Unifying the $\mathbf{p}\cdot\mathbf{A}$ and $\mathbf{d}\cdot\mathbf{E}$ interactions in photodetector theory

P. D. Drummond
Physics Department, University of Auckland, Private Bag, Auckland, New Zealand
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A new technique is introduced in which a canonical transformation is used to transform the atom-radiation Hamiltonian to one without counter-rotating terms. As well as allowing an exact treatment of a one-mode dipole-coupled photodetection model, the theory predicts the existence of squeezing induced by the detector coupling. A feature of the canonical transformation used here is that the resulting interaction Hamiltonian has a symmetric combination of the $\mathbf{p}\cdot\mathbf{A}$ and $\mathbf{d}\cdot\mathbf{E}$ interactions. This allows the resolution of long-standing questions about the correct interaction Hamiltonian to use in photodetector theory. The new theory is able to handle dipole-coupled radiative transitions over ultrafast time scales of the order of a single oscillation period, where the rotating-wave approximation is invalid. The coupling of the detector to the radiation field is predicted to vary as $f^{-1/2}$ at high frequencies, and as $f^{+1/2}$ at low frequencies.

I. INTRODUCTION

The detection of quanta of the radiation field (or other quantum fields) has an important position in the development of quantum theory. The photoelectric effect is perhaps the earliest example of photodetection, in the sense that this effect was interpreted as resulting from discrete quanta.[1] Of course, at this stage it was not possible to detect individual photons. More recent developments have largely centered around Glauber's treatment of photodetection,[2] with its emphasis on photon statistics. This has led to surprising predictions like that of photon antibunching[3,4] and squeezing,[5,6] which were later observed in resonance fluorescence experiments.

While Glauber's treatment is intuitively clear, it is also perturbative. It is therefore less useful in treating *efficient photodetection, in which essentially all the photons are observed.* An obvious result of efficient photodetection is that two photodetectors placed side by side will have the same count rate in a plane-wave field; while a photodetector placed behind another one will have a near-zero count rate. This effect is not allowed for in Glauber photodetector theory, but is true in practice for laboratory photodetectors. For this reason, the current paper will focus on Heisenberg equation techniques which allow for efficient photodetection, rather than on perturbative scattering theory.

The problem of efficient photodetection or absorption was treated by Mollow[7] in a nonperturbative treatment of a one-mode model, using the rotating-wave approximation. This approximation, which is not necessary in scattering theory, has a strong effect in photodetection. It essentially removes all of the detector-radiation coupling from the ground state of the interacting system, when the usual Hamiltonian is employed. The rotating-wave approximation therefore alters the treatment of time-varying fields, in that counter-rotating terms (which are the terms omitted) can significantly change the predicted excitation rates on short time scales. Short-time-scale effects were analyzed by Louisell and others[8] in theories of the photoexcitation of an isolated atom. The result obtained, using standard calculational techniques, is that counter-rotating terms cause divergences in the photons counted on short time scales. This effect is normally disregarded by calculating the steady-state *count rate* rather than the total excitation probability. The count-rate argument is clearly less helpful when detecting an input of only a few cycles duration, as produced in a femtosecond laser.

Another problem in traditional photodetection theory is the question of frequency dependence with a broadband input, which is the subject of current debate in the literature.[9] What is the frequency response of a photodetector—to take the simplest case, with a one-atom detector? The answer obtained apparently depends on the form chosen for the Hamiltonian, which can be written either using a $\mathbf{p}\cdot\mathbf{A}$ interaction or a $\mathbf{d}\cdot\mathbf{E}$ interaction. In the first case the interaction term diverges at low frequencies, scaling as $(\omega_0/\omega)^{1/2}$, where ω_0 is the resonant frequency. In the second case the interaction diverges at high frequencies, scaling as $(\omega/\omega_0)^{1/2}$. This problem is to a large extent related to the choice of physical input states. In the usual Power-Zienau treatment[10] of the $\mathbf{d}\cdot\mathbf{E}$ interaction, the $\mathbf{E}$ operator is the electric *displacement,* not the transverse electric field. Accordingly, the field states corresponding to the vacuum, one-photon, two-photon, ... excitations obtained with this operator are different from those obtained using the minimal-coupling annihilation and creation operators. Another way of stating this is that the free parts of the Hamiltonians obtained using these two interactions, and hence the states corresponding to the respective noninteracting vacuums, are distinct. This implies that the term photon, as normally used, is not unique when a detector atom is present. Unfortunately, in neither the $\mathbf{p}\cdot\mathbf{A}$ nor the $\mathbf{d}\cdot\mathbf{E}$ Hamiltonian is the free vacuum state identical to the ground state of the *full* Hamiltonian, so neither form of interaction is preferable from this point of view.

In this paper, a technique of unifying the two approaches to photodetection is presented. The technique is extremely simple in principle, although nontrivial in prac-

tice. It is an alternative to the Power-Zienau transform used to obtain the multipolar form of a Hamiltonian. In photodetection theory, the Power-Zienau transform is only moderately useful: The alternative transform given here alters the minimal-coupling Hamiltonian into a form without counter-rotating terms. This allows the use of the methods pioneered by Mollow, without requiring the rotating-wave approximation. Photon-counting divergences are also removed, since the free vacuum and the interacting vacuum are identical in the new form of the Hamiltonian.

Earlier treatments of radiative coupling using canonical transformations include Heitler's technique,[11] and the Davidovich-Nussenzweig theory of the natural line shape.[12] The canonical transformation employed here is related to the Davidovich-Nussenzweig method, but is more general, as it eliminates counter-rotating terms in the self-energy as well as in all of the atom-field dipole-coupled transitions. This is achieved through a combination of a squeezing transformation for renormalization, and a recoupling transformation similar to that of Power and Zienau. The final Hamiltonian is a hybrid form that combines the $\mathbf{p}\cdot\mathbf{A}$ and $\mathbf{d}\cdot\mathbf{E}$ interactions symmetrically.

II. PHOTODETECTION PHYSICS

The usual model of photodetection[2,7] is an ionizable quantum system with a dipole coupling to the radiation field. The ionizing transition can be regarded as involving a set of levels, in a discrete approximation. Instead of measuring the field, the measuring process takes place on the ionizable system. Photodetection is thus reduced to the detection of an electron in the excited states of the detector quantum system, allowing an analysis of the coupling of the detector to the radiation field.

For the present purpose, the detailed structure of the upper levels in the detector system is relatively superfluous. Instead, only one upper level really needs to be taken into account. It is necessary to treat this case in a physical way. For this reason, as true two-level electric-dipole-coupled detectors seem to be nonexistent, a harmonic oscillator model will be used. This does allow the occurrence of multiple photon absorption, but the model can be extended to a large number of weakly coupled harmonic oscillators in order to reduce multiple excitations. The technique is also easily generalized to other multilevel quantum systems with localized dipole interactions.

The photodetection Hamiltonian to be treated has the standard, nonrelativistic, minimal-coupling form[8] in the Coulomb gauge:

$$\hat{H}=\tfrac{1}{2}\int[\epsilon_0\hat{\mathbf{E}}_T(\mathbf{x})^2+\hat{\mathbf{B}}^2(\mathbf{x})/\mu_0]d^3\mathbf{x} + \sum_{n=1}^{N}\frac{1}{2m}[\hat{\mathbf{p}}_n-e\hat{\mathbf{A}}(\hat{\mathbf{q}}_n)]^2+V(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_n)\,. \tag{2.1}$$

This describes N scalar particles of mass m, charge e interacting with the electromagnetic field. Throughout this paper, the dipole-coupling approximation is used, in which $\hat{\mathbf{A}}(\mathbf{x}_0)$ replaces $\hat{\mathbf{A}}(\hat{\mathbf{q}}_n)$, where $\mathbf{x}_0$ corresponds to the center of mass of the (stationary) detector atom.

