of the electromagnetic field will be reviewed in this section. Its advantage over the wave equation is that it deals with observables (such as energy and momentum) whose quantum-mechanical equivalents can describe temporal evolution and spatial progression in the quantized electromagnetic field. However, the energy and momentum functions have to be chosen appropriately so that their use in the quantum-mechanical description of propagation gives the right results: When a light beam traverses a nonabsorbing dielectric, the number of photons contained in the beam, their frequency, and the total energy of the beam must remain unchanged, while its wave vector and the total momentum must increase proportionally to the refractive index.

The material polarization P is a key concept in classical electromagnetism, in that it can store part of the electromagnetic energy: The energy density u within the volume of a dielectric is greater than that of the free field, and this increase is attributed to the deformation of the material

$$u=\frac{1}{8\pi}(E^2+H^2+4\pi PE)\ . \tag{2.3}$$

In quantum optics, however, the bookkeeping is different. The energy is carried by the photons and by the material excitations. Far from all resonances, the material undergoes transitions only to virtual states which generally live less than 10^{-15} sec for a transparent material in the optical region. Thus, for time scales of experimental interest, energy is carried only by the field. This implies that for an experiment that is not fast enough to resolve the processes associated with the virtual excitations, the increase in energy density in a transparent dielectric is regarded as an increase in the photon density. This partition between electromagnetic and mechanical energies suggests the Minkowski definition for the momentum density[10]

$$\mathbf{g}=\frac{1}{4\pi c}\mathbf{D}\times\mathbf{B} \tag{2.4}$$

(which includes both the momentum of the field and the momentum associated with the material polarization) as the proper momentum function for quantum optics. The usual (Abraham—von Laue) definition for the momentum density

$$\mathbf{g}'=\frac{1}{4\pi c}\mathbf{E}\times\mathbf{H} \tag{2.5}$$

refers to the momentum that can be attributed to the field alone, and is independent of the medium in which light propagates. Its use would require an explicit treatment of the momentum associated with the virtual excitations of the material. For this reason, we shall adopt the definition (2.4), even though its use in classical electrodynamics is subject to controversy on thermodynamic grounds. We may also define the Poynting vector $\mathbf{S}$, which gives a measure of the energy flux as

$$\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\ . \tag{2.6}$$

The energy flux is conserved when a light beam goes through a dielectric; thus, an energy function proportional to $\mathbf{S}$ would be the proper quantum-mechanical Hamiltonian.

For the classical field, and in the absence of an external driving agent, the energy-flow equation may be written as

$$-\nabla\cdot\mathbf{S}=\frac{\partial}{\partial t}\left[\frac{E^2+H^2}{8\pi}\right]+E\frac{\partial P}{\partial t}\ , \tag{2.7a}$$

while the momentum-flow equation is

$$-\frac{\partial}{\partial t}\mathbf{g}=\nabla\cdot\overleftrightarrow{\mathbf{T}}\ , \tag{2.7b}$$

where $\overleftrightarrow{\mathbf{T}}$ is the unsymmetrical Maxwell stress tensor,[3,10] whose elements are

$$T_{\alpha\beta}=\frac{1}{4\pi}[E_\alpha D_\beta+H_\alpha B_\beta-\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})\delta_{\alpha\beta}] \tag{2.7c}$$

with $\alpha,\beta=x,y,z$. For a field that decays at infinity, Eqs. (2.7) give energy and momentum conservation upon integration over a volume *totally enclosing* the field,

$$\frac{\partial U}{\partial t}=0\ , \tag{2.8a}$$

$$\frac{\partial G}{\partial t}=0\ , \tag{2.8b}$$

where

$$U=\int_V u\,dv \tag{2.9a}$$

is the total energy in the closed volume V, while

$$G=\int_V g\,dv \tag{2.9b}$$

is the total momentum. To obtain the actual values of U and G, however, we have to consider an *open* finite volume through which flow can occur. For one-dimensional propagation Eqs. (2.7) reduce to scalar form

$$-\frac{\partial S}{\partial z}=\frac{\partial u}{\partial t}\ , \tag{2.10a}$$

$$-\frac{\partial g}{\partial t}=\frac{\partial u}{\partial z}\ , \tag{2.10b}$$

where we have used $T_{zz}=u$, in Eq. (2.10b).

Equations (2.10) are usually integrated over a fixed volume, in order to relate the energy flux to the change in energy content in the volume. Here, however, in order to make contact with the energy and momentum calculations for the quantized electromagnetic field, we shall solve Eqs. (2.10) for the case of a finite light pulse, rather than for a fixed volume. For simplicity, we may consider a "square" pulse of length L (in free space), consisting of a plane wave packet. When the pulse enters a dielectric of index n, its velocity changes to $v=c/n$ while all the wavelengths that compose it become $\lambda'=\lambda/n$. Thus, a pulse of length L in free space shortens to $L'=L/n$. However, the duration of the pulse remains constant,

$$t'=\frac{L'}{v}=\frac{L}{c}\ , \tag{2.11}$$

and is therefore independent of the medium. The energy and momentum carried by this pulse may be calculated through Eqs. (2.10), by considering a surface A placed downstream from the pulse. Since the electric and magnetic fields vary as $z-vt$, the total energy and momentum crossing the surface may be obtained by integrating Eqs. (2.10) over the time and over space, giving

$$U=SAt'=\frac{SAL}{c}=\frac{SV}{c}\ , \tag{2.12a}$$

$$G=\frac{uV}{c}\ , \tag{2.12b}$$

where V is the volume occupied by the pulse in free space. For a general state of the radiation field, which is a superposition of forward- and backward-going waves (varying as $z-vt$ and $z+vt$, respectively) Eqs. (2.12) become

$$U=\frac{V}{c}(S_+-S_-)\ , \tag{2.13a}$$

$$G=\frac{V}{c}(u_+-u_-)\ , \tag{2.13b}$$

where the $+$ $(-)$ index, the energy density and flux due to, the forward (backward) waves alone. Clearly, Eqs.

(2.12) and (2.13) are independent of the medium in which the pulse propagates, and permit us to keep track of the energy and the momentum of the pulse as it traverse different media.

