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Conjecture on the dimensions of chaotic attractors of delayed-feedback dynamical systems

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The Lyapunov dimension of chaotic attractors is found to be almost equal to the delay time divided by the correlation time of the feedback driving force for three dynamical systems: Mackey-Glass model for white-cell production, optical bistable hybrid system, and nonlinear ring cavity. This discovery will enable experimentalists to estimate the complexity of a high-dimension system much more easily than by time-series methods, as illustrated by a hybrid experiment.

Three dynamical dissipative systems driven by a force proportional to a delayed nonlinear feedback were found numerically to exhibit finite-dimensional chaotic attractors with similar behaviors of their metric entropies h and their Lyapunov dimensions d_L, when the delay τ_R is large enough with respect to the dissipative time γ^{-1}.[1-4] Explicitly, h was found to be nearly independent of τ_R, and d_L was found to increase linearly with the delay. Farmer[1] was the first to observe this for a system modeling white-blood-cell production attributable to Mackey and Glass.[5] He foresaw that the behavior of both ergodic quantities, h and d_L, should be related to a sampling time other than τ_R. The other two examples are optical systems, the plane-wave ring cavity[2,3] and the electro-optic hybrid,[4] which may exhibit either bistability[6] or periodic behavior followed by a chaotic regime,[7] depending on the delay and the strength of the driving force. These systems have an asymptotic solution

$$x(t)=\gamma\int_0^t du\, e^{-\gamma u} f[x(t-u-\tau_R)] \ , \qquad (1)$$

where $x(t)$ is an observable and $f[x(t)]$ the deterministic driving force. The results of the theoretical study of the statistical properties[4] of the chaotic regime of a hybrid device have led us to generalize our results to other delayed-feedback systems.

The correlation time δ of the driving force f, which is a signature of the sensitivity to initial conditions, must characterize the dynamics of the system and the strange attractor. The correlation time has been found to be independent of and much smaller than τ_R in the limit $\gamma\tau_R \gg 1$, leading to an insightful and intuitive understanding of chaos. The interaction between the system and its feedback can be seen as a set of "kicks" of mean duration δ; after each time interval δ the system is ready to undergo a new kick of the feedback and so on during the time interval τ_R. Loosely speaking, the τ_R/δ kicks are independent events; the nth kick during the kth round trip is only correlated with the nth kick of the $(k+1)$st round trip, as confirmed by the secondary extrema of the feedback correlation function at approximately τ_R, $2\tau_R$, $3\tau_R, \ldots$. It follows intuitively that the number of independent kicks should be the effective number of degrees of freedom. Furthermore, the description of the dynamics as a series of independent events during an interval τ_R allows us to conjecture that the entropy is just the average amount of information stored during the memory time δ, and thus is independent of τ_R.

We conjecture that the dimension of the attractor is equal to τ_R/δ. This conjecture is supported by numerical calculations for all three delayed-feedback systems; the correlation time was determined from the correlation function of the feedback, and the ratio τ_R/δ was compared with d_L for different strengths of the driving force.

For high-dimensional attractors, the Lyapunov dimension d_L is, among all the approaches proposed for theoretically determining the information dimension, the easiest to calculate accurately.[8-10] The Lyapunov dimension, which is an upper bound to the information dimension, is defined with the help of the Lyapunov exponents, λ_i, by $d_L=j+|\lambda_j+1|^{-1}\sum\lambda_i$, where the summation is from 1 to j, where the integer j (defined such that $\sum\lambda_i\geq 0$) represents the number of degrees of freedom. It displays two advantages: Its integer part has a clear meaning, and it can be obtained exactly because the computation of j converges fast enough.

