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Phase dynamics with a material derivative due to a flow field

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We show that a flow field, which can be either externally imposed or caused by the geometrical configuration, gives rise to a convective term or a material derivative in the resulting phase equation. The stationary solutions of this equation are in agreement with recent experimental results on convection with a through flux and on the flow between two rotating concentric cones (a modified Taylor configuration). We also discuss the fact that there is no additional independent dynamic degree of freedom associated with these flow fields.

I. INTRODUCTION

The theory of phase dynamics—the analog of hydrodynamics for pattern-forming nonequilibrium systems in large-aspect-ratio cells, e.g., for Rayleigh-Benard convection (in which a thin layer of a simple fluid is heated from below) or for the Taylor instability (in which the fluid in the gap between two concentric cylinders is subjected to a torque by rotating the inner or both cylinders)—has been developed gradually over the last few years. Shortly after it was recognized[1] that a phase diffusion equation is also applicable to describe the long-wavelength relaxation of a set of convective roles well above the onset of the instability, this prediction was confirmed experimentally for large-Prandtl-number fluids.[2,3] Since then the approach has been generalized to various systems. The influence of vertical vorticity on convective phase dynamics was elucidated,[4] propagating modes were predicted to occur for a number of systems,[5-10] nonlinearities in the phase equations were included,[6,7,9-11] and the influence of defects was investigated.[12-14] In addition, it has been discussed[15] that the concept of phase dynamics should also be applicable to pattern-forming systems under the influence of an external field or an external load such as, e.g., the Rosenzweig instability in ferrofluids or buckling in metal plates.

Here we extend the previous lines of thought in a different direction. We investigate to what extent there exists an analog of the convective term or material derivative which arises in all nonlinear hydrodynamic equations for variables close to local thermodynamic equilibrium. Evidently this question is quite subtle for pattern-forming nonequilibrium systems, since there is no reason to expect that the density of linear momentum arises naturally as an additional quasihydrodynamic variable in a phase-dynamic description. Therefore we will discuss in Sec. II the influence of an externally imposed flow field and establish close contact between some of the predictions and recent experimental results.[16] In Sec. III we discuss a simple model for the case of an internally generated flow field and compare its implications with results of experiments which have been carried out for the flow between concentric rotating cones.[17,18] This is followed by a short section containing conclusions and a perspective.

II. THE APPEARANCE OF A MATERIAL DERIVATIVE IN PHASE DYNAMICS

In this section we consider the effect of an external flow field on a one-dimensional set of rolls. We were stimulated by early experimental observations[19] to consider a situation where this superimposed flow leads to a change in the wave vector along the cell and to a stationary situation.

Since the superimposed flow field v is not an internally generated quantity, it clearly does not obey an equation by itself, so that we can focus on the question of how to incorporate the effect of this field into the phase equation, which reads without flow for spatial variations in one dimension:[1]

$$\dot{\phi}=D_{\parallel}\phi_{xx} \; . \tag{2.1}$$

Clearly the fact that a stationary situation arises is incompatible with a driving term proportional to v in Eq. (2.1), as this would correspond to a propagation of the roll pattern which is not observed.[19] It is obvious, however, that such a term would be allowed by symmetry arguments. Since the new term has to break the $x \rightarrow -x$, $\phi \rightarrow -\phi$ symmetry, the lowest-order nontrivial term satisfying these requirements and not giving rise to a propagation of the pattern takes the form $v\partial_x\phi$. Formally this term looks like a material derivative or a convective term giving rise to the phase equation

$$(\partial_t + v\partial_x)\phi = D_{\parallel}\phi_{xx} \; , \tag{2.2}$$

thus describing a transformation of Eq. (2.1) to a situation with advection due to the imposed flow field. It is important to note that there is no additional equation for v, as is the case for hydrodynamic systems close to equilibrium, since v is completely controlled externally.

In the case of a stationary situation the implication of Eq. (2.2) is easy to see. One obtains a variation Δk of the wave number k or cell size as one moves along the cell

$$\frac{\Delta k}{k}=\frac{v}{D_{\parallel}} \; . \tag{2.3}$$

Relation (2.3) ties in nicely with the detailed experimental results and their thorough discussion by Pocheau, Croquette, LeGal, and Poitou,[16] which have been carried out for convection in an annular geometry with a through flux. Fairly close to the onset of the instability and for small flow rates the authors find that $k/D_{\parallel}$ is independent of the flow rate and that Δk varies linearly with x along the azimuthal direction of the cell over most of its length.

Starting from Eq. (2.2) it is straightforward to incorporate higher-order effects and nonlinearities along the lines of Refs. 6 or 11. We obtain

$$\dot{\phi}+v\partial_x\phi+\alpha v\partial_{xxx}\phi+\beta v(\partial_x\phi)\partial_{xx}\phi$$
$$=D_{\parallel}\partial_{xx}\phi+D_1\partial_{xxxx}\phi+E_1(\partial_x\phi)\partial_{xx}\phi\ , \quad (2.4)$$

or, equivalently,

$$\dot{\phi}+v[1+\alpha\partial_{xx}+\beta\partial_x\phi]\partial_x\phi$$
$$=(D_{\parallel}+D_1\partial_{xx}+E_1\partial_x\phi)\partial_{xx}\phi\ . \quad (2.5)$$

The higher-order gradient terms and the nonlinearities become more important as the variations in ϕ become more rapid as they do, e.g., close to the filling holes.