One of the basic results of classical optics is that the energy flux is conserved when an electromagnetic wave encounters a dielectric. Thus, for normal incidence on a sharp vacuum-dielectric interface, we have the well-known relationship among the incident (I), reflected (R), and transmitted (T) waves

$$S_I = S_R + S_T \tag{2.14a}$$

and, in particular,

$$S_R = \left[\frac{n-1}{n+1}\right]^2 S_I \ , \tag{2.14b}$$

$$S_T = \frac{4n}{(n+1)^2} S_I \ , \tag{2.14c}$$

while the electric fields follow

$$E_R = \left[\frac{n-1}{n+1}\right] E_I \ , \tag{2.14d}$$

$$E_T = \frac{2}{n+1} E_I \ , \tag{2.14e}$$

since the electric field in each medium is related to the energy flux through

$$S = nE^2 \ . \tag{2.14f}$$

If the refractive index at the surface changes gradually from 1 to n so that there are no reflections, or if the dielectric is entered through an "antireflective" coating, then the flux in the dielectric equals the flux in free space,

$$S_T = S_I \ . \tag{2.15a}$$

Equation (2.15a) together with Eq. (2.12a) imply that the energy content of a pulse is not affected by the medium in which the pulse is traveling, as long as there are no reflection losses. In the absence of reflections, the electric and magnetic fields of the transmitted wave in the dielectric are related to the corresponding incident fields (in free space) by

$$E_T = E_I / \sqrt{n} \ , \tag{2.15b}$$

$$H_T = \sqrt{n} H_I \ . \tag{2.15c}$$

The energy density inside the dielectric is therefore increased with respect to the free-space energy density by a factor of n,

$$u_T = n u_I \ , \tag{2.15d}$$

which implies a similar increase for the total momentum of the pulse

$$G_T = n G_I \ . \tag{2.15e}$$

That is, the momentum carried by the pulse, defined according to Eqs. (2.4) and (2.9b), is always proportional to the wave vector of its photons, which in turn is proportional to the refractive index.

We are now in a position to apply the above discussion to the quantized electromagnetic field. For this, we note that the periodic boundary conditions used in quantum optics for the cavity of quantization are equivalent to considering the cavity as an open volume through which flow occurs cyclically: The flux exiting on the right reenters on the left. Thus, a propagating field may be considered as a "square" pulse that fills exactly the volume of quantization at $t=0$. Propagation causes the energy (and the momentum) of the square pulse to flow through the boundary of the cavity, while the periodic boundary conditions replenish the cavity through the opposite boundary. The energy and momentum of the cavity of quantization correspond to the energy and momentum of that pulse. They are therefore given by Eqs. (2.13) (with V being now the volume of the cavity of quantization) and exhibit the proper behavior for the description of temporal evolution and spatial progression: The energy of the cavity is independent of the medium it contains, while its momentum is directly proportional to the wave vector of the excited field modes, and depends on the medium. Equations (2.13) will therefore be used as the starting point for the quantum-mechanical treatment of light propagation. Their use to describe quantum-mechanical light propagation through a dielectric implies that the optical path length of the cavity of quantization (i.e., the number of wavelengths it contains for each mode) remains constant upon introduction of a dielectric. This insures that the periodic-boundary conditions of the cavity are the same irrespective of whether the cavity is empty or contains a dielectric, and establishes a correspondence between the free-field and refracted-wave modes of the cavity. At the same time, it means that the physical length of the cavity of quantization shortens in the presence of a dielectric, so that energy may be conserved. This result may seem paradoxical when thinking of the cavity as a finite box of fixed size and set boundary conditions. However, it should be remembered that the periodic-boundary cavity is simply a device for treating the propagating modes in infinite space where energy conservation (as light traverses different media) is insured by Eq. (2.8a).

We note that in this treatment of quantum optics energy is conserved while *momentum is not conserved* when entering a dielectric, a problem that exists also in the conventional treatment of classical optics. This apparent discrepancy arises from the phenomenological treatment of the material response through the polarization (or susceptibility) rather than through the explicit treatment of the transitions to virtual states undergone by the atoms when the dielectric is traversed by a light beam. Thus, processes associated with these transitions, such as the recoil of the atoms and the resulting Doppler shift of the field, are neglected. However, we shall retain this level of approximation in our formalism, since its assumptions are compatible with the times scales and experimental conditions of quantum optics.

III. QUANTUM PROPAGATION IN FREE SPACE

Following any classic textbook,[1] we can quantize the electromagnetic field in a box large compared with experimental dimensions under periodic-boundary conditions. For simplicity, we look only at propagation along the $\pm z$ axis and consider only the set $\{\pm j\}$ of longitudinal modes having the lowest-order transverse structure (e.g., TEM_{00} or plane waves), with the electric field polarized along x, and the magnetic field along y. The electromagnetic vector potential operator $\hat{A}$ is usually written as

$$\hat{A}(z,t)=c\sum_j\left[\frac{2\pi}{V\omega_j}\right]^{1/2}(\hat{a}_j^\dagger e^{i\omega_j t-ik_jz+i\phi}+\hat{a}_je^{-i\omega_jt+ik_jz-i\phi})\ , \quad (3.1a)$$

where $\hat{a}_j^\dagger,\hat{a}_j$ are the creation, annihilation operators for a photon in the jth mode of wave vector k_j (with $k_{-j}=-k_j$) and frequency $\omega_j=c\,|k_j|$. To simplify notation, in all equations we use $\hbar=1$, and omit unit vectors (x, y, and z) since the directions of all vector operators are fixed. The $\hat{a}_j^\dagger,\hat{a}_j$ operators follow Bose commutation relations. It is convenient to rearrange Eq. (3.1a) in a manner that is familiar to solid-state physics, by grouping together the creation and the annihilation operators of two counterpropagating waves,

$$\hat{A}(z,t)=c\sum_j\left[\frac{2\pi}{V\omega_j}\right]^{1/2}(\hat{a}_j^\dagger e^{i\omega_jt}+\hat{a}_{-j}e^{-i\omega_jt})e^{-ik_jz}\ , \quad (3.1b)$$

where we have chosen the initial phase $\phi=0$ for simplicity. In this paper we shall not treat ϕ explicitly, since linear-propagation phenomena do not depend on the initial phase of the field. Thus, in all our results, a nonzero initial phase can be treated simply by considering the temporal evolution of the relevant operators, and translating the time origin appropriately. The individual j components thus defined correspond to the complex "coefficients" of the spatial Fourier expansion of $\hat{A}$. The electric and magnetic field operators may then be obtained as

$$\hat{E}(z,t)=-\frac{1}{c}\frac{\partial}{\partial t}\hat{A}=\sum_j\hat{e}_j=\sum_j -i\left[\frac{2\pi\omega_j}{V}\right]^{1/2}(\hat{b}_j^\dagger-\hat{b}_{-j}) \quad (3.2a)$$

and

$$\hat{H}(z,t)=\frac{\partial}{\partial z}\hat{A}=\sum_j\hat{h}_j=\sum_j -is_j\left[\frac{2\pi\omega_j}{V}\right]^{1/2}(\hat{b}_j^\dagger+\hat{b}_{-j})\ . \quad (3.2b)$$

Throughout the paper, the caret distinguishes a Hilbert-space operator from the corresponding classical quantity. In Eqs. (3.2) $s_j=\pm1$ is the sign of j, and the t and z dependence is understood to be implicit in the $\hat{b}_j^\dagger,\hat{b}_j$ operators. That is,

$$\hat{b}_j^\dagger=\hat{a}_j^\dagger e^{i\omega_jt-ik_jz}\ , \quad (3.3a)$$

$$\hat{b}_j=\hat{a}_je^{-i\omega_jt+ik_jz}\ . \quad (3.3b)$$

We note that the individual components $\hat{e}_j$ and $\hat{h}_j$ of the electric and magnetic fields are not Hermitian,

$$\hat{e}_j^\dagger=\hat{e}_{-j}\ , \quad (3.4a)$$

$$\hat{h}_j^\dagger=\hat{h}_{-j}\ , \quad (3.4b)$$

and thus do not constitute observable quantities. They are simply a convenience to facilitate bookkeeping when performing a spatial integration. The electric field of the jth mode on the other hand, is an operator of the form

$$\hat{E}_j=-i\left[\frac{2\pi\omega_j}{V}\right]^{1/2}(\hat{b}_j^\dagger-\hat{b}_j)\ , \quad (3.5)$$

which is Hermitian, and thus constitutes an observable. Similarly, this applies for the magnetic field operator $\hat{H}_j$.

