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Higher-order squeezing from an anharmonic oscillator

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We show that ordinary coherent light, interacting with a nonabsorbing nonlinear medium modeled as an anharmonic oscillator, is highly squeezed, at least up to the sixth order.

Recently Hong and Mandel[1] have introduced the notion of higher-order squeezing of quantized electromagnetic fields as a generalization of the much-discussed second-order squeezing.[2] By higher-order squeezing one means that the Nth-order moments of the fields may take on values less than their coherent-state values. Since the squeezing effect is uniquely nonclassical only for even moments, one may be restricted to calculate only moments for which $N=2n$. In Ref. 1 it was shown that a number of systems exhibit higher-order squeezing; in some cases the degree of higher-order squeezing is even greater than in second order. Systems discussed in Ref. 1 are degenerate parametric down conversion, harmonic generation, and resonance fluorescence. Kozierowski[3] has recently shown that higher-order squeezing is obtainable in kth harmonic generation. In this paper we show that coherent light interacting with a nonabsorbing nonlinear medium modeled as an anharmonic oscillator becomes highly squeezed at least to sixth order. Previously, Tanas[4] has shown that second-order squeezing is produced by this system. Milburn[5] has also discussed, among other things, the second-order squeezing for this system. We show here that the fractional squeezing increases with order.

The higher-order squeezing is defined in the following way.[1] We let $\hat{E}_1$ and $\hat{E}_2$ stand for the Hermitian quadratures of the field such that

$$\begin{aligned}\hat{E}_1&=\hat{E}^{(+)}e^{i(\omega t-\varphi)}+\hat{E}^{(-)}e^{-i(\omega t-\varphi)}\ ,\\ \hat{E}_2&=\hat{E}^{(+)}e^{i(\omega t-\varphi-\pi/2)}+\hat{E}^{(-)}e^{-i(\omega t-\varphi-\pi/2)}\ ,\end{aligned} \tag{1}$$

where φ is an arbitrary phase. If $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ satisfy the commutation relations

$$[\hat{E}^{(+)},\hat{E}^{(-)}]=C\ , \tag{2}$$

where C is an ordinary number, then

$$[\hat{E}_1,\hat{E}_2]=2iC \tag{3}$$

so that $\hat{E}_1$ and $\hat{E}_2$ are the conjugate in and out of phase quadratures of the field. The variances of second order obey the uncertainty relation

$$\langle(\Delta\hat{E}_1)^2\rangle\langle(\Delta\hat{E}_2)^2\rangle\geq C^2\ , \tag{4}$$

where $A\hat{E}=\hat{E}-\langle\hat{E}\rangle$. For a coherent state $\langle(\Delta\hat{E}_{1,2})^2\rangle=C$ so that the equality holds in Eq. (4). If some phase φ, $\langle(\Delta\hat{E}_1)^2\rangle<C$, then the $\hat{E}_1$ quadrature is squeezed to second order. Now, according to Ref. 1 the $\hat{E}_1$ quadrature is squeezed to the Nth order (N even) if

$$\langle(\Delta\hat{E}_1)^N\rangle<(N-1)!!C^{N/2} \tag{5}$$

for some angle φ. The degree to which the state is squeezed to the Nth order is conveniently given by

$$q_N=\frac{\langle(\Delta\hat{E}_1)^N\rangle-(N-1)!!C^{N/2}}{(N-1)!!C^{N/2}}\ . \tag{6}$$

q_N is negative for Nth-order squeezing and $|q_N|$ has a maximum value of 1. In our calculations we write

$$\begin{aligned}\hat{E}_1(t)&=\hat{a}(t)e^{i(\omega t-\varphi)}+\hat{a}^{\dagger}(t)e^{-i(\omega t-\varphi)}\ ,\\ \hat{E}_2(t)&=\hat{a}(t)e^{i(\omega t-\varphi-\pi/2)}+\hat{a}^{\dagger}(t)e^{i(\omega t-\varphi-\pi/2)}\ ,\end{aligned} \tag{7}$$

so that $C=1$.