Let us focus the discussion on the plane-wave ring cavity because its dynamics has a simple physical explanation. The equations[2,11] that describe a plane-wave ring cavity containing a cell of gaseous two-level atoms are a boundary condition for the electric field amplitude $E(t)$ and a differential rate equation for the phase ϕ:

$$E(t)=E_0+RE(t-\tau_R)\exp\{i\chi[\phi(t-\tau_R)-1]\} \ , \qquad (2)$$

$$\dot{\phi}=d\phi(t)/dt=-\gamma[\phi(t)+A(\phi)\,|E(t)|^2] \ , \qquad (3)$$

$$A(\phi)=\{\exp[\alpha l(\phi-1)]-1\}/\alpha l \ . \qquad (4)$$

The intensity $|E(t)|^2$ is scaled to the off-resonance atomic saturation intensity, E_0 is the internal input field amplitude, $\phi(t)$ is proportional to the energy stored by the atoms and $\phi(0)=0$, and R is the reflectivity of both input and output mirrors. The complex linear index is $\chi=\alpha l(1-i\Delta)/2$, where α^{-1} and l are the off-resonance absorption and cell lengths, respectively; $\Delta=2\pi T_2(\nu_a-\nu_f)$ is the detuning between the atomic and the light frequencies scaled to the homogeneous half-width $(2\pi T_2)^{-1}$.

The correlation function $\Gamma(\tau)$ of the field amplitude is given by the time average

$$\Gamma(\tau)=\frac{1}{2}\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,E^*(t+\tau)E(t)+\text{c.c.}\right]$$
$$=\frac{1}{2}[\langle E^*(t+\tau)E(t)\rangle+\text{c.c.}]\ . \quad (5)$$

To obtain $\Gamma(\tau)$, 1 or 1.5 million points have been accumulated, after the transients have died away, corresponding to about a thousand τ_R. The initial conditions were $E(t)\equiv E_0$ during the first τ_R and all the atoms in the lower state at time $t=0$. Figures 1(a)–1(c) display the central portion of the normalized correlation functions

$$\Gamma_f(\tau)=\frac{\Gamma(\tau)-|\langle E\rangle|^2}{\Gamma(0)-|\langle E\rangle|^2} \quad (6)$$

for three chaotic solutions. Each Γ_f exhibits a peak about $\tau=0$ whose shape was found to be nearly independent of τ_R. For $\gamma\tau_R=20$, the curves reproduce very well the corresponding ones in Fig. 1, with a slight change in the tail of the central peak. Figure 1(d) shows the secondary extrema which occur at approximately τ_R, $2\tau_R$, ... Let us define the correlation time δ by the half-width of the peak at $\Gamma(\delta)=e^{-1}$. This definition is somewhat arbitrary, but the uncertainty in the numerical values are less than 2%. For $E_0=1.1$, Γ_f is obviously not a decreasing exponential while it nearly is for $E_0=2.5$.

Table I shows that the conjecture $d_L\cong\tau_R/\delta$ works very well in spite of the roughness of the determination of δ. The Lyapunov exponents were calculated using the Farmer algorithm.[1] It has also been pointed out[3] that $\tau_R/d_L\cong\delta$ is nearly equal to the power-broadening relaxation time $\tau_c=[\gamma(1+I_0)]^{-1}$ of a single atom driven by an intense continuous electric field ($I_0\equiv E_0^2\gg 1$). The relationship $d_L\cong\tau_R/\delta$ has the satisfying interpretation that the dimension of the attractor is simply the number of active modes of the cavity with frequency spacing τ_R^{-1}.

The relationship $d_L\cong\tau_R/\delta$ seems quite general and applies even in the limiting case $\gamma\tau_R\lesssim 1$ where d_L is no longer linearly proportional to τ_R.

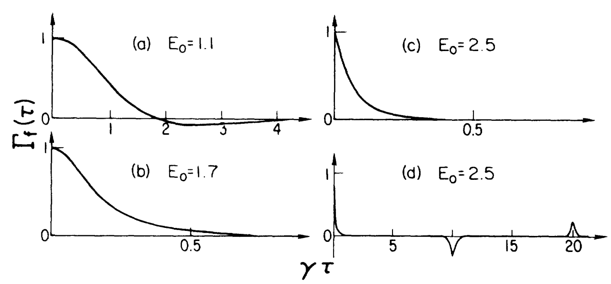

FIG. 1. $\Gamma_f(\tau)$ as defined by Eqs. (5) and (6) for the ring cavity with $\gamma\tau_R=10$, $R=0.95$, $\alpha l=4$, $\Delta=3\pi$. Note that $\Gamma_f(\tau)$ in (d) is drawn on a compressed time scale to exhibit the τ_R memory. For this set of parameters, two regions of chaotic behavior appear as E_0 is increased; $E_0=1.1$ lies in the lower region and $E_0=1.7$ and 2.5 in the upper region.