Another way of rearranging the electric and magnetic field operators is to distinguish a positive- and a negative-frequency part. The positive-frequency part of the electric field operator corresponds to the sum of all the annihilation-operator terms in Eq. (3.2a), while the negative-frequency part corresponds to the sum of all the creation-operator terms.

The implicit spatial and temporal dependence of the $\hat{b}_j^\dagger,\hat{b}_j$ operators constitutes the propagation of the electromagnetic field. This propagation (spatial progression and temporal evolution) can be recalculated quantum mechanically by evaluating the total momentum and energy (Hamiltonian) operators for the electromagnetic field, and setting up the corresponding differential equations. The calculation of these operators can be done by using Eqs. (3.2), which permit us to convert all classical quantities associated with the electromagnetic field to the corresponding quantum-mechanical operators, simply by replacing the electric and magnetic fields in a classical expression by Eqs. (3.2).

Thus, the energy-density operator can be written as

$$\hat{u}=\frac{1}{8\pi}(\hat{E}^2+\hat{H}^2)=\frac{1}{8\pi}\sum_{j,l}(\hat{e}_j\hat{e}_l+\hat{h}_j\hat{h}_l)\ . \quad (3.6)$$

This operator can be used in the calculation of the total momentum of the field, through the quantum-mechanical equivalent of Eq. (2.13b). However, since this latter equation results from integration over space of the spatial derivative of $\hat{u}$, the cross terms with $l\neq -j$ may be eliminated in Eq. (3.6). The reason is that these terms are subject to spatial oscillations of the form $e^{i(k_j+k_l)z}$, and thus cancel out upon spatial integration, while the terms with $l=-j$ which do not oscillate, survive. Equation (3.6) can then be written as

$$\hat{u}=\sum_j\hat{u}_j\ , \quad (3.7a)$$

where

$$\hat{u}_j=\frac{1}{8\pi}(\hat{e}_j\hat{e}_{-j}+\hat{h}_j\hat{h}_{-j})=\frac{\omega_j}{2V}(\hat{b}_j^\dagger\hat{b}_j+\hat{b}_{-j}^\dagger\hat{b}_{-j}+1)\ . \quad (3.7b)$$

Thus, the energy-density operator is given by

$$\hat{u}=\sum_j \frac{\omega_j}{V}\hat{b}_j^\dagger \hat{b}_j \ . \tag{3.7c}$$

The total momentum $\hat{G}$ then is

$$\hat{G}=\frac{V}{c}(\hat{u}_+-\hat{u}_-)=\sum_j k_j \hat{b}_j^\dagger \hat{b}_j \ . \tag{3.8}$$

This is the operator that describes the spatial characteristics of the electromagnetic field: As is known from elementary quantum mechanics, the momentum operator gives the spatial derivative of any wave function to which it is applied,

$$\hat{G}\psi=-i\frac{\partial \psi}{\partial z} \ , \tag{3.9}$$

while the spatial derivative of any operator $\hat{Q}$ is given by a Heisenberg-like equation involving the momentum,

$$\frac{\partial \hat{Q}}{\partial z}=-i[\hat{G},\hat{Q}] \ . \tag{3.10}$$

Equation (3.10) is a differential equation that permits the calculation of the spatial dependence of the field operators, once the momentum operator is known. It thus gives the spatial progression of an electromagnetic wave, just like the ordinary Heisenberg equation gives its temporal evolution.

For free-space propagation of plane waves, the momentum operator of Eq. (3.8) gives the differential equation (e.g., for the operator $\hat{b}_j$),

$$\frac{\partial \hat{b}_j}{\partial z}=-i[\hat{G},\hat{b}_j]=ik_j\hat{b}_j \ . \tag{3.11}$$

Integration of Eq. (3.11) gives the spatial progression of the annihilation operator for the jth plane wave as

$$\hat{b}_j(z)=\hat{b}_j e^{ik_j z} \ , \tag{3.12}$$

where the phase at $z=0$ is chosen as $\phi=0$. The spatial progression of all other operators may be calculated in a similar fashion.

Similarly, we may calculate the Poynting vector operator as

$$\hat{S}=\frac{c}{4\pi}\hat{E}\,\hat{H}=\frac{c}{4\pi}\sum_j \hat{e}_j \hat{h}_{-j} \tag{3.13a}$$

giving

$$\hat{S}=\sum_{j(>0)} \frac{c\omega_j}{V}(\hat{b}_j^\dagger \hat{b}_j-\hat{b}_{-j}^\dagger \hat{b}_{-j}) \ . \tag{3.13b}$$

The total energy of the free field inside the cavity of quantization is thus

$$\hat{\mathscr{H}}=\hat{U}=\frac{V}{c}(\hat{S}_+-\hat{S}_-)=\sum_j \omega_j \hat{b}_j^\dagger \hat{b}_j \ . \tag{3.14}$$

This operator corresponds to the Hamiltonian of the free field, and describes the oscillatory temporal evolution of each mode through the Heisenberg equation, for example,

$$\frac{\partial \hat{b}_j}{\partial t}=i[\hat{\mathscr{H}},\hat{b}_j]=-i\omega_j\hat{b}_j \ , \tag{3.15a}$$

to give

$$\hat{b}_j(t)=\hat{b}_j e^{-i\omega_j t} \ . \tag{3.15b}$$

This concludes the quantum-mechanical calculation of light propagation in free space. The results are, of course, trivial, as they simply reproduce the spatial progression and temporal evolution given in the definition of the $\hat{b}_j$ operators [Eq. (3.3)]. They demonstrate, however, the properties of the different quantum-mechanical operators pertaining to the electromagnetic field that are used in this paper.