Now, for the system of interest here the Hamiltonian is

$$\hat{H}=\hbar\omega\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}K\hat{a}^{\dagger 2}\hat{a}^2\ , \tag{8}$$

where K is the anharmonicity parameter related to the third-order susceptibility of the medium and is known to lead to optical bistability.[6] All nonenergy conserving terms have been dropped and we consider only a single mode of the electromagnetic field. Heisenberg's equation for a reads

$$\dot{\hat{a}}=-\frac{i}{\hbar}[\hat{a},H]=-i(\omega+K\hat{a}^{\dagger}\hat{a})\hat{a}\ . \tag{9}$$

Since $\hat{a}^{\dagger}\hat{a}$ can be shown to be a constant of the motion, Eq. (9) has the simple solution

$$\hat{a}(t)=\exp\{-it[\omega+K\hat{N}(0)]\}\hat{a}(0)\ , \tag{10}$$

where $\hat{N}(0)=\hat{a}^{\dagger}(0)\hat{a}(0)$.

Now the variances we wish to calculate may be expressed directly in terms of the expectation values of the field and its powers. We consider only the $\hat{E}_1$ quadrature. Thus the even-order moments of E_1 are (to sixth order)

$$\langle(\Delta\hat{E}_1)^2\rangle=\langle\hat{E}_1^2\rangle-\langle\hat{E}_1\rangle^2\ , \tag{11a}$$

$$\langle(\Delta\hat{E}_1)^4\rangle=\langle\hat{E}_1^4\rangle-4\langle\hat{E}_1^3\rangle\langle\hat{E}_1\rangle+6\langle\hat{E}_1^2\rangle\langle\hat{E}_1^2\rangle-3\langle\hat{E}_1\rangle^4, \tag{11b}$$

$$\langle(\Delta\hat{E}_1)^6\rangle=\langle\hat{E}_1^6\rangle-6\langle\hat{E}_1^5\rangle\langle\hat{E}_1\rangle+15\langle\hat{E}_1^4\rangle\langle\hat{E}_1\rangle^2-20\langle\hat{E}_1^3\rangle\langle\hat{E}_1\rangle^3+15\langle\hat{E}_1^2\rangle\langle\hat{E}_1\rangle^4-5\langle\hat{E}_1\rangle^6\ . \quad (11c)$$

For squeezing to the respective orders we must have

$$\langle(\Delta\hat{E}_1)^2\rangle<1,\ \ \langle(\Delta\hat{E}_1)^4\rangle<3,\ \ \langle(\Delta\hat{E}_1)^6\rangle<15\ .$$

With the initial state a coherent state $|\alpha\rangle$, we calculate $\langle\hat{E}_1^m\rangle$ $m=1,\ldots,6$. We are required to calculate terms of the form

$$\langle\alpha|(\hat{a}^\dagger e^{i\tau\hat{N}})^l(e^{-i\tau\hat{N}}\hat{a})^m|\alpha\rangle=(\alpha^*)^l(\alpha)^m\exp\left[\frac{i\tau}{2}[l(l-1)-m(m-1)]\right]\times\exp(|\alpha|^2\{\exp[-i\tau(m-l)]-1\})\ , \quad (12)$$

where $\hat{a}$ and $\hat{a}^\dagger$ stand for the operators at $t=0$ and $\tau=Kt$. With $n=|\alpha|^2$ and using Eqs. (7), (10), and (12), we obtain

$$\langle\hat{E}_1\rangle=2\,\mathrm{Re}\{\alpha e^{-i\varphi}\exp[n(e^{-i\tau}-1)]\}\ , \quad (13)$$

$$\langle\hat{E}_1^2\rangle=1+2n+2\,\mathrm{Re}\{\alpha^2e^{-i(\tau+\varphi)}\exp[n(e^{-2i\tau}-1)]\}\ , \quad (14)$$

$$\langle\hat{E}_1^3\rangle=2\,\mathrm{Re}\{\alpha ne^{-3i(\tau+\varphi)}\exp[n(e^{-3i\tau}-1)]+3\alpha e^{-i\varphi}(1+ne^{-i\tau})\exp[n(e^{-i\tau}-1)]\}\ , \quad (15)$$

