Inhomogeneously broadened laser with a saturable absorber

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A quantum theory for a laser with inhomogeneously broadened active and absorber atoms is presented. Photon-number distributions are derived for both on- and off-resonance operation of the laser, and the results are compared with the results of earlier investigations.

I. INTRODUCTION

It is well known that a nonlinear absorber inside a laser cavity imparts new characteristics to the laser which are of interest both theoretically and experimentally.[1-8] Examples of such lasers are the laser with a saturable absorber (LSA) and the dye laser. Both systems have been investigated in some detail. The treatments for the LSA with inhomogeneously broadened atoms are, however, valid either only for small intensities or do not fully take into account the effect of quantum noise. We would like to present a quantum-mechanical treatment that allows us to derive the photon-number distribution for an inhomogeneously broadened LSA. We also incorporate the effects of detuning and atomic motion.[9-11]

II. EQUATION OF MOTION

Consider a single-mode electromagnetic field at a frequency Ω inside a laser cavity interacting with a set of amplifying and another set of absorbing two-level atoms in gas phase. The distribution of atomic speeds along the resonator axis will be assumed to be Gaussian with root-mean-square speeds u_1 and u_2,

$$D_i(v)=\frac{1}{u_i\sqrt{\pi}}\exp\left[-\left[\frac{v}{u_i}\right]^2\right],\quad i=1,\,2\,. \tag{1}$$

Throughout this paper the subscripts 1 and 2 will refer to active and absorber atoms, respectively. The single atom-field Hamiltonian is given by[2]

$$\hat{H}=\hbar\Omega\hat{a}^{\dagger}\hat{a}+\sum_{i=1}^{2}[\hbar\omega_{0i}\hat{b}_i^{\dagger}\hat{b}_i+(\hbar g_i\hat{a}\hat{b}_i^{\dagger}+\text{H.c.})]\,. \tag{2}$$

Here $\hat{a}^{\dagger}(\hat{a})$ is the creation (annihilation) operator for the field, $\hat{b}_i^{\dagger}(\hat{b}_i)$ is the raising (lowering) operator for the atoms of type i, and ω_{0i} is the transition frequency. The coupling constant is given by $g_i=ex_i\sqrt{\Omega/2\epsilon_0\hbar V}$, where x_i is atomic transition dipole moment, Ω is the field frequency, and V is the quantization volume. The spatial variation of the field inside the cavity is taken into account by multiplying g_i by an appropriate (traveling or standing wave) mode function.

The equation of motion for the field density matrix is derived following the work of Scully, Kim, and Lamb[10] and Riska and Stenholm.[11] The details of this derivation can be found in the work of Roy.[2] Here we merely outline the various steps involved and point out differences that arise due to inhomogeneous broadening. We first calculate the change in the density matrix due to an atom introduced in state $|\alpha\rangle$. When this contribution is multiplied by r_α, the number of atoms introduced per second in $|\alpha\rangle$ and averaged over the distribution of detunings, we get the coarse-grained rate of change of the density matrix due to atoms introduced in state $|\alpha\rangle$. Similarly, passive losses are calculated by introducing a fictitious set of two-level atoms that absorb laser radiation. Adding the contribution from atoms and losses we obtain the following equation of motion for the probability $p(n)$ for finding n photons in the laser:[2]

$$\begin{aligned}\dot{p}(n)=&-[(n+1)R_1(n)+nR_2(n-1)+Cn]p(n)\\&+nR_1(n-1)p(n-1)\\&+[(n+1)R_2(n)+C(n+1)]p(n+1)\,,\end{aligned} \tag{3}$$

where

$$R_1(n)=\frac{2r_a|g_1|^2\gamma_{ab}}{\gamma_a}\int_{-\infty}^{\infty}dv\frac{D_1(v)}{(\omega_{01}-\Omega+kv)^2+\gamma_{ab}{}^2\left[1+\dfrac{4|g_1|^2}{\gamma_a\gamma_b}(n+1)\right]}\,, \tag{4}$$

$$R_2(n)=\frac{2r_d|g_2|^2\gamma_{cd}}{\gamma_d}\int_{-\infty}^{\infty}dv\frac{D_2(v)}{(\omega_{02}-\Omega+kv)^2+\gamma_{cd}{}^2\left[1+\dfrac{4|g_2|^2}{\gamma_c\gamma_d}(n+1)\right]}\,, \tag{5}$$