IV. THE QUANTUM-MECHANICAL MAXWELL EQUATIONS

The individual components of the rearranged electric and magnetic field operators defined in Eq. (3.2),

$$\hat{e}_j=-i\left[\frac{2\pi\omega_j}{V}\right]^{1/2}(\hat{b}_j^\dagger-\hat{b}_{-j}) \ , \tag{4.1a}$$

$$\hat{h}_j=-is_j\left[\frac{2\pi\omega_j}{V}\right]^{1/2}(\hat{b}_j^\dagger+\hat{b}_{-j}) \ , \tag{4.1b}$$

have a structure similar to that of the momentum and position operators (respectively) for the harmonic oscillator, however, with a very important difference: Eqs.(4.1) are linear combinations of a creation and an annihilation operator of *opposite* indices ($\pm j$). This gives rise to a modified operator algebra (with respect to that of the harmonic oscillator) in that it is the $\hat{e}_j$ and $\hat{h}_j$ operators of *opposite* index that give a nonzero commutator,

$$[\hat{e}_j,\hat{e}_l]=[\hat{h}_j,\hat{h}_l]=0 \ , \tag{4.2a}$$

$$[\hat{e}_j,\hat{h}_l]=s_{-j}\left[\frac{4\pi\omega_j}{V}\right]\delta_{-j,l} \ , \tag{4.2b}$$

where $\delta_{j,l}$ is the Kronecker δ symbol. The structure of Eqs. (4.1) implies that $\hat{e}_j$ and $\hat{h}_j$ follow equations of motion similar to those of the harmonic oscillator. We shall show in this section that these harmonic-oscillator-like equations of motion are the quantum-mechanical equivalent of the Maxwell equations (2.1a) and (2.1b), which for one-dimensional propagation through a linear isotropic medium can be written as

$$\frac{\partial H}{\partial z}=-\frac{\epsilon}{c}\frac{\partial E}{\partial t} \ , \tag{4.3a}$$

$$\frac{\partial E}{\partial z}=-\frac{1}{c}\frac{\partial H}{\partial t} \ . \tag{4.3b}$$

The temporal derivatives of $\hat{e}_j$ and $\hat{h}_j$ can be calculated through the Hamiltonian (2.13a),

$$\frac{\partial \hat{e}_j}{\partial t}=i[\hat{\mathscr{H}},\hat{e}_j]=i\frac{V}{c}([\hat{S},\hat{e}_j]_+-[\hat{S},\hat{e}_j]_-) \ , \tag{4.4a}$$

where the $+$ $(-)$ indices of the commutators refer to the forward- (backward-) going waves as in the definition of $\hat{\mathscr{H}}$. Evaluating the commutator of $\hat{e}_j$ with the Poynting vector operator $\hat{S}$,

$$\hat{S}=\frac{c}{4\pi}\sum_j \hat{e}_j\hat{h}_{-j}\ , \tag{4.4b}$$

we have

$$\frac{\partial \hat{e}_j}{\partial t}=i\omega_j s_j\{(\hat{e}_j)_+-(\hat{e}_j)_-\}=i\omega_j s_j\hat{h}_j \tag{4.4c}$$

and similarly

$$\frac{\partial \hat{h}_j}{\partial t}=i\omega_j s_j\{(\hat{h}_j)_+-(\hat{h}_j)_-\}=i\omega_j s_j\hat{e}_j\ , \tag{4.4d}$$

where, according to the definition (4.1),

$$(\hat{e}_j)_+=-i\left[\frac{2\pi\omega_j}{V}\right]^{1/2}\hat{b}_j^\dagger \ \text{ if } j>0 \tag{4.5a}$$

$$=i\left[\frac{2\pi\omega_j}{V}\right]^{1/2}\hat{b}_{-j} \ \text{ if } j<0\ , \tag{4.5b}$$

and so on, of $(\hat{e}_j)_-$, $(\hat{h}_j)_+$, and $(\hat{h}_j)_-$. Equations (4.4) show that $\hat{e}_j$ and $\hat{h}_j$ are each other's derivative, and are time harmonic with frequency ω_j, both in free space and inside matter, since the Poynting vector operator has the same form (4.4b) irrespective of the medium in which light is propagating.

The spatial derivatives are given through the momentum operator (2.13b). To evaluate them, we first calculation the commutators of $\hat{e}_j$ and $\hat{h}_j$ with the energy-density operator inside a dielectric

$$[\hat{u},\hat{e}_j]=\frac{1}{8\pi}\sum_l[(\epsilon\hat{e}_l\hat{e}_{-l}+\hat{h}_l\hat{h}_{-l}),\hat{e}_j]=\frac{s_j\omega_j}{V}\hat{h}_j \tag{4.6a}$$

and

$$[\hat{u},\hat{h}_j]=\frac{s_j\omega_j}{V}\epsilon\hat{e}_j\ . \tag{4.6b}$$

The spatial derivatives are then given by

$$\frac{\partial \hat{e}_j}{\partial z}=-i[\hat{G},\hat{e}_j]=-is_j\frac{\omega_j}{c}\{(\hat{h}_j)_+-(\hat{h}_j)_-\}$$

$$=-\frac{1}{c}\frac{\partial}{\partial t}\hat{h}_j \tag{4.7a}$$

and

$$\frac{\partial \hat{h}_j}{\partial z}=-i[\hat{G},\hat{h}_j]=-is_j\frac{\omega_j\epsilon}{V}\{(\hat{e}_j)_+-(\hat{e}_j)_-\}$$

$$=-\frac{\epsilon}{c}\frac{\partial}{\partial t}\hat{e}_j\ . \tag{4.7b}$$

Clearly, Eqs. (4.7) have a form identical to that of the classical Maxwell equations (4.3), indicating that the Maxwell equations are implicit in the structure of the operators (4.1), and correspond simply to their equations of motion. This conclusion is, of course, trivial when applied to the free field, since the electric and magnetic field operators were defined by solving the Maxwell equations in free space. However, Eqs. (4.7) indicate that this conclusion is much more general, and applies also to propagation through matter. That is, although the operators $\hat{e}_j$ and $\hat{h}_j$ are initially defined for the free field, their equations of motion under the Hamiltonian and momentum operators of the form (2.13) are valid also *inside a material system,* since they reproduce the Maxwell equations. It is therefore possible to describe the temporal evolution and spatial progression of the electromagnetic field inside a material within the framework of the quantum-mechanical equations of motion. Alternatively, the Maxwell equations inside the material may be solved quantum mechanically through the same algebraic techniques of second quantization that permit the solution of the equations of motion for Bose operators.

To this end we note that the commutators of $\hat{e}_j$ and $\hat{h}_j$ with the energy-density operator (4.6) have the same form as the equations of motion for the momentum and position operators under the Hamiltonian of a harmonic oscillator with a change in its mass, from m to m/ϵ. The solution to this problem is well known: Diagonalization of the Hamiltonian yields the renormalization of the oscillation frequency due to the mass change. The same method can be applied to the solution of the Maxwell equations (4.7). Diagonalization of the energy-density operator should give the renormalization of the wave vector due to refraction.