To a large extent, this formulation of interacting atoms and radiation has been replaced in the recent literature by formulations of quantum electrodynamics in terms of gauge-invariant fields.[10] The most commonly used one is the multipolar Hamiltonian of Power and Zienau, which expresses the Hamiltonian as a series in terms of dipole and higher-order multipole interactions with the radiation field. More general formulations also exist, including arbitrary integration paths and particle spin.[13] The Power-Zienau form of the Hamiltonian can be obtained from Eq. (2.1) by a unitary transformation on the Hamiltonian or on the operators. The simplest way of treating the transformation is by regarding it as a canonical transformation. According to Dirac,[14] this is a unitary transformation on the dynamical variables (operators in the case of a quantum theory). The Hamiltonian is then re-expressed in terms of the new dynamical variables but is itself invariant under the transformation, as are the physical states of the system. The alternative view, in which the Hamiltonian is transformed, leads to the complication that the *same* operators correspond to *different* physical observables, before and after the transformation. In this paper, Dirac's canonical viewpoint will be used throughout.

For later comparison, $\hat{H}$ can be rewritten with the transformed variables $\hat{\pi},\hat{\mathbf{D}}$ using the Power-Zienau transformation. The new operators are still Hermitian (and hence are observables):

$$\begin{aligned}\hat{\pi}_n&=\hat{S}\hat{\mathbf{p}}_n\hat{S}^{-1}\,,\\ \hat{\mathbf{D}}(\mathbf{x})&=\epsilon_0\hat{S}\,\hat{\mathbf{E}}_T(\mathbf{x})\hat{S}^{-1}\,,\end{aligned} \tag{2.2}$$

Here $\hat{S}$ is a unitary transformation defined (for a detector atom with its center of mass at $\mathbf{x}_0=\mathbf{0}$) by

$$\hat{S}=\exp\left[\frac{-i}{\hbar}\int\hat{\mathbf{P}}(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})d^3\mathbf{x}\right], \tag{2.3}$$

where

$$\hat{\mathbf{P}}(\mathbf{x})=e\delta^{(3)}(\mathbf{x})\sum_{n=1}^{N}(\mathbf{q}_n)+\cdots\,.$$

The physical interpretation of the new operators $\hat{\pi}_n,\hat{\mathbf{D}}(\mathbf{x})$ is that $\hat{\pi}_n$ is the mechanical momentum of the nth particle [i.e., $m\hat{\mathbf{v}}$], while $\hat{\mathbf{D}}(\mathbf{x})$ is the electric displacement operator [i.e., $\epsilon_0\hat{\mathbf{E}}_T(\mathbf{x})+\hat{\mathbf{P}}_T(\mathbf{x})$]. By comparison, in the minimal-coupling case, the canonical momentum $\hat{\mathbf{p}}_n$ includes an admixture of field properties; just as the transformed operator $\hat{\mathbf{D}}(\mathbf{x})$ now includes some of the particle position properties. The end result of the transformation is

$$\hat{H}=\frac{1}{2\epsilon_0}\int[\hat{\mathbf{D}}(\mathbf{x})-\hat{\mathbf{P}}_T(\mathbf{x})]^2d^3\mathbf{x}+\frac{1}{2\mu_0}\int\hat{\mathbf{B}}^2(\mathbf{x})d^3\mathbf{x} + \frac{1}{2m}\sum_{n=1}^{N}\hat{\pi}_n^2+V(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_n)+\cdots\,. \tag{2.4}$$

In Eq. (2.1), interactions occur via the $\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}$ coupling. In Eq. (2.4), interactions occur via the coupling of the polarization to the electric displacement: Since the polarization is proportional to the electric dipole moment in the dipole

approximation, this is termed a $\hat{\mathbf{d}}\cdot\hat{\mathbf{E}}$ interaction. The complete forms of Eq. (2.1) and Eq. (2.4) are equivalent. However, the "free" and "interacting" parts of the Hamiltonian are *not* equivalent under this transformation. Thus, when using approximations, it is possible to obtain results that differ between the Hamiltonians of Eq. (2.1) and Eq. (2.4), even though these describe identical physical systems. The reason for this difference is that the coupling terms in the interaction part of the Hamiltonian vary as $(1/\omega)^{1/2}$ in the $\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}$ Hamiltonian and as $(\omega)^{1/2}$ in the $\hat{\mathbf{d}}\cdot\hat{\mathbf{E}}$ Hamiltonian, where ω is the mode frequency. It is obviously desirable to write down the Hamiltonian in a form that requires the least approximation.

III. MEASURING $\hat{\mathbf{D}}$ AND $\hat{\mathbf{A}}$

Having written the Hamiltonian in two possible ways, with different field operators, the question arises as to which of these alternatives is relevant to photodetection. Does a photodetector measure $\hat{\mathbf{D}}, \hat{\mathbf{A}}$, or some combination of the two? In the case of resonant interactions on long time scales, either formulation is equivalent, to lowest order in perturbation theory. However, it is necessary to include nonresonant interactions when detecting photons on fast time scales, in which case the frequency dependence of the coupling becomes significant. In addition, the usual approximation that $\hat{\mathbf{D}}=\epsilon_0\hat{\mathbf{E}}_T$ is only correct to lowest order in the coupling, and cannot be used in calculations of a nonperturbative type. In fact, $\hat{\mathbf{D}}$ and $\epsilon_0\hat{\mathbf{E}}_T$ differ by a term equal to $\hat{\mathbf{P}}_T(\mathbf{x})$, which is localized at the detector in the dipole approximation. This is obviously a problem: after all, the detector is intended to measure the field at its own location, which is exactly the place where $\epsilon_0\hat{\mathbf{E}}_T$ and $\hat{\mathbf{D}}$ are (nearly) infinitely different from each other.

In fact calculations exist in the literature which indicate just how troublesome these questions really are. Using standard techniques, it is straightforward to calculate completely different photodetection probabilities using the two forms of the Hamiltonian. Not surprisingly, the differences—and the photodetection probabilities—diverge as the ultraviolet cutoff of the theory is taken to infinity.