$$\langle\hat{E}_1^4\rangle=3+12n+6n^2+2\,\mathrm{Re}\{n^2e^{-i(6\tau+4\varphi)}\exp[n(e^{-4i\tau}-1)]+(4n^2e^{-3i\tau}+6ne^{-i\tau})e^{-2i\varphi}\exp[n(e^{-2i\tau}-1)]\}\ , \quad (16)$$

$$\langle\hat{E}_1^5\rangle=2\,\mathrm{Re}\{\alpha n^2e^{-i(5\tau+10\varphi)}\exp[n(e^{-5i\tau}-1)]+(10\alpha ne^{-3i\tau}+5\alpha n^2e^{-6i\tau})e^{-3i\varphi}\exp[n(e^{-3i\tau}-1)]+(15\alpha+30\alpha ne^{-i\tau}+10\alpha n^2e^{-2i\tau})e^{-i\varphi}\exp[n(e^{-i\tau}-1)]\}\ , \quad (17)$$

and finally

$$\langle\hat{E}_1^6\rangle=15+90n+90n^2+20n^3+2\,\mathrm{Re}\{n^3e^{-i(6\varphi+15\tau)}\exp[n(e^{-6i\tau}-1)]+(45ne^{-i\tau}+15n^3e^{-5i\tau}+60n^2e^{-3i\tau})e^{-2i\varphi}\exp[n(e^{-2i\tau}-1)]+(15n^2e^{-6i\tau}+6n^3e^{-10i\tau})e^{-4i\varphi}\exp[n(e^{-4i\tau}-1)]\}\ . \quad (18)$$

In our calculations the phase of α is chosen so that α is real.

Due to the complicated nature of Eqs. (13)–(18) it is convenient to write out the variance only for the second order

$$\langle(\Delta\hat{E}_1)^2\rangle=1+2n\{1-\exp[2n(\cos\tau-1)]\}+2n\,\mathrm{Re}\{e^{-2i\varphi}e^{-i\tau}\exp[n(e^{-2i\tau}-1)]-e^{-2i\varphi}\exp[2n(e^{-i\tau}-1)]\}\ . \quad (19)$$

For the phase $\varphi=0$ this coincides with the results of Tanas.[4] For crystal of a few centimeters τ is typically on the order of 10^{-6}. As shown by Tanas[4] with $\tau=10^{-6}$, the second-order variance is squeezed for the average number of photons n of the order of 10^6. This can be obtained from the direct numerical evaluation of Eq. (19) using high precision computation. On the other hand, for $n\sim10^6$, $\tau\sim10^{-6}$ one may expand Eq. (19) as a function of $n\tau$ to obtain[7]

$$\langle(\Delta\hat{E}_1)^2\rangle\simeq1+2n^2\tau^2[1-\cos(2n\tau)]-2n\tau\sin(2n\tau)\ . \quad (20)$$

In any case the squeezing obtained can be quite high as shown in Fig. 1. We obtain a first minimum at $n\tau=0.59$ and a second deeper minimum at $n\tau=3.29$. The q factor at these minima are, respectively, $q_2=-0.6600$ and $q_2=-0.9778$.

Now for the fourth- and sixth-order variances it is obvious that expanding in terms of $n\tau$ is extremely laborious. Instead we resorted to calculating Eqs. (11b) and (11c) using Eqs. (13)–(16) with double precision on a Cyber 720. The results are displayed in Figs. 2 and 3. We note that

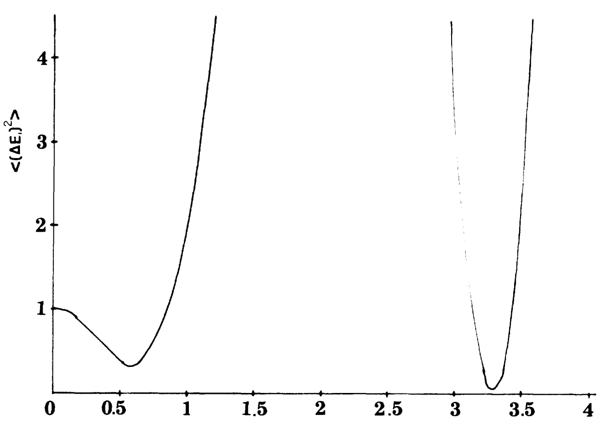

FIG. 1. The second-order variance.