V. REFRACTION AND REFLECTION

We consider the free-space electromagnetic field quantized inside a large volume under periodic-boundary conditions. If the cavity of quantization is filled with a linear dielectric material, the electromagnetic energy-density operator is given by Eqs. (2.3), (2.2), and (3.2) as

$$\hat{u}=\frac{1}{8\pi}(\hat{E}^2+\hat{H}^2+4\pi\chi\hat{E}^2)$$

$$=\frac{1}{8\pi}\sum_j(\hat{e}_j\hat{e}_{-j}+\hat{h}_j\hat{h}_{-j}+4\pi\chi\hat{e}_j\hat{e}_{-j}) \tag{5.1}$$

where, as discussed in Sec. III, the cross terms are eliminated when dealing with the field inside a large volume. Substituting $\hat{e}_j$ and $\hat{h}_j$ in terms of the free-field Bose operators, the jth component of the energy-density operator can be written as

$$\hat{u}_j=\frac{\omega_j}{2V}\{\hat{b}_j^\dagger\hat{b}_j+\hat{b}_{-j}^\dagger\hat{b}_{-j}-2\pi\chi(\hat{b}_j^\dagger-\hat{b}_{-j})(\hat{b}_{-j}^\dagger-\hat{b}_j)\}\ . \tag{5.2}$$

Thus, inside a dielectric, the energy-density operator $\hat{u}$ is not diagonal when expressed in terms of the free-field operators. This implies that the free-field states are not momentum eigenstates in the dielectric, since the linear polarization introduces a coupling between the free-field $\pm j$ waves.

The energy-density operator $\hat{u}$ may be diagonalized through a Bogolyubov transformation which permits us to express Eq. (5.2) in terms of refracted-wave operators. This transformation consists of the application of a unitary operator of the form

$$e^{\gamma\hat{R}} , \tag{5.3a}$$

where γ is a real number, while $\hat{R}$ is an anti-Hermitian operator of the form

$$\hat{R}=\sum_j (\hat{b}_j\hat{b}_j-\hat{b}_j^\dagger\hat{b}_{-j}^\dagger)=\sum_j (\hat{B}_j\hat{B}_{-j}-\hat{B}_j^\dagger\hat{B}_{-j}^\dagger) . \tag{5.3b}$$

It relates the refracted-wave creation (annihilation) operators $\hat{B}_j^\dagger/(\hat{B}_j)$ to the corresponding free-field operators through

$$\hat{B}_j^\dagger=e^{-\gamma\hat{R}}\hat{b}_j^\dagger e^{\gamma\hat{R}}=(\cosh\gamma)\hat{b}_j^\dagger-(\sinh\gamma)\hat{b}_{-j} , \tag{5.4a}$$

$$\hat{B}_j=e^{-\gamma\hat{R}}\hat{b}_j e^{\gamma\hat{R}}=(\cosh\gamma)\hat{b}_j-(\sinh\gamma)\hat{b}_{-j}^\dagger , \tag{5.4b}$$

and

$$\hat{b}_j^\dagger=e^{\gamma\hat{R}}\hat{B}_j^\dagger e^{-\gamma\hat{R}}=(\cosh\gamma)\hat{B}_j^\dagger+(\sinh\gamma)\hat{B}_{-j} , \tag{5.5a}$$

$$\hat{b}_j=e^{\gamma\hat{R}}\hat{B}_j e^{-\gamma\hat{R}}=(\cosh\gamma)\hat{B}_j+(\sinh\gamma)\hat{B}_{-j}^\dagger . \tag{5.5b}$$

Substituting Eqs. (5.5) into Eq. (5.2) we can eliminate the off-diagonal terms (the cross products of $+j$ and $-j$ operators) for

$$\gamma=\frac{1}{4}\ln(1+4\pi\chi)=\tfrac{1}{2}\ln n \tag{5.6}$$

to obtain

$$\hat{u}_j=\frac{n\omega_j}{2V}(\hat{B}_j^\dagger\hat{B}_j+\hat{B}_{-j}^\dagger\hat{B}_{-j}) , \tag{5.7}$$

where n is the refractive index. The eigenvalues of the energy-density inside the dielectric are thus increased by a factor of n with respect to the corresponding values in free space, as in the classical case (2.15d). The momentum operator is then given by

$$\hat{G}=\frac{V}{c}(\hat{u}_+-\hat{u}_-)=\sum_j K_j\hat{B}_j^\dagger\hat{B}_j \tag{5.8}$$

with $K_j=nk_j$. The spatial progression of the refracted wave is given by Eqs. (3.10) and (5.8) as

$$\hat{B}_j(z)=\hat{B}_j e^{iK_jz} \tag{5.9}$$

as expected from classical considerations. We note that in this formalism the exact form of the refractive index as a square root arises from the nonperturbative treatment of the antiresonant terms in Eq. (5.2). These terms have the form $\hat{b}_j^\dagger\hat{b}_{-j}^\dagger+\hat{b}_j\hat{b}_{-j}$, oscillate rapidly at a frequency of $2\omega_j$, and do not conserve energy to first order. However, if these terms are neglected, the refractive index becomes $n=1+2\pi\chi$ [by inspection of Eq. (5.2)], which is the result of first-order perturbation theory, and is valid for small χ.

The unitary operator (5.3) that diagonalizes the energy density relates all refracted-wave operators to their free-field equivalents, through Eqs. (5.4) and (5.5). Thus, the electric and magnetic field operators inside the dielectric can be obtained by inserting Eqs. (5.5) into Eqs. (3.2) as

$$\hat{E}(z,t)=\sum_j -i\left[\frac{2\pi\omega_j}{nV}\right]^{1/2}(\hat{B}_j^\dagger-\hat{B}_{-j}) \tag{5.10a}$$

and

$$\hat{H}(z,t)=\sum_j -is_j\left[\frac{2\pi n\omega_j}{V}\right]^{1/2}(\hat{B}_j^\dagger+\hat{B}_{-j}) . \tag{5.10b}$$

Similarly, the Poynting vector operator inside the dielectric is given by

$$\hat{S}=\sum_{j(>0)}\frac{c\omega_j}{V}(\hat{B}_j^\dagger\hat{B}_j-\hat{B}_{-j}^\dagger\hat{B}_{-j}) \tag{5.11}$$

so that the Hamiltonian can be calculated through Eq. (2.13a) as

$$\hat{\mathscr{H}}=\sum_j \omega_j\hat{B}_j^\dagger\hat{B}_j . \tag{5.12}$$

Thus, the temporal evolution of the refracted wave is given by

$$\hat{B}_j(t)=\hat{B}_j e^{-i\omega_j t} . \tag{5.13}$$

Equations (5.8) and (5.12) show that both the Hamiltonian and the momentum operator are diagonal in the refracted-wave basis set. The refracted-wave Hamiltonian retains the same eigenvalues as the free-field Hamiltonian, whereas the eigenvalues of the refracted-wave momentum operator are renormalized by a factor of n with respect to the corresponding free-field eigenvalues. This is, of course, in accord with the well-known result of classical optics, that inside a linear dielectric the frequency of a time-harmonic electromagnetic wave retains its free-space value, whereas its wave vector is changed by the refractive index. Thus, Eqs. (5.8) and (5.12) describe completely the propagation of the electromagnetic field inside a dielectric, yielding the results expected from classical optics.