In order to illustrate this, a harmonic oscillator example will be used. The minimal-coupling dipole approximation Hamiltonian is written after subtracting the free Hamiltonian ground-state energies, as

$$\hat{H}=\tfrac{1}{2}\int :[\epsilon_0\hat{\mathbf{E}}_T^2(\mathbf{x})+\hat{\mathbf{B}}^2(\mathbf{x})/\mu_0]:d^3\mathbf{x} + \tfrac{1}{2}:[\hat{\mathbf{p}}^2/m+m\omega_0^2\hat{\mathbf{q}}^2]:-\frac{e}{m}\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}(0)+\frac{e^2}{2m}\hat{\mathbf{A}}^2(0)\ . \tag{3.1}$$

Here colons indicate normal ordering. To simplify the problem of a spherically symmetric three-dimensional oscillator as in Eq. (3.1), this can be rewritten, for oscillators in the direction of unit vectors $\mathbf{u}_j$ and for mode polarizations $\mathbf{e}_k$ as

$$\hat{H}/\hbar=\sum_k\omega_k\hat{a}_k^\dagger\hat{a}_k+\sum_j\omega_0\hat{b}_j^\dagger\hat{b}_j + \sum_k\sum_j\mathbf{g}_k\cdot\mathbf{u}_j(\hat{b}_j^\dagger+\hat{b}_j)(\hat{a}_k+\hat{a}_k^\dagger) + \left[\sum_k\mathbf{g}_k(a_k+a_k^\dagger)\right]^2\Big/\omega_0\ , \tag{3.2}$$

where

$$\mathbf{g}_k=-e\left[\frac{\omega_0}{4\epsilon_0 V m\omega_k}\right]^{1/2}\mathbf{e}_k\ ,$$

$$\hat{\mathbf{p}}=\sum_j(\hat{b}_j+\hat{b}_j^\dagger)\left[\frac{\hbar\omega_0 m}{2}\right]^{1/2}\mathbf{u}_j\ ,$$

$$\hat{\mathbf{A}}=\sum_k\left[\frac{\hbar}{2\epsilon_0 V\omega_k}\right]^{1/2}(\mathbf{e}_k\hat{a}_k+\mathbf{e}_k^*\hat{a}_k^\dagger)\ ,$$

$$\hat{\mathbf{q}}=i\sum_j(\hat{b}_j-\hat{b}_j^\dagger)\left[\frac{\hbar}{2\omega_0 m}\right]^{1/2}\mathbf{u}_j\ .$$

The general situation is more complex, as there are three independent harmonic oscillator operators, with characteristic frequencies that could be different. The Hamiltonian of Eq. (3.2) does take into account the different mode polarization directions through $\mathbf{g}_k\cdot\mathbf{u}_j$, which for simplicity is taken to be real. It is obviously no worse than two-level atom theory, and will be utilized here to give an example of a photodetection divergence problem.

The normal procedure in published accounts of photodetector theory is to start with the detector in its free ground state, which is then allowed to evolve in time. The probability of photodetection is the excited-state population after a defined interaction time T. The term corresponding to $\hat{\mathbf{A}}^2$ is dropped here, in order to compare directly with calculations in the literature.[8] Using the Heisenberg picture, the time-evolution equations are

$$\frac{\partial}{\partial t}\hat{a}_k=-i\omega_k\hat{a}_k-i\sum_j g_{kj}(\hat{b}_j+\hat{b}_j^\dagger)\ ,$$
$$\frac{\partial}{\partial t}\hat{b}_j=-i\omega_0\hat{b}_j-i\sum_k g_{kj}(\hat{a}_k+\hat{a}_k^\dagger)\ , \tag{3.3}$$

where $g_{kj}=\mathbf{g}_k\cdot\mathbf{u}_j$.

At this stage it is clear that the counter-rotating terms severely complicate the solution of these equations. These are the terms in the Hamiltonian of form $\hat{a}\,\hat{b}$ and $\hat{a}^\dagger\hat{b}^\dagger$, which cause the mixing of annihilation and creation operator terms on the right-hand side of Eq. (3.3). A common procedure is to approximate the time evolution of $\hat{a}_k$ by its known free-field behavior. This gives the result

$$\hat{b}_j(t)=\hat{b}_j(0)e^{-i\omega_0 t} - \sum_k\int_0^t ig_{kj}[\hat{a}_k^0 e^{-i\omega_k\tau}+(\hat{a}_k^0)^\dagger e^{i\omega_k\tau}]e^{-i\omega_0(t-\tau)}d\tau\ , \tag{3.4}$$

where $\hat{a}_k(t)\approx\hat{a}_k^0\exp(-i\omega_k t)$.

It is now possible to obtain the apparent photodetection probability, $P_{\mathbf{p}\cdot\mathbf{A}}(T)$, in a time T. This is defined as the

expectation value of the Heisenberg operator $\mathbf{b}^\dagger(T)\cdot\mathbf{b}(T)$. As a first step, suppose the field was initially in its free vacuum state $|0\rangle_a$; here one might reasonably expect to detect no photons. In fact, the probability of a photon being detected in time T is

$$P_{\mathrm{p\cdot A}}(T)=\sum_j \langle \hat{b}_j^\dagger(T)\hat{b}_j(T)\rangle = \sum_k \frac{4|\mathbf{g}_k|^2}{(\omega_0+\omega_k)^2}\sin^2\left[\frac{(\omega_0+\omega_k)T}{2}\right] . \tag{3.5}$$

Replacing the sum over discrete k's by an integral, which gives the large mode volume limit of $\sum \to V\int \omega^2 d\omega/(\pi^2 c^3)$, gives

$$P_{\mathrm{p\cdot A}}(T)\propto \int_0^{\omega_M}\frac{\omega}{(\omega_0+\omega)^2}\sin^2\left[\frac{(\omega_0+\omega)T}{2}\right]d\omega . \tag{3.6}$$

Here the (trivial) angular integral is omitted, and use is made of the fact that $|\mathbf{g}_k|^2\propto 1/\omega$. Despite this diminished coupling at large ω, the overall photodetection probability is logarithmically divergent as $\omega_M\to\infty$. In fact, ω_M has a natural limit: An ultraviolet cutoff at $\omega_M\approx c/r$ is necessary owing to the use of the dipole approximation, where r is an atomic radius. In summary, not only is there an apparently nonzero photodetection probability, but the calculation is divergent in the large cutoff limit.

It is possible to repeat this calculation using the alternate Hamiltonian, Eq. (2.4). In this case the transformed Hamiltonian is

$$\hat{H}/\hbar=\sum_k \omega_k\hat{\alpha}_k^\dagger\hat{\alpha}_k+\sum_j\omega_0\hat{\beta}_j^\dagger\hat{\beta}_j - \sum_k\sum_j\gamma_{kj}(\hat{\beta}_j^\dagger-\hat{\beta}_j)(\hat{\alpha}_k^\dagger-\hat{\alpha}_k) - \sum_k\left[\sum_j\gamma_{kj}(\hat{\beta}_j^\dagger-\hat{\beta}_j)\right]^2\Big/\omega_k , \tag{3.7}$$

where

$$\gamma_{kj}=\left[\frac{\omega_k}{\omega_0}\right]g_{kj} ,$$

$$\hat{\alpha}_k=\hat{S}\,\hat{a}_k\,\hat{S}^{-1}=\hat{a}_k+\sum_j g_{kj}(\hat{b}_j^\dagger-\hat{b}_j)/\omega_0 ,$$

$$\hat{\beta}_j=\hat{S}\,\hat{b}_j\hat{S}^{-1}=\hat{b}_j+\sum_k g_{kj}(\hat{a}_k+\hat{a}_k^\dagger)/\omega_0 .$$

For comparison with Eq. (3.4), the time evolution of $\hat{\beta}(t)$ can be computed, without the self-energy $(\hat{\beta}^\dagger-\hat{\beta})^2$ terms. In this truncated form, the Hamiltonian is formally similar to Eq. (3.2), except for the coupling term signs. Again, suppose the field has an initial vacuum state. This time the photodetection probability is $P_{\mathrm{d\cdot E}}(T)$, which is defined as

$$P_{\mathrm{d\cdot E}}(T)=\sum_j\langle\hat{\beta}_j^\dagger(t)\hat{\beta}_j(t)\rangle . \tag{3.8}$$

An identical approximation to the earlier case is used. The result is now, after replacing the sum over discrete k's by an integral:

$$P_{\mathrm{d\cdot E}}(T)\propto\int_0^{\omega_M}\frac{\omega^3}{\omega_0^2(\omega_0+\omega)^2}\sin^2\left[\frac{(\omega_0+\omega)T}{2}\right]d\omega . \tag{3.9}$$

This expression is quadratically divergent, scaling as ω_M^2. The presence of the $\sin^2$ term does not prevent this divergence, as its average value is nonzero. The divergence arises even for an initial free vacuum in the calculation; of course other differences occur in fields of finite photon number due to the different dependence of g_k and γ_k on ω_k.

