# Multiphoton pair creation

Marvin H. Mittleman
*Physics Department, The City College of the City University of New York, New York, New York 10031*
(Received 31 October 1986)

Pair creation by an intense linearly polarized pulse in the vicinity of a nucleus is considered. It is shown to be negligibly small for all laser intensities for essentially all frequencies. Some novel theoretical effects are obtained from the relativistic generalization of the ponderomotive potential.

## I. INTRODUCTION

An ultra-intense laser pulse propagating in a near vacuum experiences a variety of absorptive processes which degrade its energy. One of these is the creation of an electron-positron pair in the vicinity of a bare nucleus. However, the cross section for this process at optical frequencies or below is so small at any laser intensity as to make it completely negligible. It may be the smallest (nonzero) cross section on record. It does however exhibit some very interesting features for the theorist which are probably reflected in other processes which couple relativistic charged particles to intense laser fields. These are detailed below.

If the laser is circularly polarized then the pair must be created with a very large relative angular momentum since a very large number of photons, each with angular momentum $\hbar$, will be absorbed. This will create a large angular momentum barrier between the particles which will make the cross section even smaller. We therefore restrict ourselves to a linearly polarized laser and at the end compare with previously obtained results for circular polarization.[1]

The first problem in the calculation is the creation process itself. Perturbation theory (for about $10^6$ photons) is out of the question. It is avoided by a transformation of the QED description of the field to a classical one via the "phase representation[2]" of the laser model. The interaction with the Coulomb field of the nucleus is treated in first order and the resulting "Fermi golden rule" form is obtained. This exhibits some very peculiar forms largely due to relativistic propagation of the charged particles in the laser field in Volkov[3] states. For example, the relativistic generalization of the ponderomotive potential causes some very surprising distortion of the particle kinematics so that absorption of extra photons (in some cases) actually reduces the energy of the emerging particles. The details of this and other effects are presented in the next section. The reader who wishes to avoid the details of the calculations is urged to skip to the discussion following (2.26) and to the final result following (2.39).

## II. OUTLINE OF THE CALCULATION

Our starting point is the QED Hamiltonian of the electron-positron field coupled to only the laser mode of the electromagnetic field ($\hbar=c=1$)

$$H=\hat{\mathbf{N}}\int d^3r\,\psi^\dagger(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m+V)\psi+H_{\rm em}+H_{\rm int}\,. \tag{2.1}$$

Here, $V=-Ze^2/r$ is the interaction of the charge particles with the (infinitely massive) nucleus and $\hat{N}$ is the normal ordering operator. The laser field energy is

$$H_{\rm em}=\omega N=\omega a^\dagger a\ , \tag{2.2}$$

where $N$ is the number operator for the laser mode and the interaction energy is

$$H_{\rm int}=e\hat{N}\int d^3r\,\psi^\dagger\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r})\psi\ , \tag{2.3}$$

where the vector potential is

$$\mathbf{A}=\left[\frac{2\pi}{\omega V}\right]^{1/2}\hat{\boldsymbol{\epsilon}}(e^{i\mathbf{k}\cdot\mathbf{r}}a+e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger)\ . \tag{2.4}$$

The remaining symbols have their usual meaning. The occupation number of the laser mode ($N$) is assumed to be very large and the unitary transformation to the phase representation[2] for the laser mode gives

$$a=e^{-i\phi}\left[N+\frac{1}{i}\frac{\partial}{\partial\phi}\right]^{1/2}$$
$$\simeq e^{-i\phi}N^{1/2}\left[1-\frac{i}{2N}\frac{\partial}{\partial\phi}+\cdots\right] \tag{2.5}$$

and its Hermitian conjugate so that

$$H_{\rm em}=N\omega-i\omega\frac{\partial}{\partial\phi}\ . \tag{2.6}$$

All but the leading term of (2.6) is neglected with the result that

$$H=\hat{N}\int d^3r\,\psi^\dagger\{\boldsymbol{\alpha}\cdot[\mathbf{p}+e\,\mathbf{A}(\mathbf{r},\phi)]+\beta m+V\}\psi-i\omega\frac{\partial}{\partial\phi} \tag{2.7}$$

where

$$\mathbf{A}(\mathbf{r},\phi)=\mathbf{E}/\omega\cos(\mathbf{k}\cdot\mathbf{r}-\phi) \tag{2.8}$$

and

$$\mathbf{E}=(8\pi\omega N/V)^{1/2}\hat{\boldsymbol{\epsilon}} \tag{2.9}$$

is the equivalent classical field strength. The $c$ number $N\omega$ has been dropped from (2.7) since it has no dynamic effect. The Schrödinger equation