One of the basic problems of classical optics is the treatment of a vacuum-dielectric interface. This problem has been addressed through the conventional formulation of quantum optics,[5] by first redefining the radiation modes so that the new modes consist each of three segments, representing the incident, reflected, and transmitted (possibly evanescent) waves; the new modes are then quantized in a manner analogous to that of the free-space modes. However, the mode redefinition requires the *full classical solution* of the wave equation at the interface, implying that the spatial progression of the light waves inside the dielectric is still addressed within a classical framework. We shall show that the relationship between the free-field and refracted-wave operators permits a rigorous quantum-mechanical treatment of the vacuum-dielectric interface. We note, however, that since in this paper we consider only one-dimensional propagation (and consequently normal incidence on the interface) we do not get evanescent wave solutions except, of course, when the refractive index is purely imaginary as in the polariton problem.

We now consider that the volume of quantization of the electromagnetic field consists of two half-spaces separated by a plane interface at $z=0$, perpendicular to the direction of propagation of the plane-wave modes of the field. The $-z$ half-space is empty (free space) so that the free-field energy and momentum operators (3.8 and 14) are applicable to it, while the $+z$ half-space is filled with a linear dielectric, so that the refracted-wave energy and momentum operators describe propagation in it. We consider a forward-going wave packet of finite extent, so that it is initially totally within the empty half-space, and can thus be described completely in terms of free-field operators.

If the interface is such that the refractive index varies gradually between 1 and n, so that reflections are eliminated, then the forward-going wave packet enters the dielectric with no loss. The smooth variation of the refractive index can be formalized as an "adiabatic switching" of the interaction term proportional to χ in Eqs. (5.1) and (5.2), so that when the wave packet enters the dielectric all free-field eigenstates in the $-z$ half-space go over to the corresponding refracted-wave eigenstates in the $+z$ half-space. Equivalently, the free-field creation (annihilation) operators for each mode go over to the corresponding refracted-wave operators. We may thus consider the behavior of each mode across the interface separately. In particular we examine the case in which each mode in the empty half-space is in a coherent state (in the $\phi=0$ quadrature). To simplify notation in what follows we examine only the $\pm j$ modes. When the wave packet enters the refractive half-space, it will still consist of a superposition of modes excited to a coherent state,

$$|\alpha_j\rangle = e^{-i\alpha(\hat{b}_j+\hat{b}_j^\dagger)}|0\rangle \rightarrow |\tilde{\alpha}_j\rangle = e^{-i\alpha(\hat{B}_j+\hat{B}_j^\dagger)}|0\rangle , \tag{5.14}$$

where $|0\rangle$ is the free-space vacuum in the $-z$ half-space, and the refracted-wave vacuum in the $+z$ half-space. Such states have quasiclassical properties in that they present a nonzero expectation value for the electric and magnetic field operators; in particular, in the refractive medium their values may be obtained through Eqs. (5.10) as

$$E_T = \langle\tilde{\alpha}|\hat{E}|\tilde{\alpha}\rangle = 2\alpha\left[\frac{2\pi\omega_j}{nV}\right]^{1/2} \tag{5.15a}$$

and in free space as

$$E_I = \langle\alpha|\hat{E}|\alpha\rangle = 2\alpha\left[\frac{2\pi\omega_j}{V}\right]^{1/2} , \tag{5.15b}$$

where the subscripts I and T refer to the incident and transmitted fields. Equations (5.15) imply that

$$E_T = E_I/\sqrt{n} \tag{5.16a}$$

as in the classical equations (2.15b). Similarly, for the magnetic field Eq. (5.14) gives

$$H_T = H_I\sqrt{n} \tag{5.16b}$$

identical to the classical Eqs. (2.15c). It is easy to verify that the mean number of photons in the mode, the energy, and the energy flux do not change, while the energy density and momentum are multiplied by the refractive index.

Ordinarily, a sharp vacuum-dielectric interface entails an abrupt discontinuity in refractive index from 1 to n across the interface. When the wave packet reaches the interface, its jth component is reflected into the $-j$ free-field wave and/or is transmitted into the $+j$ refracted wave. The behavior of the wave packet due to the sudden change in the momentum operator at $z=0$ can be treated within the "sudden approximation": The free-field waves get projected onto the refracted waves. We may calculate this projection through Eqs. (5.4), which can be rewritten as

$$\hat{b}_j^\dagger = \frac{2\sqrt{n}}{n+1}\hat{B}_j^\dagger + \left[\frac{n-1}{n+1}\right]\hat{b}_{-j} , \tag{5.17a}$$

$$\hat{b}_j = \frac{2\sqrt{n}}{n+1}\hat{B}_j + \left[\frac{n-1}{n+1}\right]\hat{b}_{-j}^\dagger , \tag{5.17b}$$

where we have used the explicit expressions for $\cosh\gamma$ and $\sinh\gamma$ through Eq. (5.6). When a coherent wave packet reaches the interface, it gets split into a transmitted and a reflected coherent wave packet,

$$|\alpha_j\rangle = e^{-i\alpha(\hat{b}_j+\hat{b}_j^\dagger)}|0\rangle \rightarrow |\tilde{\alpha}_T,\alpha_R\rangle = e^{-i\alpha\{[(2\sqrt{n})/(n+1)](\hat{B}_j+\hat{B}_j^\dagger)+[(n-1)/(n+1)](\hat{b}_{-j}+\hat{b}_{-j}^\dagger)\}}|0\rangle . \tag{5.18}$$

Clearly, the corresponding expectation values of the electric field satisfy

$$E_T = \frac{2}{n+1}E_I , \tag{5.19a}$$

$$E_R = \frac{n-1}{n+1}E_I \tag{5.19b}$$

(where the subscript R denotes the reflected wave), while the mean photon numbers (or energies) verify

$$N_T = \frac{4n}{(n+1)^2}N_I , \tag{5.20a}$$

$$N_R = \left[\frac{n-1}{n+1}\right]^2 N_I , \tag{5.20b}$$

giving transmission and reflection coefficients for the field and for the energy identical to the corresponding classical coefficients (2.14).

Equations (5.17) imply that the negative-frequency part of the incident electric field wave projects onto the negative-frequency part of the transmitted wave and onto the *positive*-frequency part of the reflected wave. Consequently, as in Eq. (5.18), both the negative- and the positive-frequency parts of the electric field operator are necessary to describe reflection within this formalism, even though the positive-frequency part (which corresponds to the annihilation operators) may give zero contribution to the description of the incident wave in terms of photons.

A single photon, that is, a wave packet created by a single excitation of the electric field (e.g., dipole emission of one quantum) in the jth free-field wave,

$$(\hat{b}_j+\hat{b}_j^\dagger)|0\rangle=\hat{b}_j^\dagger|0\rangle , \tag{5.21a}$$

will give at the interface

$$\left[\frac{2\sqrt{n}}{n+1}(\hat{B}_j+\hat{B}_j^\dagger)+\frac{n-1}{n+1}(\hat{b}_{-j}+\hat{b}_{-j}^\dagger)\right]|0\rangle = \frac{2\sqrt{n}}{n+1}\hat{B}_j^\dagger|0\rangle+\frac{n-1}{n+1}\hat{b}_{-j}^\dagger|0\rangle , \tag{5.21b}$$

that is, it will be transmitted with a probability given by the classical energy-transmission coefficient, or will be reflected with a probability given by the classical energy-reflection coefficient.