Thus while the two approaches give different results, each causes divergences to appear in this type of calculation. The reason that this occurs is partly that the initial free vacuum states are inequivalent in the two calculations: The "free Hamiltonians" in the two calculations are physically distinct, and therefore have different ground states. It is, of course, possible to force the two approaches to give identical results by choosing the initial and final states so that they are in fact physically identical in the two calculations.[15] These would no longer correspond to free vacuum states in one of the formulations, leaving the problem of which initial and final states to choose still unresolved.

In addition, part of the problem outlined in this section is due to the approximation of removing the self-energy terms from the calculation. This was done to show the difficulties inherent in the standard calculation of photodetection probabilies. Another approach is to attempt to solve the Heisenberg equations of motion rigorously, including the self-energy terms. This has been carried out by Rzazewski,[15] who finds some cancellation of the worst part of the divergences when the self-energy term is included in Eq. (3.7). The solutions obtained in this way have the structure of inverse Laplace transform expressions for the Heisenberg operators, and so do not have a direct interpretation in terms of observables. Nevertheless, the problem of different definitions of initial states remains. Both calculations still predict divergences in the Heisenberg operator time developments. The actual photodetection probability would then depend on the choice of initial state for the detector atom. This choice is not made by Rzazewski, leaving open the question of suitable initial states and detector output states.

IV. SINGLE-MODE CANONICAL TRANSFORMATIONS

The solution to the paradox outlined in Sec. III is simple. It is to use an initial state which does not depend on an artificial splitting of the Hamiltonian into "free" and "interacting" parts. An example is just the ground state of the total Hamiltonian. Since this ground state is an eigenvector of the complete Hamiltonian, it is time invariant, and does not lead to any time-dependent photodetection probabilities in the case of a vacuum. States of finite photon number should then be defined relative to this true ground state.

In order to implement this, it would be advantageous to have a Hamiltonian without the counter-rotating terms that cause divergences. One approach, called the rotating-wave approximation,[7] is to omit these terms com-

pletely. Since this leads to an *infinite* (in the limit $\omega_M \to \infty$) change in the results predicted, it scarcely seems to be a good approximation to use in general. Accordingly, the normal rotating-wave approximation (RWA) will not be used here. An alternative approach is to canonically transform the original operators to obtain a Hamiltonian without counter rotating terms. In some cases, this provides a simple way of computing the true (interacting) ground state, as well as handling the time development.

Suppose that a harmonic oscillator detector is to be treated, as before. A unitary transformation $\hat{U}$ is applied to the operators:

$$\begin{aligned} \bar{a}_k &= \hat{U}\,\hat{a}_k\,\hat{U}^{-1}\,, \\ \bar{b}_j &= \hat{U}\,\hat{b}_j\,\hat{U}^{-1}\,, \end{aligned} \tag{4.1}$$

where

$$\hat{U}^{-1} = \hat{U}^{\dagger}\,.$$

This transformation leaves the commutation relations invariant, i.e.,

$$\begin{aligned} [\bar{b}_k, \bar{b}_j^{\dagger}] &= [\bar{a}_k, \bar{a}_j^{\dagger}] = \delta_{kj}\,, \\ [\bar{a}_k, \bar{b}_j] &= [\bar{a}_k, \bar{b}_j^{\dagger}] = 0\,. \end{aligned} \tag{4.2}$$

Thus the transformed operators can still be regarded as creating and annihilating elementary excitations in the coupled quantum system of interest. In order that the new operators are still identifiable as detector and radiation operators, the transformation should be minimal, in the sense that the transformed operators should differ from the original ones by terms of $O(g/\omega_0)$. This rules out the use of a diagonalization transformation, which changes the operators by terms of $O(1)$ on resonance. A suitable transformation can now be evaluated for the case of one mode coupled to a detector atom, with the requirement of no counter-rotating terms. The initial form of the Hamiltonian is (with just one oscillator direction included):

$$\begin{aligned} \hat{H}/\hbar = {} & \omega \hat{a}^{\dagger}\hat{a} + \omega_0 \hat{b}^{\dagger}\hat{b} + g(\hat{a}+\hat{a}^{\dagger})(\hat{b}+\hat{b}^{\dagger}) \\ & + g^2(\hat{a}+\hat{a}^{\dagger})^2/\omega_0\,, \end{aligned} \tag{4.3}$$

and the required form, in terms of $\bar{a}$ and $\bar{b}$, is

$$\hat{H}/\hbar = \bar{\omega}\,\bar{a}^{\dagger}\bar{a} + \bar{\omega}_0 \bar{b}^{\dagger}\bar{b} + \bar{g}\,\bar{a}\,\bar{b}^{\dagger} + \bar{g}\,\bar{a}^{\dagger}\bar{b} + \Delta\omega\,. \tag{4.4}$$

In this form it is clear that $\overline{H} = \hat{H} - \hbar\Delta\omega$ is positive semidefinite if $\bar{\omega}, \bar{\omega}_0 > \bar{g} > 0$. This follows since each of the following terms is non-negative:

$$\begin{aligned} \langle \Psi \mid \overline{H} \mid \Psi \rangle = \hbar[& (\bar{\omega}-\bar{g})\langle \Psi \mid \bar{a}^{\dagger}\bar{a} \mid \Psi \rangle \\ & + (\bar{\omega}_0 - \bar{g})\langle \Psi \mid \bar{b}^{\dagger}\bar{b} \mid \Psi \rangle \\ & + \bar{g}\langle \Psi \mid (\bar{a}+\bar{b})^{\dagger}(\bar{a}+\bar{b}) \mid \Psi \rangle]\,. \end{aligned} \tag{4.5}$$

The state $|\overline{\Psi}_0\rangle = |0\rangle_{\bar{a}}\,|0\rangle_{\bar{b}}$ has a zero energy eigenvalue relative to $\overline{H}$. It is therefore the true ground state.

It is now necessary to evaluate a suitable transformation that leaves $\hat{H}$ in the form of Eq. (4.4). This is a question of *rewriting* $\hat{H}$ using new operators $\bar{a}, \bar{b}$ instead of the old set $\hat{a}, \hat{b}$. In order to achieve this, it is instructive to examine how the usual Power-Zienau transformation [see Eq. (2.2) and (2.3)] alters the functional form of $\hat{H}$ in Eq. (4.3). In this case, the Power-Zienau transformation has the form

$$\hat{S} = \exp[g(\hat{a}+\hat{a}^{\dagger})(\hat{b}-\hat{b}^{\dagger})/\omega_0]\,.$$

Hence,

$$\begin{aligned} \hat{\alpha} &= \hat{a} + g(\hat{b}^{\dagger}-\hat{b})/\omega_0\,, \\ \hat{\beta} &= \hat{b} + g(\hat{a}+\hat{a}^{\dagger})/\omega_0\,. \end{aligned} \tag{4.6}$$