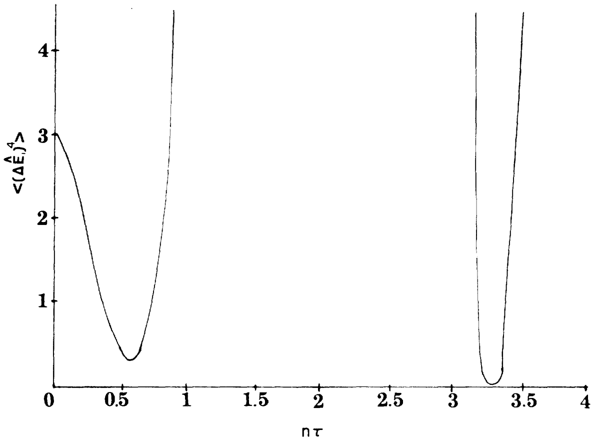

FIG. 2. The fourth-order variance.

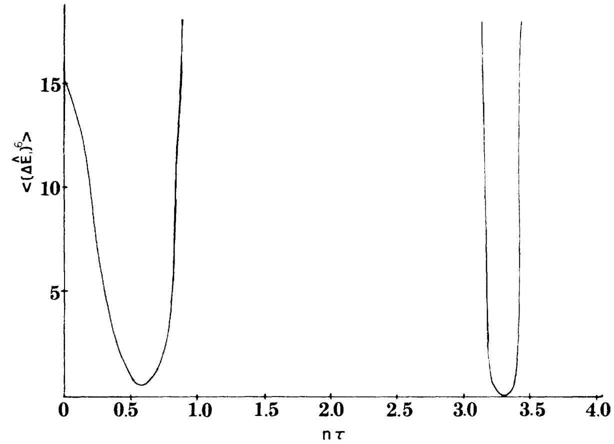

FIG. 3. The sixth-order variance.

minima occur at the same locations as for second order. At $n\tau=0.59$ we obtain $q_4=-0.884$ and $q_6=-0.9667$, while at $n\tau=3.29$ we have $q_4=-0.9995$ and $q_6=-0.9999$. Evidently the fractional squeezing increases with the order of the variance.

Finally we wish to point out that the higher-order squeezing from the anharmonic oscillator is not intrinsic, i.e., dominated by the second-order variance. In order to see this the even-order variances may be written in terms of the normally ordered variances as

$$\langle(\Delta\hat{E}_1)^N\rangle=\sum_{r=0}^{\frac{N}{2}-1}\left[\begin{pmatrix}N\\2r\end{pmatrix}\frac{(2r)!}{r!2^r}\langle:(\Delta\hat{E}_1)^{N-2r}:\rangle+(N-1)!!\right]. \tag{21}$$

For the fourth- and sixth-order squeezing it is required that, respectively,

$$\langle:(\Delta\hat{E}_1)^4:\rangle+6\langle:(\Delta\hat{E}_1):\rangle<0\ , \tag{22}$$

$$\langle:(\Delta\hat{E}_1)^6:\rangle+15\langle:(\Delta\hat{E}_1)^4:\rangle+45\langle:(\Delta\hat{E}_1)^2:\rangle<0\ , \tag{23}$$

where $\langle:(\Delta\hat{E}_1)^2:\rangle=\langle(\Delta\hat{E}_1)^2\rangle-1$. The squeezing is called intrinsic if $\langle:(\Delta\hat{E}_1)^N:\rangle$ is not negative beyond $N=2$ but that the $N=2$ term of the series in Eqs. (22) and (23) dominates. However, for the case at hand, using the numerical results obtained from Eq. (11b) and (11c) we find that at the point of high squeezing $\langle:(\Delta\hat{E}_1)^4:\rangle$ and $\langle:(\Delta\hat{E}_1)^6:\rangle$ are indeed negative indicating that the higher-order squeezing is not intrinsic.

In summary then we have shown in this paper that a nonabsorbing nonlinear medium modeled as an anharmonic oscillator is capable of producing higher-order squeezing and that the degree to which the squeezing occurs increases with order.

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