Similarly, a wave packet created by a double excitation of the electric field (e.g., formally by a term quadratic in the electric field),

$$(\hat{b}_j+\hat{b}_j^\dagger)^2|0\rangle=\hat{b}_j^\dagger\hat{b}_j^\dagger|0\rangle+|0\rangle , \tag{5.22a}$$

may split in three different ways at the interface,

$$\frac{4n}{(n+1)^2}\hat{B}_j^\dagger\hat{B}_j^\dagger|0\rangle+\left[\frac{n-1}{n+1}\right]^2\hat{b}_{-j}^\dagger\hat{b}_{-j}^\dagger|0\rangle + \frac{2(n-1)\sqrt{n}}{(n+1)^2}\hat{B}_j^\dagger\hat{b}_{-j}^\dagger|0\rangle+|0\rangle , \tag{5.22b}$$

that is, both photons may be reflected or both transmitted, or one photon may be reflected and one transmitted, each process occurring with the probability of the corresponding product of the classical reflection and transmission coefficients. A more thorough examination of two-photon (nonlinear) processes will be given elsewhere.[9]

These results demonstrate that the diagonalization of the energy-density operator for a linear medium is equivalent to the solution of the quantum-mechanical Maxwell equations, and permits a quantum-mechanical description of linear propagative phenomena, reproducing the results of classical optics. In particular, our formalism gives a quantum-mechanical theory of the refractive index as the parameter that renormalizes the momentum eigenvalues in the medium and describes the spatial progression of the refracted waves. The quantum-mechanical description of propagation through the momentum operator permits us also to address the problem of a medium interface through the standard techniques of quantum-perturbation theory, and gives a successful quantum-mechanical description of reflection and transmission of the electromagnetic field at an interface. Within this quantum-mechanical formalism, the entity reflected or transmitted at an interface *is not the photon,* but rather the *electric and magnetic fields* that create it, as in classical optics. This seemingly paradoxical result is due to the fact that in the conventional picture of the photon as an eigenstate of the number operator, only the negative-frequency part of the field is considered, while both in the classical and in the quantum versions of the Maxwell equations, propagation and boundary conditions are formulated for both the positive and the negative parts of the fields simultaneously.

Before closing this section we shall point out the close relationship that exists between the classical phenomenon of refraction, and the recent considerations on quantum mechanical "squeezing," that is, on the creation of states of the electromagnetic field which exhibit a quantum-mechanical noise level (in one quadrature) below that of the vacuum fluctuations. The unitary transformation (5.3) that diagonalizes the energy-density operator inside a refractive material is precisely the "two-mode squeeze operator" introduced for describing a class of two-photon coherent states[7] that exhibit nondegenerate squeezing.[11] This transformation relates free-space waves to refracted waves through Eqs. (5.4) and (5.5), and this means that free-space number eigenstates (photons) are related to refracted wave eigenstates through

$$|N_j\rangle_{\text{refr}}=e^{-\gamma\hat{R}}|N_j\rangle_{\text{free}} , \tag{5.23a}$$

$$|N_j\rangle_{\text{free}}=e^{\gamma\hat{R}}|N_j\rangle_{\text{refr}} , \tag{5.23b}$$

where N_j is a positive integer or zero. In other words, refracted photon states are two-mode squeezed states of the corresponding free-space photons, and vice versa. In particular, the refracted-wave vacuum ($N_j=0$), when expressed in terms of free-space eigenstates, consists of a superposition of all two-mode number eigenstates where both modes are equally excited,

$$|0\rangle_{\text{refr}}=e^{-\gamma(\hat{b}_j\hat{b}_{-j}-\hat{b}_j^\dagger\hat{b}_{-j}^\dagger)}|0\rangle_{\text{free}} = \frac{1}{\cosh\gamma}\sum_{N=0}^{\infty}\tanh^N\gamma\,|N_j,N_{-j}\rangle_{\text{free}} , \tag{5.24}$$

and has thus a structure identical to that of the output of an ideal two-photon device. However, Eq. (5.24) has no physical significance in the time scales of quantum optics, since the free-field photons that appear in it are produced by energy nonconserving terms, and constitute therefore virtual states. In other words, inside a dielectric there is no experiment that can detect free-space photons and thus there is no way to investigate the consequences of Eqs. (5.23) and (5.24) regarding squeezing. To clarify this point, we look at the quantum noise characteristics of the refracted-wave vacuum as measured in a hypothetical experiment that mixes the $+j$ and $-j$ modes with a local oscillator on the surface of a photodetector in the dielectric (notwithstanding the geometrical restrictions of one-dimensional propagation) and detects the two quadratures of the resulting photocurrent. If free-space photons were detectable inside the dielectric, then the two-mode quadrature-phase amplitude operators relevant to such an experiment would be

$$\hat{\alpha}_{1j}=\frac{-i}{\sqrt{2}}(\hat{b}_j^\dagger-\hat{b}_{-j}) \tag{5.25a}$$

and

$$\hat{\alpha}_{2j}=\frac{1}{\sqrt{2}}(\hat{b}_j^\dagger+\hat{b}_{-j}) , \tag{5.25b}$$

whose variances for the refracted-wave vacuum are

$$_{\mathrm{refr}}\langle 0 | \Delta\hat{\alpha}_{1j} | 0 \rangle_{\mathrm{refr}} = \frac{1}{2n} < \tfrac{1}{2} , \qquad (5.26a)$$

$$_{\mathrm{refr}}\langle 0 | \Delta\hat{\alpha}_{2j} | 0 \rangle_{\mathrm{refr}} = \frac{n}{2} > \tfrac{1}{2} , \qquad (5.26b)$$

where n is the refractive index. Similar variances would be obtained for refracted coherent states, indicating that if the free-field could be measured inside a dielectric the noise that is in phase with the signal would exhibit, fluctuations below those of free-space vacuum ($\Delta=\frac{1}{2}$), while the noise level in quadrature with the signal would be higher than that of the vacuum fluctuations. However, inside a dielectric we can only detect refracted waves whose electric field operators are given by Eq. (5.10). Since a photodetector is a photon number or energy measuring device, Eq. (2.14f) implies that the relevant quadrature-phase amplitude operators inside a refractive medium are

$$\hat{\beta}_{1j} = -\frac{i}{\sqrt{2}}(\hat{B}_j^{\dagger} - \hat{B}_{-j}) , \qquad (5.27a)$$

$$\hat{\beta}_{2j} = \frac{1}{\sqrt{2}}(\hat{B}_j^{\dagger} + \hat{B}_{-j}) , \qquad (5.27b)$$

which give identical variances for both quadratures,

$$_{\mathrm{refr}}\langle 0 | \Delta\hat{\beta}_{1j} | 0 \rangle_{\mathrm{refr}} = {}_{\mathrm{refr}}\langle 0 | \Delta\hat{\beta}_{2j} | 0 \rangle_{\mathrm{refr}} = \tfrac{1}{2} , \qquad (5.28)$$

meaning that squeezing due to refraction is not observable.