The Hamiltonian $\hat{H}$ from Eq. (4.3), written in terms of operators $\hat{\alpha}, \hat{\beta}$ becomes

$$\begin{aligned} \hat{H}/\hbar = {} & \omega\hat{\alpha}^{\dagger}\hat{\alpha} + \omega_0\hat{\beta}^{\dagger}\hat{\beta} - \gamma(\hat{\beta}-\hat{\beta}^{\dagger})(\hat{\alpha}-\hat{\alpha}^{\dagger}) \\ & - \frac{\gamma^2}{\omega}(\hat{\beta}^{\dagger}-\hat{\beta})^2\,, \end{aligned} \tag{4.7}$$

where

$$\gamma = g(\omega/\omega_0)\,.$$

This transformation has the effect of producing a Hamiltonian with symmetrically interchanged terms relative to Eq. (4.3). The self-energy now appears in the $(\hat{\beta}^{\dagger}-\hat{\beta})^2$ term. While the counter-rotating terms are still present in the interaction, they have the opposite sign to those previously obtained. A hybrid transformation, midway between the minimal coupling and the Power-Zienau forms, could be expected to eliminate these terms completely. A transformation of a suitable type is

$$\hat{U}(\eta,\hat{a},\hat{b}) = \exp[\eta(\hat{a}+\hat{a}^{\dagger})(\hat{b}-\hat{b}^{\dagger})]\,,$$

where (4.8)

$$\hat{\alpha}_{\eta} = \hat{U}\,\hat{a}\,\hat{U}^{-1}, \quad \hat{\beta}_{\eta} = \hat{U}\,\hat{b}\,\hat{U}^{-1}\,.$$

Here η defines the nature of the resulting operators $\hat{\alpha}_{\eta}, \hat{\beta}_{\eta}$. With $\eta = 0$ these operators have a $\mathbf{p}\cdot\mathbf{A}$ interaction; with $\eta = g/\omega_0$ they have a $\mathbf{d}\cdot\mathbf{E}$ interaction in the final Hamiltonian, and $\hat{U}$ has the Power-Zienau form. The Hamiltonian obtained by choosing $\eta = g/(\omega_0+\omega)$ has the following symmetric form:

$$\begin{aligned} \hat{H}/\hbar = {} & \omega\hat{\alpha}_{\eta}^{\dagger}\hat{\alpha}_{\eta} + \omega_0\hat{\beta}_{\eta}^{\dagger}\hat{\beta}_{\eta} \\ & + \frac{\overline{\gamma}}{2}(\hat{\alpha}_{\eta}-\hat{\alpha}_{\eta}^{\dagger})(\hat{\beta}_{\eta}^{\dagger}-\hat{\beta}_{\eta}) + \frac{\overline{\gamma}}{2}(\hat{\alpha}_{\eta}+\hat{\alpha}_{\eta}^{\dagger})(\hat{\beta}_{\eta}^{\dagger}+\hat{\beta}_{\eta}) \\ & + \frac{\overline{\gamma}^2}{4\omega_0}(\hat{\alpha}_{\eta}+\hat{\alpha}_{\eta}^{\dagger})^2 - \frac{\overline{\gamma}^2}{4\omega}(\hat{\beta}_{\eta}-\hat{\beta}_{\eta}^{\dagger})^2\,, \end{aligned} \tag{4.9}$$

where

$$\overline{\gamma} = 2g\omega/(\omega_0+\omega)\,.$$

The obvious property of this Hamiltonian is the combination of $\mathbf{p}\cdot\mathbf{A}$ and $\mathbf{d}\cdot\mathbf{E}$ interactions, together with self-energy terms in $(\hat{\alpha}_{\eta}+\hat{\alpha}_{\eta}^{\dagger})^2$ and $(\hat{\beta}_{\eta}-\hat{\beta}_{\eta}^{\dagger})^2$. The $\mathbf{p}\cdot\mathbf{A}$ interaction [i.e., $(\hat{\alpha}_{\eta}+\hat{\alpha}_{\eta}^{\dagger})(\hat{\beta}_{\eta}^{\dagger}+\hat{\beta}_{\eta})$] has an exactly equal weighting to the $\mathbf{d}\cdot\mathbf{E}$ interaction [i.e., $(\hat{\alpha}_{\eta}-\hat{\alpha}_{\eta}^{\dagger}) \times (\hat{\beta}_{\eta}^{\dagger}-\hat{\beta}_{\eta})$]. This means that, with this choice of η, the counter-rotating terms of order g—which have an opposite sign in the two different types of interaction

Hamiltonian—now cancel out of the Hamiltonian completely.

It is possible to further improve on the Hamiltonian of Eq. (4.9) by noting that one can also remove the self-energy corrections, thus removing all of the counter-rotating terms. This is a renormalization problem, which could be treated by adding suitable counter terms to the Hamiltonian in Eq. (4.3). Rather than using this procedure, it is more straightforward to keep the original Hamiltonian and transform the operators. The frequency ω_0 is therefore the unrenormalized frequency, which is later used to calculate an interacting frequency $\bar{\omega}_0$. The interacting frequency includes the complete nonrelativistic renormalization or Lamb shift of the ground-state energy, provided m is regarded as the bare (unrenormalized) mass.[8]

The initial transformation required is a squeezing transformation[5,6] on $\hat{\alpha},\hat{\beta}$ which produces additional self-energy terms that are equal and opposite in sign to those of Eq. (4.9). The recoupling transformation that alters the interaction to the symmetric form then cancels these self-energy terms. This leaves the final Hamiltonian without counter-rotating or self-energy terms, as desired. In order to achieve this result, define new operators $\bar{a},\bar{b}$:

$$\begin{aligned}\bar{a}&=\hat{U}\hat{R}\hat{a}\hat{R}^{-1}\hat{U}^{-1}\,,\\ \bar{b}&=\hat{U}\hat{R}\hat{b}\hat{R}^{-1}\hat{U}^{-1}\,,\end{aligned}\tag{4.10}$$

where

$$\hat{R}\equiv\exp\{\Theta[\hat{a}^2-(\hat{a}^\dagger)^2-\hat{b}^2+(\hat{b}^\dagger)^2]/2\}\,,$$

$$\hat{U}\equiv\exp[\eta(\bar{a}+\bar{a}^\dagger)(\bar{b}-\bar{b}^\dagger)]\,,$$

hence,

$$\hat{a}=\bar{a}\cosh(\Theta)-\bar{a}^\dagger\sinh(\Theta)+\eta\exp(\Theta)(\bar{b}-\bar{b}^\dagger)\,,$$

$$\hat{b}=\bar{b}\cosh(\Theta)+\bar{b}^\dagger\sinh(\Theta)-\eta\exp(\Theta)(\bar{a}+\bar{a}^\dagger)\,.$$

The ground-state energy $\hbar\Delta\omega$ is

$$\hbar\Delta\omega=\hbar(\omega+\omega_0)\sinh(\Theta)\exp(\Theta)\tag{4.11}$$

where

$$\Theta=\tfrac{1}{4}\ln\left[1+\frac{4\omega g^2}{\omega_0(\omega_0+\omega)^2}\right]\,,$$

$$\bar{\omega}/\omega=\bar{\omega}_0/\omega_0=\exp(2\Theta)\,,$$

$$\eta=\frac{g}{\bar{\omega}_0+\bar{\omega}}\,,$$

$$\bar{g}=\frac{2g\omega}{\omega_0+\omega}\,.$$

The result of rewriting the Hamiltonian of Eq. (4.3) using this transformation gives the required form specified in Eq. (4.4), without counter-rotating terms. This is an exact transformation. Apart from the frequency renormalization of the bare frequencies ω,ω_0 to the interacting frequencies $\bar{\omega},\bar{\omega}_0$, the Hamiltonian has a similar form to Eq. (4.9). The counter-rotating terms in the detector coupling as well as the self-energy terms are all canceled by the operator transformation. This exact canonical transformation extends an approximate calculation published previously.[16]