 

$$\left[i\frac{\partial}{\partial t}-H\right]\Psi=0 \tag{2.10}$$

can be written, with the transformation $\theta=\phi-\omega t$, in the form

$$\left[i\frac{\partial}{\partial t}-\tilde{H}\right]\Psi=0\ , \tag{2.11}$$

where

$$\tilde{H}=\hat{N}\int d^3r\,\psi^\dagger\{\boldsymbol{\alpha}\cdot[\mathbf{p}+e\,\mathbf{A}(\mathbf{r},\omega t+\theta)]+\beta m+V\}\psi\ . \tag{2.12}$$

Here, $\theta$ enters only as an unobservable phase and may be eliminated by a shift in time.

The electron-positron field can be expanded in the complete set of Volkov states[3] which satisfy

$$\left[i\frac{\partial}{\partial t}-\boldsymbol{\alpha}\cdot[\mathbf{p}+e\,\mathbf{A}(\mathbf{r},\omega t)]-\beta m\right]\chi_{qi}(\mathbf{r},t)=0 \tag{2.13}$$

which are orthonormal,

$$\int d^3r\,\chi^\dagger_{qi}(\mathbf{r},t)\chi_{q'i'}(\mathbf{r},t)=\delta_{ii'}\delta(\mathbf{q}-\mathbf{q}')\ . \tag{2.14}$$

These states can be written

$$\chi_{qi}=(2\pi)^{-3/2}[1+i\lambda_{qi}C(\nu)g^\dagger]\times\exp\{i[\mathbf{q}\cdot\mathbf{r}-E_{qi}t-S_{qi}(\nu)]\}Z_{qi}\ , \tag{2.15}$$

where $E_q=\pm(m^2+q^2)^{1/2}$, the plus sign being used for the spin states labeled $i=1,2$ and the minus sign for $i=3,4$. $\mathbf{q}$ is the momentum of the particle *outside* the pulse. The other factors are

$$\lambda_{qi}=m(E_{qi}-\hat{\mathbf{k}}\cdot\mathbf{q})^{-1}\ , \tag{2.16a}$$

where $\hat{\mathbf{k}}$ is the $(\hat{\mathbf{z}})$ propagation direction of the laser pulse.

$$g^\dagger=\boldsymbol{\sigma}\cdot(\hat{\mathbf{k}}\times\mathbf{x})-i\boldsymbol{\alpha}\cdot\mathbf{x}\ , \tag{2.16b}$$

$$\mathbf{x}=e\mathbf{E}/2m\omega,\quad x^2=\tfrac{1}{4}\alpha_F^2\left[\frac{\mathrm{Ry}}{\hbar\omega}\right]^2\frac{I}{I_0}\ , \tag{2.16c}$$

where $\alpha_F=e^2/\hbar c\simeq(137)^{-1}$ and $I_0=0.88\times10^{16}$ W/cm$^2$. The phase term is

$$S_{\mathbf{q},i}(\nu)=\lambda_{qi}\left[\frac{2m}{\omega}x^2\int_0^\nu d\nu' C^2(\nu')+\frac{2}{\omega}\mathbf{x}\cdot\mathbf{q}\int_0^\nu d\nu' C(\nu')\right]\ , \tag{2.16d}$$

where

$$\nu=\omega(t-\hat{\mathbf{k}}\cdot\mathbf{r})\ . \tag{2.16e}$$

The function $C(\nu)$ is a generalization of the cosine occurring in (2.8). It is supposed to include the envelope of the laser pulse. We shall eventually go to the limit of a very long pulse in which case $C(\nu)$ will become $\cos\nu$. Finally, $Z_{qi}$ is the matrix part of the plane-wave Dirac state.[4]