and C is the rate at which the field intensity decays. γ_α is the decay rate of level $|\alpha\rangle$ and $\gamma_{\alpha\beta}$ is the decay rate for the transition dipole moment between levels $|\alpha\rangle$ and $|\beta\rangle$. We have assumed that the active atoms are introduced in their upper state $|a\rangle$ and the absorber atoms are introduced in their lower state $|d\rangle$. The velocity integrals in Eqs. (4) and (5) appear because the atoms moving with different speeds will have different resonance frequencies due to the Doppler shift $kv=(\Omega/c)v$. The velocity integrals can be expressed in terms of plasma dispersion function.[12] Equations (3)—(5) are the basic equations of this paper.

III. PHOTON-NUMBER DISTRIBUTIONS

We first consider the on-resonance operation ($\omega_{01}=\Omega=\omega_{02}$). In the limit of extreme Doppler broadening $ku_i \gg \gamma_{ab},\gamma_{cd}$, a condition which is satisfied for many gas lasers, the velocity integrals can be evaluated[12] to give the following equation of motion for $p(n)$:

$$\dot{p}(n)=-\frac{A(n+1)}{\left[1+\frac{B}{A}(n+1)\right]^{1/2}}p(n)-\frac{Gn}{\left[1+\frac{H}{G}n\right]^{1/2}}p(n)-Cnp(n)$$
$$+\frac{An}{\left[1+\frac{B}{A}n\right]^{1/2}}p(n-1)+\frac{G(n+1)}{\left[1+\frac{H}{G}(n+1)\right]^{1/2}}p(n+1)+C(n+1)p(n+1)\ , \tag{6}$$

where

$$A=\frac{2\sqrt{\pi}r_a|g_1|^2}{\gamma_a ku_1},\quad B=\frac{4|g_1|^2}{\gamma_a\gamma_b}A\ , \tag{7}$$

$$G=\frac{2\sqrt{\pi}r_d|g_2|^2}{\gamma_d ku_2},\quad H=\frac{4|g_2|^2}{\gamma_c\gamma_d}G\ . \tag{8}$$

This equation satisfies detailed balance in the steady-state situation so that the probability $p_s(n)$ for n photons in the laser field becomes

$$p_s(n)=p_s(0)\prod_{m=1}^{n}\frac{\left[\frac{A}{C}\right]\left[1+\frac{H}{G}m\right]^{1/2}}{\left[1+\frac{B}{A}m\right]^{1/2}\left[\frac{G}{C}+\left[1+\frac{H}{G}m\right]^{1/2}\right]}\ . \tag{9}$$

We have derived Eq. (9) for a traveling-wave field mode for which the spatial structure of the field plays no essential role since all the atoms see the same field. For a standing-wave field mode atoms are subject to spatial hole burning. In the limit of large Doppler broadening, however, the effects of spatial holes are washed out because the atoms sample the field over several wavelengths before decaying. This is the so-called rate-equation approximation[4,13] (REA) which is known to be an excellent approximation even for relatively large excitations. This means that on resonance Eq. (9) is valid for a standing-wave gas laser also in the REA.

The expression for $p_s(n)$ simplifies under certain circumstances. In the nonoscillating regime $A<C$, the mean photon number is small so that saturation effects are not important and we obtain

$$p_s(n)=\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}},\quad \langle n\rangle=\frac{A}{G+C}\ , \tag{10}$$

which is the Bose-Einstein distribution for thermal photons. The role of the absorber in this regime is simply to increase losses. With increasing gain the mean photon number increases causing saturation effects to come into play. In the regime where $A>C$ but $(A-C)/C\ll 1$ so that both $(B/A)n$ and $(H/G)n$ are small we can use the expansion $\ln(1+x)\simeq x-x^2/2$ to obtain

$$p_s(n)=k\exp\left[\left[\frac{A}{G+C}-1\right]n+\frac{1}{4}\frac{B}{A}\left[\frac{s\alpha}{1+\alpha}-1\right]n^2 -\frac{1}{12}\frac{B^2}{A^2}\left[\frac{s^2\alpha(\alpha+3/2)}{(1+\alpha)^2}-1\right]n^3\right]\ , \tag{11}$$