From Eq. (4.11), the true ground-state energy (in $|0\rangle_{\bar{a}}|0\rangle_{\bar{b}}$) is $\hbar\omega g^2/[\omega_0(\omega+\omega_0)]$ for weak coupling. It is instructive to compare this with the energy in either of the two bare vacua. In the case of the $\mathbf{p}\cdot\mathbf{A}$ coupling, this energy is $\hbar(g^2/\omega_0)$, which varies as $1/\omega$. Thus the $\mathbf{p}\cdot\mathbf{A}$ bare vacuum state $|0\rangle_a|0\rangle_b$ approaches the true ground-state energy at high mode frequencies. With the $\mathbf{d}\cdot\mathbf{E}$ coupling, the energy of the bare vacuum state $|0\rangle_\alpha|0\rangle_\beta$ is $\hbar(\omega g^2/\omega_0^2)$, which varies as $1/\omega_0$. This gives a better approximation to the true ground-state energy at low mode frequencies.

The above results demonstrate that it is possible to remove all the counter-rotating terms from the Hamiltonian of a harmonic oscillator interacting with a single field mode, with a nonresonant transformation having minimal mixing between the "detector" and "radiation" operators. It is still possible to distinguish atom and field observables, in the same way that the minimal coupling momentum $\hat{\mathbf{p}}$ is regarded as a particle momentum; even though it includes some field properties.[10] Here the transformed momentum $\bar{\mathbf{p}}$ is an operator with hybrid characteristics, having a combination of minimal coupling ($\hat{\mathbf{p}}$) and mechanical ($\hat{\boldsymbol{\pi}}$) properties.

The new Hamiltonian has an extremely simple time development for its Heisenberg operators. The exact solution for an initial state $|\Psi\rangle_{\bar{a}}|0\rangle_{\bar{b}}$ gives the photon detection probability:

$$P(t)=\langle\bar{b}^\dagger(t)\bar{b}(t)\rangle=[\bar{g}\sin(\Omega t)/\Omega]^2\langle\bar{a}^\dagger(0)\bar{a}(0)\rangle\,,\tag{4.12}$$

where

$$\Omega^2=\bar{g}^2+\Delta^2\,,$$

$$\Delta=\tfrac{1}{2}|\bar{\omega}-\bar{\omega}_0|\,.$$

Thus the detector excitation corresponds in a straightforward way to the input excitation. Completely efficient photodetection is possible: This occurs only for exact resonance of the renormalized frequencies. In this case it is necessary to observe for a known time interval, $T=\pi/(2\bar{g})$, in order to complete the transfer of energy from the mode to the detector. At the end of this time, the mode is left in its ground state, with the excitation in the detector system. This allows all the information present originally in the state of the field to be transferred to the detector state: thus higher-order correlations could be inferred also. While this procedure is similar to the pointer basis technique[17] developed recently in the analysis of quantum measurements, it leaves the original, measured field in a completely different state (i.e., its ground state) from its initial one.

In this measurement, there is no reduction of the wave packet in the usual sense. Nevertheless, an efficient measurement of this type is different from the case of an inefficient photodetection measurement. In inefficient photodetection the final state is never an eigenstate of the photon number operator. In the efficient case, the new state of the field is a well-defined eigenstate of the number operator, and can be used as the starting point for subse-

quent measurements. Thus an efficient photodetection measurement is a type of quantum nondemolition measurement.[17,18]

This procedure justifies Mollow's[7] treatment of single-mode photodetection. The actual operators and fields do not correspond to those used previously, however, except in an approximation valid near the resonance frequency ω_0. The measured creation and annihilation operators are the transformed operators $\bar{a}^\dagger,\bar{a}$. The canonically transformed field operators, denoted $\bar{\mathbf{A}},\bar{\mathbf{E}}$ for simplicity, couple to the detector symmetrically with frequency weighted $\bar{\mathbf{p}}\cdot\bar{\mathbf{A}}$ and $\bar{\mathbf{d}}\cdot\bar{\mathbf{E}}$ interactions, respectively. These fields can be regarded as interpolating between the minimal-coupling and the multipolar forms of the operators.

V. SQUEEZING

The Hamiltonian generated by the canonical transformation is of the structure of Eq. (4.4). Since the $\bar{a}^\dagger$ operator is only a slight modification [i.e., of order (g/ω_0)] of the original $\hat{a}^\dagger$ operator, it is reasonable to refer to it as creating photons. This terminology is just as logical as the accepted practice of referring to the field operator $\hat{\mathbf{D}}(\mathbf{x})/\epsilon_0$ in the multipolar Hamiltonian as being an electric field operator. It too differs from the minimal coupling operator $\hat{\mathbf{E}}_T(\mathbf{x})$ by terms of $O(g/\omega_0)$. In fact, one usually regards the lowest energy state of the Hamiltonian as being the state of zero photon number. This is clearly true of $\bar{a}^\dagger,\bar{a}$, although not of $\hat{a}^\dagger\hat{a}$, since the *bare* minimal-coupling vacuum state is not an eigenstate of the full Hamiltonian.

While it is simple enough to regard the operators $\bar{a},\bar{a}^\dagger$ as creating dressed photons, it is necessary to also recall that the definition of $\bar{a},\bar{a}^\dagger$ is not unique in terms of giving a Hamiltonian of the required structure. The idea of a dressed photon does, however, provide a useful insight into the effect that a detector has on the physical system it is measuring. One *unique* property of *any* $\bar{a}$ operator as defined according to Eq. (4.4) is its vacuum state: which is the (unique) ground state of the full Hamiltonian. It is useful to examine the difference between this and the ground state of the free minimal-coupling Hamiltonian.

In order to do this, quadrature operators for the minimal-coupling electromagnetic field will be defined as

$$\begin{aligned}\hat{X}_1&=(\hat{a}+\hat{a}^\dagger)\ ,\\ \hat{X}_2&=i(\hat{a}-\hat{a}^\dagger)\ .\end{aligned}\tag{5.1}$$

In terms of the bare vacuum ($|\Psi_0\rangle=|0\rangle_a|0\rangle_b$) one has

$$\langle\Psi_0|\hat{X}_1^2|\Psi_0\rangle=\langle\Psi_0|\hat{X}_2^2|\Psi_0\rangle=1\ .\tag{5.2}$$

In the interacting vacuum ($|\bar{\Psi}_0\rangle=|0\rangle_{\bar{a}}|0\rangle_{\bar{b}}$), the quadrature fluctuations can be calculated simply on re-expressing $\hat{X}_1,\hat{X}_2$ in terms of the new operators $\bar{a},\bar{b}$, using the results of Eq. (4.11):

$$\begin{aligned}\langle\bar{\Psi}_0|\hat{X}_1^2|\bar{\Psi}_0\rangle&=\exp(-2\Theta)\ ,\\ \langle\bar{\Psi}_0|\hat{X}_2^2|\bar{\Psi}_0\rangle&=(1+4\eta^2)\exp(2\Theta)\ .\end{aligned}\tag{5.3}$$