VI. CONCLUSIONS

The conventional theory of quantum optics rests on a set of basic assumptions, such as the following.

(1) The field is quantized as a set of quasidiscrete modes inside an empty (vacuum) cavity with periodic boundary conditions.

(2) The total energy of the field (its Hamiltonian) is given by the integral of the electromagnetic energy density over the physical volume of the cavity in assumption (1).

(3) The dynamical behavior of the field may be calculated by applying the Hamiltonian obtained in (2) to the proper superposition of modes, corresponding to the initial conditions of a given problem.

These assumptions permit a successful treatment of the electromagnetic field in vacuum, as it interacts with material points representing emitters, absorbers, or detectors. At the same time, however, these same assumptions are at the root of the shortcomings of conventional quantum optics. In particular, it is impossible to treat directly the spatial progression of a wave under these assumptions, since the Hamiltonian formalism in (3) can only deal with the time dimension. The calculation of spatial progression is avoided by fixing the spatial characteristics of the modes defined in (1) to the proper form, and then calculating only the temporal evolution of a wave packet composed of these modes, traveling like a train on predetermined tracks. This is in contrast to the treatment of propagation in classical optics, where spatial differential equations can be set up to calculate explicitly the spatial progression of the field, including the determination of its mode structure. The difficulties associated with spatial calculations in quantum optics are even more pronounced when examining propagation through bulk matter, since in that case its basic assumptions break down. In particular, there is no prescription on how the periodic-boundary conditions in (1) get modified upon introduction of a medium, while at the same time assumption (2) gives a change in total energy of a traveling wave when a polarizable material is present, a result that is at variance with the actual behavior of the electromagnetic field.

These difficulties may be overcome by qualifying or reformulating these assumptions, and thus extending the scope of quantum optics to include the description of propagative phenomena inside a medium. The new assumptions are the following.

(1′) The periodic-boundary cavity constitutes an *open volume* permitting flow through its boundaries, and is a device for dealing with traveling waves in infinite space (where the modes are standing waves vanishing at infinity). We may postulate that, upon introduction of a linear medium in infinite space, the dimensions of the periodic-boundary cavity get modified so that the number of wavelengths it contains remains unchanged. This insures that periodic-boundary conditions still hold, and establishes the right correspondence between the free-space and refracted-wave modes of infinite space.

(2′) Since quantization is done in a "flow cavity," the total energy of the field corresponds to the *energy flux*. The same holds for the total momentum of the field, the momentum flux being given by the energy density.

(3′) The temporal evolution of the field is given by the Hamiltonian (2′), while its spatial progression can be calculated through the momentum operator given in (2′). This permits us to treat on the same footing both the temporal and the spatial coordinates of an electromagnetic wave.

The simultaneous use of the Hamiltonian and the momentum operators defined according to (2′) yields spatial-temporal equations of motion for the electric and magnetic fields, having a form identical to that of the classical Maxwell equations. This confirms the validity of assumptions (1′)–(3′) and implies that the formalism based on them is not subject to the limitations of conventional quantum optics, but can also treat medium-dependent propagative phenomena quantum mechanically. Thus its use should permit a rigorous description of quantum propagative phenomena, as for example spontaneously initiated (noise-driven) stimulated processes, which are usually treated through classical propagation theory. In classical theory initiation is produced by a statistical noise field, which in essence simulates quantum-mechanical uncertainty.

For propagation through a linearly polarizable medium, the quantum-mechanical Maxwell equations can be solved exactly through second quantization operatorial techniques. In particular, diagonalization of the energy density (or momentum) operator inside a refractive medium gives directly the refractive index, the renormalization of the wave vectors (i.e., the momentum eigenvalues) and the corresponding change in the spatial structure of the field modes. This treatment can readily describe all refractive phenomena, such as reflection and transmission at an interface, as well as the propagation (i.e., spatial progression and temporal evolution) of an electromagnetic wave in a refractive medium. In the diagonalization of the momentum (energy-density) operator, *all the terms* resulting from the different products of positive- and negative-frequency parts of the field have to be taken into account to describe refraction: In the expansion of the refractive index as a power series of the susceptibility, only the first-order term is due to the energy-conserving (nonoscillating) field terms. All higher-order terms result from rapidly oscillating, energy-nonconserving products of positive- or of negative-frequency-field terms.

To obtain a correct description of propagative phenomena, the contributions of both the positive- and the negative-frequency parts of the electromagnetic field have to be treated on the same footing, even if one of the two parts may be redundant in some situations. In particular, in the description of the field in terms of photons, the positive-frequency part is usually dropped, since it corresponds to negative-energy photons, or equivalently to photon annihilation in the vacuum. However, in the description of reflection and transmission of the field at an interface, if the positive-frequency part of the incident wave is neglected, the reflection of photons from a dielectric surface cannot be described.

The fidelity with which this quantum-mechanical treatment of linear propagation reproduces all the results known from the classical treatment of refraction demonstrates the validity of our approach, based on the momentum operator. In the next paper in this series we shall use the momentum-operator approach to treat propagation through a nonlinearly polarizable medium.

[1]R. Loudon, *The Quantum Theory of Light,* (Oxford University Press, New York, 1983); W. H. Louisel, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[2]This viewpoint is used, for example, in C. K. Hong and L. Mandel, Phys. Rev. A **31**, 2409 (1985).

[3]J. M. Jauch and K. M. Watson, Phys. Rev. **74**, 950 (1948).

[4]Y. R. Shen, Phys. Rev. **155**, 921 (1967); in *Quantum Optics,* edited by R. J. Glauber (Academic, New York, 1969), p. 473.

[5]C. K. Carniglia and L. Mandel, Phys. Rev. D **3**, 280 (1971).

[6]S. A. Akhmanov, A. P. Sukhurukov, and A. S. Chirkin, Zh. Eksp. Teor. Fiz. **55**, 1430 (1968) [Sov. Phys.—JETP **28**, 748 (1969)].

[7]D. Stoler, Phys. Rev. D **1**, 3217 (1980); **4**, 1925 (1971); H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[8]H. P. Yuen and J. Shapiro, IEEE Trans. Info. Theory **IT-24**, 657 (1978); **IT-26**, 78 (1980); H. P. Yuen, J. Shapiro, and J. A. Machado Mata, *ibid.* **IT-25**, 179 (1979).

[9]I. Abram and R. Raj (unpublished).

[10]J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975), Sec. 6.9; J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941); R. Becker, *Electromagnetic Fields and Interactions* (Dover, New York, 1982).

[11]C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); **31**, 3093 (1985).