where k is a constant, $\alpha=G/C$ and $s=(A/B)/(G/H)$ is the ratio of the saturation number of photons $n_s=A/B$ and $\bar{n}_s=G/H$ for the active and absorber atoms. Both the numbers n_s and $\bar{n}_s$ are of order 10^6–10^7 for gas lasers. Note that the coefficient of the n^2 term in the exponent may become positive. If this is the situation the cubic term, whose coefficient is then necessarily negative, is needed to ensure a normalized distribution. Photon-number distribution [Eq. (11)] was also derived by Kazantsev and Surdutovich[3] by using perturbative approach to the density matrix equation of motion. Under suitable approximations we can also derive the distribution given

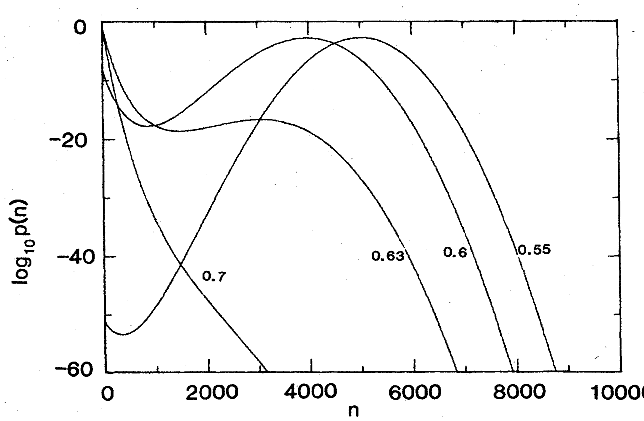

FIG. 1. Forms of the probability distribution for $A/C=1.5$, $B/A=10^{-4}$, $s=10$, and several different values of G/C as derived from Eq. (9).

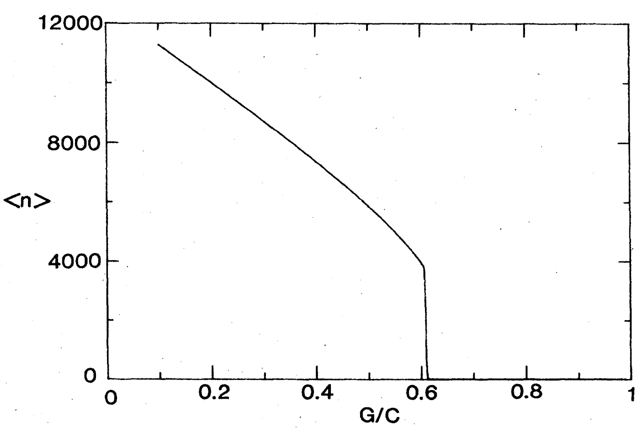

FIG. 2. Variation of the mean photon number $\langle n\rangle$ with G/C for $A/C=1.5$, $s=10$ in the transition region as derived from Eq. (9).

by Salomaa.[4] If $(A/C)[1+(B/A)m]^{-1/2}-1$ and $(G/C)(1+(H/G)m)^{-1/2} \ll 1$ then on exponentiating the distribution in Eq. (9), expanding the logarithmic terms, and replacing the sum by an integral we arrive at

$$p_s(n)=k\exp\{(2A^2/BC)[1+(B/A)n]^{1/2} -(2G^2/HC)[1+(H/G)n]^{1/2}-n\}\ . \qquad (12)$$

In view of the large photon numbers involved, n may be treated like a continuous variable. Then Eq. (12) is the same as that derived by Salomaa.[4]

The form of the probability distribution $p_s(n)$ as derived from Eq. (9) is shown in Fig. 1. It will be seen that $p_s(n)$ may be two peaks separated by a minimum. This means that the system may exhibit bistability for a suitable range of parameters. One of the peaks is always at $\bar{n}=0$. The position of the minimum and the nonzero maximum is determined by the roots of the equation[4]

$$A/[1+(B/A)\bar{n}]^{1/2}=G/[1+(H/G)\bar{n}]^{1/2}+C\ . \qquad (13)$$

Unfortunately the roots of this equation cannot be determined analytically. A graphical method for finding the roots of Eq. (13) has been discussed intensively in Ref. 4. In homogeneously broadened media the positions of the extrema can be determined analytically. We have compared the distribution (11) and the corresponding distribution for homogeneously broadened media.[2,5] We find that for given ratio s the region of bistability for an inhomogeneously broadened laser (in the A/C versus G/C plane) is wider than that for a homogeneously broadened system. As a consequence of this the threshold for bistable operation is lower in the former case than in the latter.