In the limit of weak coupling ($g\ll\omega,\omega_0$), this expression can be expanded to order g^2, giving a result that agrees with an earlier approximate expression.[16] This allows the frequency dependence of the squeezing[5,6] to be directly calculated:

$$\begin{aligned}\langle\bar{\Psi}_0|\hat{X}_1^2|\bar{\Psi}_0\rangle&=\left[1-\frac{2g^2\omega}{(\omega_0+\omega)^2\omega_0}\right]+O(g^4)\ ,\\ \langle\bar{\Psi}_0|\hat{X}_2^2|\bar{\Psi}_0\rangle&=\left[1+\frac{2g^2(\omega+2\omega_0)}{(\omega_0+\omega)^2\omega_0}\right]+O(g^4)\ .\end{aligned}\tag{5.4}$$

It is notable that the dressed state $|\bar{\Psi}_0\rangle$ is no longer a minimum uncertainty state relative to the $\hat{X}_1,\hat{X}_2$ quadratures. It is, of course, a minimum uncertainty state with equal fluctuations relative to the $\bar{X}_1,\bar{X}_2$ quadratures. With respect to the original quadrature operators, there is a squeezing (uncertainty reduction) in the $\hat{X}_1$ quadrature. This was expected in view of the squeezing transformation used to obtain Eq. (4.10). The nature of the squeezing is different from that occurring in recent calculations of reduced quadrature fluctuations in nonlinear optics, which usually occur in rotating frame quadrature operators.[6] Here the squeezing is in a laboratory-frame quadrature. By comparison, the squeezing calculated here would oscillate at optical frequencies if rotating frame quadrature operators were used. In fact, the quadrature with increased fluctuations is a type of electric field operator, the one with reduced fluctuations a type of magnetic field operator. Thus a squeezed state of the electromagnetic field in the laboratory frame is induced by the presence of a detector atom. Since quadrature fluctuations change the zero-point energy in the vacuum field, these results imply a change in the energy of the interacting field due to virtual transitions. Similar results to the above have been recently obtained in calculations of the energy density of the electromagnetic field near an isolated atom,[19] using low-order perturbation theory.

VI. MULTIMODE THEORY

The single-mode, single-detector theory given above can be readily extended to cover multimode cases as well as more realistic detector energy levels. In these more general cases, the transformations required for renormalization are difficult to obtain exactly, although the recoupling transformation is still reasonably straightforward. It is simpler to proceed with an approximate renormalization, as is usual in quantum electrodynamics. This will be taken to $O(g^2)$ as a first step. A complete calculation to any higher order would also require the inclusion of electric quadrupole and magnetic dipole interactions. The time evolution of the resulting equations can still be treated nonperturbatively, so this permits the treatment of efficient photodetection. In order to demonstrate the procedure, the harmonic oscillator detector will be treated here. More general cases will be treated elsewhere.

The harmonic oscillator Hamiltonian of interest has the structure given by Eq. (3.2). Although this is a relatively straightforward case, it is sufficient to illustrate the nature of the multimode calculation. A similar technique to that of Eq. (4.10) can now be employed with new multimode

transformations $\hat{R},\hat{U}$:

$$\bar{a}_k=\hat{U}\hat{R}\,\hat{a}_k\hat{R}^{-1}\hat{U}^{-1}\,,$$
$$\bar{b}_j=\hat{U}\hat{R}\,\hat{b}_j\hat{R}^{-1}\hat{U}^{-1}\,, \tag{6.1}$$

where

$$\hat{R}\equiv\exp\left[\tfrac{1}{2}\sum_k\sum_j(\Theta_{kj}\hat{a}_k\hat{a}_j-\Theta_{kj}\hat{a}_k^{\dagger}\hat{a}_j^{\dagger})-\tfrac{1}{2}\Theta\sum_j[\hat{b}_j^{2}-(\hat{b}_j^{\dagger})^2]\right],$$
$$\hat{U}\equiv\exp\left[\sum_k\eta_{kj}(\bar{a}_k+\bar{a}_k^{\dagger})(\bar{b}_j-\bar{b}_j^{\dagger})\right].$$

Here $\{\Theta_{kj},\Theta\}$ are real, symmetric terms of $O(g^2)$, and $\{\eta_{kj}\}$ is of $O(g)$. This transformation can be expressed directly, on expanding the exponentials, as follows:

$$\hat{a}_k=\sum_l\left[\cosh(\Theta)_{kl}\bar{a}_l-\sinh(\Theta)_{kl}\bar{a}_l^{\dagger}+\exp(\Theta)_{kl}\sum_j\eta_{lj}(\bar{b}_j-\bar{b}_j^{\dagger})\right],$$
$$\hat{b}_j=\cosh(\Theta)\bar{b}_j+\sinh(\Theta)\bar{b}_j^{\dagger}-\exp(\Theta)\sum_k\eta_{kj}(\bar{a}_k+\bar{a}_k^{\dagger})\,. \tag{6.2}$$

Substituting this transformation into the Hamiltonian gives, to order g^2, the following new expression:

$$\hat{H}/\hbar=\sum_k\sum_j\bar{\omega}_{kj}^{(3)}\bar{a}_k^{\dagger}\bar{a}_j+\sum_j\bar{\omega}_0\bar{b}_j^{\dagger}\bar{b}_j+\sum_k\sum_j\bar{g}_{kj}(\bar{a}_k\bar{b}_j^{\dagger}+\bar{a}_k^{\dagger}\bar{b}_j)+\Delta\omega\,. \tag{6.3}$$

Here I have utilized η_{kj} as in Eq. (4.10), so that

$$\eta_{kj}=(\mathbf{g}_k\cdot\mathbf{u}_j)\frac{1}{\omega_k+\omega_0}\,,$$

and

$$\Delta\omega=\sum_k\omega_k\Theta_{kk}+3\omega_0\Theta\,,$$
$$\bar{\omega}_{kj}=\omega_k\delta_{kj}+(\omega_k+\omega_j)\Theta_{kj}\,, \tag{6.4}$$
$$\bar{\omega}_0=\omega_0(1+2\Theta)\,,$$
$$\bar{g}_{kj}=(\mathbf{g}_k\cdot\mathbf{u}_j)\left[\frac{2\omega_k}{\omega_k+\omega_0}\right].$$

Equation (6.3) is only valid when the counter-rotating and self-energy terms cancel to the given order. This results in the following equations for the transformation parameters:

$$(\omega_k+\omega_j)\Theta_{kj}=\frac{2\mathbf{g}_k\cdot\mathbf{g}_j\omega_j\omega_k}{(\omega_k+\omega_0)(\omega_j+\omega_0)\omega_0}\,,$$
$$\Theta=\sum_k\frac{\omega_k}{3\omega_0}\left[\frac{|\,\mathbf{g}_k\,|}{\omega_k+\omega_0}\right]^2, \tag{6.5}$$

hence,

$$\Delta\omega=\sum_k\frac{\omega_k\,|\,\mathbf{g}_k\,|^{\,2}}{\omega_0(\omega_k+\omega_0)}=\frac{e^2}{4\pi^2\epsilon_0mc^3}\left[\int_0^{\omega_m}\omega\,d\omega-\int_0^{\omega_m}\frac{\omega\omega_0 d\omega}{\omega+\omega_0}\right].$$

In Eq. (6.3), the term that corresponds to the ground-state Lamb shift[20] is positive, as it was in the one-mode example. At first, this seems unusual: The standard Lamb shift calculation gives a linearly diverging negative energy shift[21] for the ground state, to order (g^2). The reason for this difference is that most calculations omit the $\hat{A}^2$ term in the Hamiltonian. This does not alter relative energy levels, but it does increase the overall energies. With this term added,[22] the ground-state Lamb shift can be checked using the minimal-coupling Hamiltonian and perturbation theory. The result is as in Eq. (6.5), where the first, quadratically diverging integral comes from the $\hat{A}^2$ term. The remaining integral can be further regularized by mass renormalization.[21]