The electron-positron field can be expanded in this complete set

$$\psi=\sum_{\mathbf{q},i=1,2}\chi_{\mathbf{q}i}a_{\mathbf{q}i}+\sum_{\mathbf{q},i=3,4}\chi_{-\mathbf{q},i}b^\dagger_{\mathbf{q}i}\ , \tag{2.17}$$

where in the usual way[4] $a_{\mathbf{q}i}$ is interpreted as an electron destruction operator and $b^\dagger_{\mathbf{q}i}$ as a positron creation operator. Then the lowest-order matrix element of $H$ (2.12) between a particle vacuum and a state with one electron $(q,i)$ and one positron $(q',i')$ is

$$\langle\,\mathbf{q}i,\mathbf{q}'i'\,|\,\tilde{H}\,|\,0\,\rangle=\int d^3r\,\chi^\dagger_{\mathbf{q}i}(\mathbf{r},t)\left[i\frac{\partial}{\partial t}+V\right]\chi^\dagger_{-\mathbf{q}',i'}(\mathbf{r},t)\ , \tag{2.18}$$

where we have used (2.13). The first term describes the transition in the absence of $V$ which will vanish since the energy-momentum conservation laws forbid this transition. The second term, proportional to $V$, gives the matrix element of interest and the time integral of this term is the lowest order (in $V$) $S$ matrix. After some algebraic calculation this can be written as

$$S_{\mathbf{q}i,\mathbf{q}'i'}=\frac{4\pi iZe^2}{(2\pi)^3}\int\frac{d\nu}{\omega}dk'_z\sum_{n=0}^{2}C^{(n)}(\nu)A^{(n)}_{ii'}(\mathbf{q},\mathbf{q}')\times\exp\left[-i\left[\frac{\nu}{\omega}(k'_z-q_z-q'_z)-4b_2\int_0^\nu d\nu' C^2(\nu')-b_1\int_0^\nu d\nu' C(\nu)\right]\right]\times[(\mathbf{q}_\perp+\mathbf{q}'_\perp)^2+(E_q+E_{q'}-q_z-q'_z)^2]^{-1}\ , \tag{2.19}$$

where the $k'_z$ arises from the Fourier transform of $V$ and $\mathbf{q}_\perp$ is the part of $\mathbf{q}$ perpendicular to the propagation direction and

$$b_1=\frac{2}{\omega}\mathbf{x}\cdot(\mathbf{q}\lambda_\mathbf{q}-\mathbf{q}'\lambda_{\mathbf{q}'})\ ,\qquad b_2=\frac{m}{2\omega}x^2(\lambda_\mathbf{q}+\lambda_{\mathbf{q}'})\ , \tag{2.20}$$

and now only positive energies occur in $\lambda$ (2.16a). We define the matrix products

$$A^{(n)}_{ii'}(\mathbf{q},\mathbf{q}')=Z^\dagger_{\mathbf{q}i}m(n)Z_{-\mathbf{q}'i'}\ , \tag{2.21a}$$

with

$$m(0)=1,\quad m(1)=-i(\lambda_\mathbf{q}g+\lambda_{\mathbf{q}'}g^\dagger)\ ,\quad m(2)=-\lambda_\mathbf{q}\lambda_{\mathbf{q}'}gg^\dagger\ . \tag{2.21b}$$