Using the technique of Newell and co-workers,[20-23] it is easy to see that an amplitude equation of the form

$$(\partial_t+v\partial_x)A=\varepsilon A-g|A|^2A+D_{\parallel}A_{xx} \quad (2.6)$$

gives near threshold rise to a phase equation of the form (2.2). That is, in the amplitude equation the flow term also appears in the form of a material derivative.

III. FLOW BETWEEN CONCENTRIC ROTATING CONES

In an interesting experiment on a generalized Taylor configuration, namely, two concentric cones with constant gap, where the outer one is fixed and the inner one is rotating, Wimmer[17,18] observed a number of interesting phenomena. First of all there is always a large-scale flow parallel to the generating line of the cone as a result of the fact that the velocity is smaller for bigger cross sections and thus for the flow which arises at the larger radius and vice versa.

For the case of a gap completely filled with vortices, which exists over a large range of values of the Reynolds number, Wimmer found that the size of the vortex pairs decreased linearly along the axis of the cones (cf. Fig. 3 of Ref. 17)—very similarly to the case of the rolls discussed in Sec. II. In addition, Wimmer found in his experiments that the vortex pairs were propagating with a constant velocity v in the direction parallel to the generating line. As the Reynolds number was increased, the velocity of propagation decreased;[17] the same feature also emerged for a cone rotating in a cylinder thus producing a variable gap.[18]

To describe these phenomena in the framework of phase dynamics we clearly need a driving force to account for the velocity with which the vortex pairs move and to obtain a linear dependence of the pair size along the generating line. A material derivative similar to that discussed in Sec. II seems mandatory, especially given the fact that there is also the possibility of a stationary state as the Reynolds number is increased.

Thus, we are led to the following "minimal" model equation

$$\dot{\phi}-v+v\partial_x\phi=D_{\parallel}\partial_x^2\phi\ , \quad (3.1)$$

where x is the coordinate parallel to the generating line. Equation (3.1) has—to the order in the gradients considered—a structure isomorphic to that obtained for the displacement field parallel to the density wave in smectic-A and cholesteric liquid crystals,[24] where in Eq. (3.1), however, v is the mean-flow velocity. From an inspection of Eq. (3.1) it is immediately clear that the first two terms on the left-hand side can account for the propagation of the vortex pairs whereas the "convective" term and phase diffusion can compensate each other to give rise to a linear variation of pair sizes as in Sec. II. Higher-order gradient terms and nonlinearities can be added to Eq. (3.1) in the same spirit. To get a closed system of equations, one can write down an equation connecting gradients of v and gradients of ϕ. As for the case of the Benard convection in simple fluids,[4] this equation is not dynamic. For the Benard convection this is in agreement with experimental results[25] and with the amplitude-equation approach.[22] Since Eq. (3.1) takes a very simple form, it is highly desirable to test its range of applicability in detail. Close to the onset of the instability the simplest possible amplitude equation leading to Eq. (3.1) takes in one dimension the form, again using the technique developed by Newell and co-workers,[20-23]

$$\dot{A}+v\partial_xA=\varepsilon A-g|A|^2A+iAv+D_{\parallel}A_{xx}\ . \quad (3.2)$$

In closing this section we note that there are also experimental results[26] on convection in a rectangular box with Poiseuille flow, which indicate that it is possible that the whole roll pattern propagates with apparently no change in the wave number, thus giving rise to a phase equation of the form

$$\dot{\phi}=D_{\parallel}\partial_x^2\phi+\alpha v\ , \quad (3.3)$$

without a material derivative. Therefore it will be necessary to understand better in which geometries and for which flows Eqs. (2.2), (3.1), and (3.3), respectively, apply. A question which goes, however, beyond the scope of the present Rapid Communication.

IV. CONCLUSIONS AND PERSPECTIVE

In the present note we have shown that both external flow and—in some cases—internally generated flow, can show up in the resulting phase equations as a convective term and we have also discussed how this can be linked with amplitude equations valid close to onset. For the case of the rotating concentric cones it is highly desirable to perform additional high-precision experiments to confirm or modify the picture suggested above, as the results presented so far have emphasized the qualitative aspects of the phenomena observed. In particular, it seems worthwhile to check in detail the twofold way the flow field enters the phase equation, as a convective and as a driving force.

It seems remarkable that in none of the examples discussed nor in any realistic physical system has there been any indication of additional dynamic degree of freedom brought about by an externally imposed flow field or by an internally generated velocity field. On the contrary, all the experiments reported so far[2,3,16–18,25] can be analyzed without a velocity field as a dynamic variable.

Finally, we note that the modifications of the phase equation discussed here are not restricted to systems showing convective rolls or Taylor-type vortices, but can also be expected to occur in pattern-forming systems producing spirals with a continuously changing pitch[27] which have been observed in precipitation[28] and solidification accompanied by convection.[29]

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