An obvious feature of this Hamiltonian that is not present in the electric dipole $(\mathbf{d}\cdot\mathbf{E})$ Hamiltonian is the presence of nondiagonal terms Θ_{kj}, coupling plane waves of different momenta. Similar terms are present in the minimal-coupling Hamiltonian, where they give rise to direct scattering processes. At high frequencies, these cause Thomson (i.e., free electron) scattering. In the usual theory of photodetection, these direct scattering terms are not present, as they are included in the dipole-coupling and self-energy terms of the multipolar $(\mathbf{d}\cdot\mathbf{E})$ Hamiltonian. In the present treatment it is clear that these processes result in a nonresonant scattering of the input field, without any corresponding excitation of the photodetector. This direct scattering occurs at both high and low frequencies in the present Hamiltonian, where it gives rise to Thomson and Raleigh scattering, respectively. It clearly presents a fundamental limitation on photodetector efficiency in the multimode situation. The natural emergence of these processes out of a single, unified Hamiltonian is a distinct advantage of the canonical transformation used here.

Despite the scattering induced by the detector, it is still possible to obtain a relatively simple expression for each oscillator mode excitation. This is achieved using the canonically transformed Heisenberg picture field operators, in a similar way to Eq. (3.4). In the Heisenberg picture each detector operator time development is

$$\bar{b}_j(t)=\bar{b}_j(0)e^{-i\bar{\omega}_0t}-\sum_k\int_0^t i\bar{g}_{kj}\bar{a}_k(\tau)e^{-i\bar{\omega}_0(t-\tau)}d\tau\,. \tag{6.6}$$

This implies that the photodetection probability is

$$P(T)=\sum_j \langle \bar{b}_j^{\dagger}(T)\bar{b}_j(T)\rangle$$
$$=\sum_j \int_0^T \int \left[\sum_k \bar{g}_{kj}^{*} \sum_m \bar{g}_{mj}\langle \bar{a}_k^{\dagger}(\tau_1)\bar{a}_m(\tau_2)\rangle \times e^{-i\bar{\omega}_0(\tau_1-\tau_2)}\right] d\tau_1 d\tau_2 \,. \tag{6.7}$$

In order to compare this with the usual form of the photodetection probability,[2] it should be noted that in perturbative treatments to lowest order, the interacting fields $\bar{a}_k$ can be replaced by the noninteracting (input) fields $\hat{a}_k^0$. To the same order, dipole-coupled operators $\hat{\alpha}_k^0$ could also be used. Since the results obtained differ with respect to the coupling of the nonresonant modes, it is clear that the theory presented here predicts different results from the usual dipole-coupled theory, for experiments on extremely fast time scales. An interesting result of these time scales is that the photodetection probability has a quadratic dependence on the time scale T for $T\bar{\omega}_0 \ll 1$, as opposed to the linear dependence assumed in axiomatic theories of continuous measurement in quantum theory.[24]

To simplify Eq. (6.7), define a new field operator $\vec{\Phi}_0(t,x)$, which is the detectable photon amplitude relative to the frequency $\bar{\omega}_0$ ($\approx\omega_0$). Using this new field, the photodetection probability in the time T, is

$$P(T)=\frac{(\pi e)^2}{4\pi\epsilon_0 m}\int_0^T \int \langle \vec{\Phi}_0^{\dagger}(\tau_1,\mathbf{x}_0)\cdot\vec{\Phi}_0(\tau_2,\mathbf{x}_0)\rangle \times e^{-i\bar{\omega}_0(\tau_1-\tau_2)} d\tau_1 d\tau_2 \,, \tag{6.8}$$

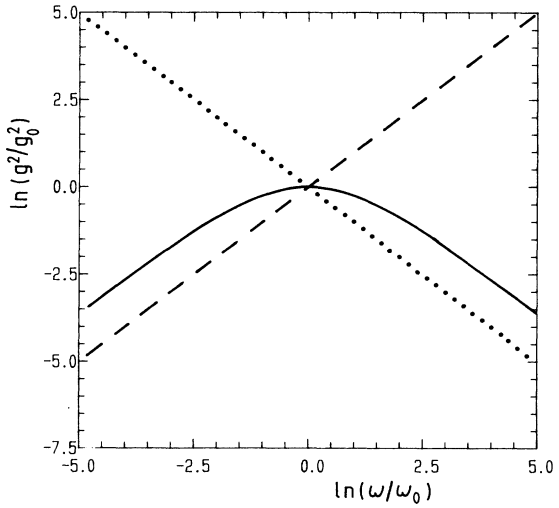

FIG. 1. Graph of coupling $(\ln[g/g_0]^2)$ vs frequency $(\ln[\omega/\omega_0])$ in the **p·A**, **d·E**, and symmetric photodetection Hamiltonians. The **p·A** interaction is denoted by the dotted line, the **d·E** interaction by the dashed line, and the solid line is the symmetric interaction.

where

$$\vec{\Phi}_0(t,\mathbf{x})=\sum_k \left[\frac{2(\omega_k\bar{\omega}_0/V)}{\omega_k+\bar{\omega}_0}\right]^{1/2} \mathbf{e}_k\bar{a}_k(t)e^{i\mathbf{k}\cdot\mathbf{x}} \,.$$

The above equation holds on all time scales. It is not dependent on cycle averaging, and it includes effects due to absorption and scattering. This requires that the fields utilized are the interacting fields, and that the frequency is the interacting (renormalized) frequency.

It can be noted that for near-resonant detection, $\vec{\Phi}_0(t,x)$ is approximately equal to one of the fields used to describe photon dynamics.[25] This shows that an atom detects photons, not the vector potential or the electric field. Equation (6.8) can be tested using a near-monochromatic field of frequency ω_k, with a time of $T<(\bar{\omega}_0-\omega_k)^{-1}$. The theory predicts that the photodetection efficiency is frequency independent for $\bar{\omega}_0\approx\omega_k$. By comparison, the **p·A** and **d·E** photodetection theories predict frequency-dependent efficiencies, proportional to ω_0/ω_k and ω_k/ω_0, respectively. The theory presented here interpolates between these two extremes, as shown in Fig. 1.

VII. CONCLUSION

In summary, the new operators defined here allow a natural treatment of photodetection in terms of a Hamiltonian without counter-rotating terms. This removes virtual corrections from the calculation, without any need for the rotating-wave approximation. Traditional methods for calculating photodetection probabilities do not give unique results when changing from the **d·E** to the **p·A** forms of the interaction Hamiltonian, as the rotating-wave approximation has different effects in the two types of Hamiltonian, except on long time scales. Unless care is taken in defining appropriate input states, these traditional methods for photodetection theory also give divergent results when the rotating-wave approximation is not employed. The unified or symmetric form of the Hamiltonian utilized here is applicable on all time scales, and gives finite results in the large cutoff limit.

The ground-state fluctuations can also be obtained from the transformation used to generate the operators. The result that the ground state is squeezed is in general agreement with field energy calculations of Persico and Power.[19] The point of interest is that a simple unitary transformation of the system operators allows these fluctuations to be included in a calculation of photodetection probability. This transformation renormalizes the Hamiltonian, as well as altering the coupling to a form without counter-rotating terms. These methods have a large potential applicability in other areas of nonperturbative and ultrafast-time-scale quantum electrodynamics.

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