If $C(\nu)$ is taken to be $\cos\nu$ then the remaining integrals can be performed

$$S_{\mathbf{q}i,\mathbf{q}'i'}=\frac{4\pi i Ze^2}{(2\pi)^3}\sum_{n=-\infty}^{\infty}2\pi\delta(E_q+E_{q'}+2\omega b_2-N\omega)\times\frac{1}{D}\sum_{n=0}^{2}A_{ii'}^{(n)}\mathcal{J}_{-N}^{(n)}\ , \tag{2.22}$$

where

$$D=(\mathbf{q}_\perp+\mathbf{q}'_\perp)^2+(E_q+E_{q'}-q_z-q'_z)^2 \tag{2.23}$$

and we define modified Bessel functions

$$\mathcal{J}_{-N}^{(n)}(b_1,b_2)=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}(\cos\theta)^n\times\exp i(N\theta+b_1\sin\theta+b_2\sin2\theta)\ , \tag{2.24}$$

which have been studied previously.[5] The $T$ matrix for the process is obtained from

$$S_{\mathbf{q}i,\mathbf{q}'i'}=-2\pi i\sum_N\delta(E_q+E_{q'}+2\omega b_2-N\omega)T_N(\mathbf{q}i,\mathbf{q}'i')\ , \tag{2.25}$$

with the result

$$T_N(\mathbf{q}i,\mathbf{q}'i')=-\frac{4\pi Ze^2}{(2\pi)^3}\frac{1}{D}\sum_{n=0}^{2}A_{ii'}^{(n)}(\mathbf{q},\mathbf{q}')\mathcal{J}_{-N}^{(n)}(b_1b_2)\ . \tag{2.26}$$

Some unusual aspects of the problem emerge at this point. First, $D^{-1}$ is the Fourier transform of the potential. In the usual cases it is evaluated at the momentum transfer to the particles. The first term in (2.23) is just the square of the perpendicular part of the momentum transfer but the second term contains the energies and so is different. These arise from the parts of $\cos(\omega t-\mathbf{k}\cdot\mathbf{r})$ which go beyond the dipole approximation. They are the dominant terms for small $\mathbf{q}$ and $\mathbf{q}'$.

The energy $\delta$ function in (2.22) gives the number of photons absorbed as a function of the final state of the particles, $\mathbf{q}$ and $\mathbf{q}'$. Note the appearance of $2\omega b_2$ (2.22), which is the relativistic generalization of the ponderomotive potential. The photon number is larger than $(E_q+E_{q'})/\omega$ because the particles are created inside the pulse and so have oscillatory kinetic energy in addition to that due to their translational motion. This effect has also been interpreted as mass shift of the electron due to dressing by the laser field.[6] The sum of these two kinetic energies is not conserved as the particles are left behind by the pulse. This phenomenon has been encountered before.[7] It is due to the fact that the amplitude of the pulse is necessarily time dependent and so is the ponderomotive potential which governs the average motion of the particle.

We may also ask for the minimum number of photons which must be absorbed to create the pair. The minimization of $E_q+E_{q'}+2\omega b_2$ with respect to $\mathbf{q}$ and $\mathbf{q}'$ occurs at $\mathbf{q}=\mathbf{q}'$ with $\mathbf{q}$ pointing in the $-\hat{\mathbf{z}}$ direction and

$$q_{\min}=mx^2(1+2x^2)^{-1/2}\ ,\quad E_{q\,\min}=m(1+x^2)(1+2x^2)^{-1/2}\ , \tag{2.27a}$$

such that

$$N_{\min}=\frac{2m}{\omega}(1+2x^2)^{1/2}\ . \tag{2.27b}$$

For small $x^2$ (2.16c), this becomes the expected result,

$$N_{\min}\simeq\frac{2}{\omega}(m+U_p)\ , \tag{2.27c}$$

where $U_p=e^2E^2/4m\omega^2$ is the nonrelativistic ponderomotive potential. However, when this expansion is not permissable the number $N$ will decrease for increasing $E_q$ for $E_q<E_{\min}$, a surprising result.