In the region of bistability the mean photon number changes rapidly and this is accompanied by large photon-number fluctuations. The behavior of the mean $\langle n\rangle$ and the normalized variance $\langle(\Delta n)^2\rangle/\langle n\rangle^2$ derived from Eq. (9) is shown in Figs. (2) and (3). These curves are similar to those for a homogeneously broadened medium. Fluctuation properties of the LSA based on distributions (11) and (12) have been discussed in Refs. 3 and 4, respectively. Based on our discussion, therefore, we conclude that the difference in the behavior of an inhomogeneously broadened LSA and a homogeneously broadened LSA is only quantitative.

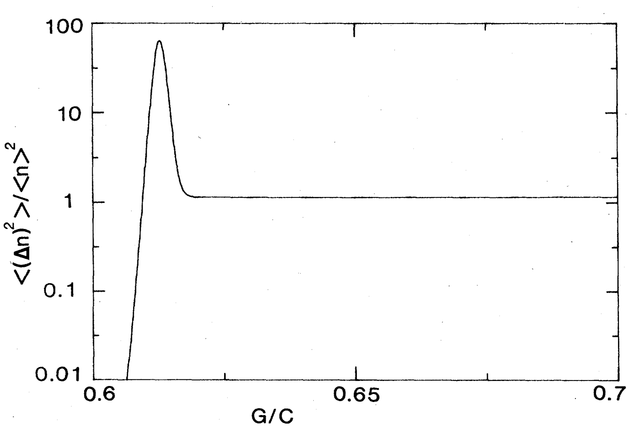

FIG. 3. Variation of the relative mean squared photon-number deviations $\langle(\Delta n)^2\rangle/\langle n\rangle^2$ in the transition region for $A/C=1.5$ and $s=10$ as derived from Eq. (9).

IV. OFF-RESONANCE OPERATION

We now consider the situation when the field frequency is detuned from the center of the Doppler profile of the atoms. Let us denote the detuning with respect to the two types of atoms by $\Delta_i=\omega_{0i}-\Omega$, $i=1,2$. For a running-wave field mode an atom moving with speed v sees the mode frequency to be Doppler shifted by an amount $(\Omega/c)v=kv$. The resulting velocity integrals are of the form (4) and (5). We will be content to evaluate them in the large Doppler-broadening $(ku \gg \gamma_{ab},\gamma_{cd})$ limit although they can be expressed in terms of the plasma-dispersion function. In the limit of large Doppler broadening we have

$$\int_{-\infty}^{\infty} dv\frac{e^{-(v/u_i)^2}}{(\Delta_i \pm kv)^2+b^2} \simeq \frac{\pi}{kb}e^{-\delta_i^2},\quad \delta_i=\frac{\Delta_i}{ku_i}\ . \qquad (14)$$

This means that for a running-wave field mode the effect of detuning is to reduce the pump rates by the factors $e^{-\delta_1^2}$ and $e^{-\delta_2^2}$ for the active and the absorber atoms. The photon-number distribution can be written down from Eq. (9) and (14) to be

$$p_s(n)=k\prod_{m=1}^{n}\left[\frac{A'}{C}\right] \times\frac{\left[1+\dfrac{H'}{G'}m\right]^{1/2}}{\left[1+\dfrac{B'}{A'}m\right]^{1/2}\left[\dfrac{G'}{C}+\left[1+\dfrac{H'}{G'}m\right]^{1/2}\right]}\ , \qquad (15)$$

where $(A',B')=(A,B)e^{-\delta_1^2}$, $(G',H')=(G,H)e^{-\delta_2^2}$. The statistics that follow from Eq. (15) and various special cases may be discussed in a manner similar to the on-resonance case.