The total transition rate for all possible numbers of photons can be gotten as

$$W=\sum_N 2\pi\int d^3q\,d^3q'\delta(E_q+E_{q'}+2\omega b_2-N\omega)\times\mathrm{tr}\left[\frac{1+\beta}{2}\right]T_N\left[\frac{1-\beta}{2}\right]T_N^\dagger\ , \tag{2.28}$$

where the spin trace arises from the sum over polarization of the particles and $T_N$ is given by (2.26). For a YAG (yttrium aluminum garnet) frequency $N_{\min}$ will be of the order of $10^6$ so we may convert the sum to an integral and the result is

$$W=\frac{2\pi}{\omega}\int\frac{d^3q\,d^3q'}{(2\pi)^6}\left[\frac{4\pi Ze^2}{D}\right]^2\sum_{n,n'=0}^{2}T^{nn'}\mathcal{J}_{-N}^{(n)}\mathcal{J}_{-N}^{(n')}\ , \tag{2.29}$$

where, Eq. (2.21),

$$T^{nn'}=\mathrm{tr}\left[\frac{1+\beta}{2}\right]A^{(n)}\left[\frac{1-\beta}{2}\right]A^{(n')}\ . \tag{2.30}$$

The large value of $m/\omega$ may be exploited for the evaluation of $\mathcal{J}$ (2.24). This has been done previously by Reiss[5] and we may simply paraphrase his results. There is no real stationary phase point in the integral of Eq. (2.24) so we resort to the method of steepest descents. This results in

$$\mathcal{J}_{-N}^{(n)}=\sum_j\frac{e^{i\Theta_j}}{(-2\pi i\Theta_j'')^{1/2}}(\cos\theta_j)^n\ , \tag{2.31}$$

where the sum runs over the two saddle points of

$$\Theta=N\theta+b_1\sin\theta+b_2\sin2\theta\ , \tag{2.32}$$

for which $I>0$. This results in

$$\mathcal{J}_{-N}^{(n)}=\left[\frac{2}{\pi}\right]^{1/2}|\cos\theta_1|^n e^{-I}(A^2+B^2)^{-1/4}\times\cos\left[R+\tfrac{1}{2}\tan^{-1}\frac{A}{B}+n\Phi\right]\ , \tag{2.33}$$

where $\cos\theta_1$ is the (complex) saddle point obtained from $\Theta'=0$ and we define

$$\cos\theta_1 = |\cos\theta_1| e^{i\Phi} , \quad (2.34a)$$

$$\Theta''|_{\theta=\theta_1} = A + iB . \quad (2.34b)$$

These may all be expressed in terms of two real parameters

$$\xi = \frac{N}{2b_2} - \frac{1}{2} = (E_q + E_{q'})/2mx^2(\lambda_q + \lambda_{q'}) , \quad (2.34c)$$

$$\eta = \frac{b_1}{8b_2} = \hat{\mathbf{x}}\cdot(\mathbf{q}\lambda_q - \mathbf{q}'\lambda_{q'})/2mx(\lambda_q + \lambda_{q'}) . \quad (2.34d)$$

Then the necessary relationships are

$$\frac{I}{2b_2} = (2\xi+1)\ln\left[\left[\frac{\xi+1}{2}+\rho\right]^{1/2} + \left[\frac{\xi-1}{2}+\rho\right]^{1/2}\right] - \frac{[\eta^2(\xi-2)+\frac{1}{2}\xi(\xi+1)+\rho(2\eta^2+\xi)]}{\left[\frac{\xi+1}{2}+\rho\right]^{1/2}\left[\frac{\xi-1}{2}+\rho\right]^{1/2}} , \quad (2.34e)$$

where

$$\rho = \left[\left[\frac{\xi+1}{2}\right]^2 - \eta^2\right]^{1/2} ,$$
$$A^2 + B^2 = 128 b_2^2 \rho(\xi - \eta^2) ,$$
$$|\cos\theta_1| = \xi ,$$
$$\tan\Phi = -\frac{1}{\eta}(\xi - \eta^2)^{1/2} . \quad (2.35)$$

A simple examination of (2.34e) and (2.20) shows that $I$ is of order $m/\omega \gg 1$. Somewhat more extensive algebraic manipulations are required to show that $I > 0$ so that (2.29) is proportional to a factor $e^{-(2m/\omega)I_m}$ where $I_m \sim 1$. This is the main reason that $W$, the pair creation rate per nucleus, is so extremely small.

We proceed with the evaluation of $W$ by using (2.33) and noting that $R$ is also of order $m/\omega$ so that the last factor in (2.33) oscillates rapidly. But $\mathscr{I}^{(n)}_{-N}$ appears bilinearly in $W$ so this factor can be averaged by $\cos^2(R \cdots) \to \frac{1}{2}$. Then $W$ can be written

$$W = \frac{1}{\pi\omega}\int \frac{d^3q\, d^3q'}{(2\pi)^3}\left[\frac{2Ze^2}{D}\right]^2 e^{-2I}(A^2+B^2)^{-1/2} \times [T^{00} + 2\eta T^{01} + \xi T^{11} + 2(2\eta^2 - \xi)T^{02} + 2\xi\eta T^{12} + \xi^2 T^{22}] . \quad (2.36)$$

This is a function of the two parameters $m/\omega \gg 1$ and $x^2$ (2.16c), which is both frequency and intensity dependent. The fact that $I > 0$ for all values of these parameters shows that $W \sim e^{-m/\omega}$ for any laser intensity, no matter how high. For lasers available now we may take $\hbar\omega \sim 1$ eV and $I \sim 10^{17}$ W/cm$^2$ which yields $x^2 = 10^{-2}$. Then it is reasonable to attempt a nonrelativistic expansion (in $q/m$) of the integrand in (2.36). The surprising result is that it is good for $\mathbf{q}_\perp$ but that such an expansion for $q_z$ is very slowly convergent. Then only the $q_\perp/m$ expansion is made in $I$ with the result

$$I = I^{(0)} + I^{(1)} + \cdots , \quad (2.37a)$$

$$I^{(0)} = \left[\frac{\mathscr{E}_q + \mathscr{E}_{q'}}{2\omega} + 1\right]\left[\ln\left[\frac{2}{x^2}\frac{\mathscr{E}_q + \mathscr{E}_{q'}}{m(\bar{\lambda}_q + \bar{\lambda}_{q'})}\right] - 1\right] , \quad (2.37b)$$

where

$$\mathscr{E}_q = (m^2 + q_z^2)^{1/2}, \quad \bar{\lambda}_q = m(\mathscr{E}_q - q_z)^{-1} . \quad (2.37c)$$

These are independent of $\mathbf{q}_\perp$. The requirement that $I^{(0)} > 0$ leads to the restriction $x^2 < 2/e$ for the validity of this expansion. We also get

$$I^{(1)} = q_\perp^2 C + q_\perp'^2 \tilde{C} - \gamma(\hat{\mathbf{x}}\cdot(\mathbf{q}\bar{\lambda}_q - \mathbf{q}'\bar{\lambda}_{q'}))^2 , \quad (2.37d)$$

where

$$C = \frac{1}{4\omega\mathscr{E}_q}\left[\ln\left[\frac{2}{x^2}\frac{\mathscr{E}_q + \mathscr{E}_{q'}}{m(\bar{\lambda}_q + \bar{\lambda}_{q'})}\right] + \frac{3}{2}\frac{\bar{\lambda}_q^2(\mathscr{E}_q + \mathscr{E}_{q'})}{m(\bar{\lambda}_q + \bar{\lambda}_{q'})} + \frac{1}{2}\right] \quad (2.37e)$$

$$\gamma = [2m\omega(\bar{\lambda}_q + \bar{\lambda}_{q'})]^{-1} , \quad (2.37f)$$

and $\tilde{C}$ is gotten from $C$ by interchanging $\mathbf{q}$ and $\mathbf{q}'$. In all the other factors of (2.36) we set $\mathbf{q}_\perp \mathbf{q}_\perp' = 0$. The remaining $\mathbf{q}_\perp, \mathbf{q}_\perp'$ integrals are straightforward. The remaining integrals take the form

$$\int\int_{-\infty}^{\infty} dq_z dq_z' e^{-2I^{(0)}} f(q_z, q_z') , \quad (2.38)$$

where $f$ is a function which does not depend upon $m/\omega$. The integral is evaluated by expansion around the minimum of $I^{(0)}$ which occurs at $q_z = q_z' = \bar{q} = m\sinh\mu$ where $\mu$ is obtained from