The case of a detuned standing-wave laser is more complicated. A standing wave may be considered to be a combination of two oppositely directed traveling waves. The two traveling-wave components of the standing-wave interact with two velocity groups of atoms such that the Doppler shift annuls the detuning, i.e., $kv_i=\pm(\omega_{0i}-\Omega)$. The presence of two velocity groups of atoms complicates the problem and one has to resort to a perturbative approach. In the conventional laser it is sufficient to include terms up to fourth order in the coupling constant, but for the LSA terms up to sixth order must be included in order to account for the correct saturation behavior. These terms can be calculated by using the approach similar to that presented by Stenholm and Lamb[13] and Riska and Stenholm.[11] The calculations are lengthy but straightforward. Here we only present the final results for $R_1(n)$ and $R_2(n)$ which read

$$R_1(n)=A'\left[1-\frac{B'}{4A'}(1+\xi_1)(n+1) +\frac{3}{32}\frac{B'^2}{A'^2}(1+\xi_1+2\xi_1^2)(n+1)^2\right]\ , \qquad (16)$$

$$R_2(n)=G'\left[1-\frac{H'}{4G'}(1+\xi_2)(n+1)\right.$$
$$\left.+\frac{3}{32}\frac{H'}{G'^2}(1+\xi_2+2\xi_2^2)(n+1)^2\right], \quad (17)$$

$$\xi_1=\frac{1}{1+(\Delta_1/\gamma_{ab})^2},\quad \xi_2=\frac{1}{1+(\Delta_2/\gamma_{cd})^2}. \quad (18)$$

In arriving at these expressions, the velocity integrals have been evaluated in the extreme Doppler limit ($ku_i \gg \gamma_{ab}, \gamma_{cd}$) and certain terms of order $(\gamma_{ab}/ku_i)^2$, $(\gamma_{cd}/ku_2)^2$ have been neglected. These ratios are of order 10^{-3}–10^{-4} for gas lasers. The steady-state photon-number distribution for the detuned laser then can be written in the form

$$p_s(n)=k\exp\left\{\left[\frac{A'}{G'+C}-1\right]n+\frac{1}{8}\left[\frac{H'}{G'+C}(1+\xi_2)-\frac{B'}{A'}(1+\xi_1)\right]n^2\right.$$
$$\left.-\frac{1}{96}\left\{\left[\frac{H'}{G'+C}\right]^2\left[\left[3+\frac{3C}{G'}\right](1+\xi_2+2\xi_2^2)-(1+\xi_2)^2\right]-\frac{B'^2}{A'^2}(2+\xi_1+5\xi_1^2)\right\}n^3\right\}. \quad (19)$$

Equations (16)–(18) are consistent with the results of Ref. (3). The distribution (19), however, was not derived there.

This formula can be used to study the frequency dependence of the fluctuation properties. It is easy to check that if we put $\delta_1=0=\delta_2$ we recover the on resonance formula (11). A case of special interest occurs when $\delta_1=\delta=\delta_2$. Even in this case, however, we cannot put $\xi_1=\xi_2$ because γ_{ab} and γ_{cd} will be different. In fact, the condition for bistability to occur requires that $\gamma_{cd}<\gamma_{ab}$. The detuning dependence can cause a large variety of shapes of the $p_s(n)$ since a large number of parameters, A', G', C, γ_{ab}, γ_{cd}, etc., are involved. Detuning affects saturation of the two types of atoms differently and as a result the region of bistability depends on the detuning. It should be mentioned that $R_1(n)$ and $R_2(n)$ can be calculated to any desired degree of accuracy at least in the REA. However, terms up to sixth order in the coupling constant are sufficient to display at least qualitatively all the interesting features of a detuned LSA with standing wave field mode.

V. SUMMARY

We have considered the operation of a laser with a saturable absorber containing inhomogeneously broadened active and absorber atoms. Both on- and off-resonance operations have been considered. For a running-wave field mode exact photon-number distributions are derived. For the standing-wave case exact on-resonance photon-number distribution can be derived in the rate-equation approximation. However, the off-resonance operation of the standing-wave laser must be considered perturbatively. We compare our results with earlier treatments and show how under appropriate operating conditions our results reduce to those derived earlier. A comparison with LSA with homogeneously broadened media is made and differences are found to be only quantitative. Curves are presented to illustrate the behavior.

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