$$x^2 = (1 + e^{-2\mu})\exp(-\coth\mu) . \quad (2.39)$$

For $x^2 \simeq 10^{-2}$ this yields $\mu = 0.2$ which substantiates the remark that $q_z$ is almost relativistic ($\sim 0.2m$) even at such small values of $x^2$. Again the integrals are straightforward with the result:

$$W = \frac{\sqrt{2}}{\pi} Z^2 \alpha_F^2 \frac{\omega^3}{m^2} e^{-2I^{(0)}_{\min}} e^{4\mu}\cosh^2\mu \sinh^3\mu \times [(2\cosh\mu - \sinh\mu)^2 - 4e^{-4\mu}\sinh^2\mu]^{-1/2} \times [2\cosh\mu + \sinh\mu + 7\sinh\mu\cosh\mu e^{\mu}]^{-1/2} \times [2\cosh\mu + \sinh\mu + 3\sinh\mu\cosh\mu e^{\mu}]^{-3/2} , \quad (2.40)$$

where

$$2I^{(0)}_{\min} = \frac{2m}{\omega} e^{-\mu}\coth\mu . \quad (2.41)$$

The result is of order $e^{-5\times10^6}$/sec for $Z=100$ for the numbers quoted above. Evidently this is not of practical interest.

In order to compare our result for a linearly polarized laser with that for the circularly polarized laser,[1] it will be sufficient to compare only the dominant factor, $\exp(-2I_{\min}^{(0)})$, with the analogous one occurring there. However, I should first point out the essential difference between the two calculations. It is that the modified Bessel functions appearing here (2.24) are replaced by the ordinary ones in the circular polarization calculation. This occurs even in the nonrelativistic domain and is due to the fact that the $A^2$ term in the Hamiltonian is a constant for circular but not for linear polarization. The dominant factor in (2.40) arises from the steepest-descent treatment of the "Bessel" functions and so it is not surprising that a difference exists.

For small $x^2$, (2.39) can be inverted as

$$\mu=\left[\ln\frac{2}{x^2}\right]^{-1}+\frac{2}{3}\left[\ln\frac{2}{x^2}\right]^{-2}+\cdots , \qquad (2.42)$$

so that

$$e^{-2I_m^{(0)}}=\exp\left[-\frac{2m}{\omega}\left[\ln\frac{2}{x^2}-\frac{5}{3}+\cdots\right]\right] . \qquad (2.43)$$

For similar conditions Yakovlev[1] gives [Eq. (33)]

$$\exp\left[-\frac{2m}{\omega}\left[\ln\frac{1}{\xi^2}+\ln\left[\ln\frac{1}{\xi^2}\right]-2\right]\right] , \qquad (2.44)$$

where $\xi^2=e^2E_0^2/m\omega^2$ in our notation. If the circularly and linearly polarized lasers have the same intensity then $\xi^2=x^2/2$. Then for small $x$, (2.43) is much larger than (2.44) because of the large value of $m/\omega$.

## ACKNOWLEDGMENT

This research was supported by a grant from the National Science Foundation and by the City University of New York (CUNY) Faculty Research Award Program No. 6-66458.


[1]V. P. Yakovlev, Zh. Eksp. Teor. Fiz. **49**, 318 (1965) [Sov. Phys.—JETP **22**, 233 (1966)].

[2]I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. A **14**, 1101 (1976).

[3]The solution of the Dirac equation in the presence of an electromagnetic unindirectional pulse was first given by D. M. Volkov, Z. Phys. **94**, 250 (1935) and in a covariant form by L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964). The form (2.15) is most readily extracted from the second of these.

[4]See, for example, J. J. Sakuri, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, MA, 1967).

[5]These functions have been discussed by many authors. For example, see L. S. Brown and T. W. B. Kibble (Ref. 3); and H. R. Reiss, J. Math Phys. 3, 59 (1962); Phys. Rev. A **22**, 1786 (1980), which contains further references.

[6]See, for example, J. H. Eberly, in *Progress in Optics*, edited by E. Wolf (Wiley, New York, 1969).

[7]E. Fiordilino and M. H. Mittleman, J. Phys. B **18**, 4425 (1985).