TABLE I. Ring-cavity numerical results supporting the conjecture that $d_L\cong\tau_R/\delta$. $\alpha l=4$, $R=0.95$, $\Delta=3\pi$, and d_L calculated for several $\gamma\tau_R$ from 1 to 20 for Eqs. (2)–(4).

E_0	$\frac{d_L}{\gamma\tau_R}$	$\frac{1}{\gamma\delta}$	$1+I_0$	Valid for $\gamma\tau_R$
1.1	0.75	0.95	2.2	$\gtrsim 5$
1.7	3.8	4.2	3.9	$\gtrsim 1$
2.5	~7.1,7.2	7.1	7.25	$\gtrsim 1$

It may be impossible to observe chaos in a ring cavity containing two-level atoms under conditions satisfying the plane-wave assumption of our theoretical analysis, but we are able to compare theory and experiment in a hybrid device.[7] The output voltage $x(t)$ is a solution of Eq. (1) with a feedback

$$f(x)=\pi\mu[1+R\cos(x+x_b)]\ , \quad (7)$$

where the bifurcation parameter μ is proportional to the product of the laser intensity and the feedback gain, R is an extinction coefficient, and x_b is the bias voltage. Because the dynamics is governed by a single differential equation with a feedback periodic in x, the mechanism leading to chaos can be understood more easily than that in a ring cavity. When $x(t)$ varies monotonically by an amount equal to (or less than) π, $f(x)$ undergoes a half-period motion. If $x(t)$ is periodic with period T, then $f(x)$ exhibits the same period. But if $x(t)$ varies monotonically by an amount just a little larger than π, dephasing between $f(x)$ and x occurs, and chaos follows. From this argument, we can predict that chaos would appear for $\mu R\cong 1$ and that the correlation time would be of the same order of magnitude as the mean time interval required for $x(t)$ to change by an amount of π.

The property of linear increase of d_L with the delay together with a constant entropy has been numerically verified for a large range of μ and R ($2\lesssim\mu\lesssim 30$; $R=\frac{2}{5}$, $\frac{2}{3}$, and 0.95). The bias x_b was set equal to $\pi/2$. Figure 2 shows that $d_L/\gamma\tau_R$ obeys the linear law $0.85(\gamma\delta)^{-1}$. Strikingly enough, this law works well with in both limits of small chaos ($\mu R\cong 2$) and large chaos ($\mu R\cong 30$).[12] The correlation function $\Gamma_f(\tau)$ was also found to be nearly invariant with τ_R for any $\tau\ll\tau_R$.

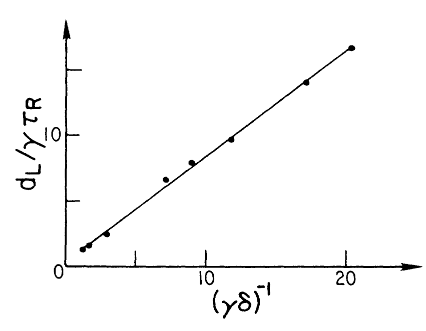

FIG. 2. The Lyapunov dimension scaled to $\gamma\tau_R$ as a function of the correlation time inverse δ^{-1} for the hybrid; δ is the half-width at e^{-1} of the autocorrelation of the feedback.

The hybrid device contains a helium-neon laser whose beam passes through an electro-optic modulator before it is coupled into a 1.1-km optical fiber. The light emerging from the fiber 6 μs later is detected by a silicon photodiode, and the resulting delayed voltage is amplified and fed back to the electro-optic modulator. The gain of the amplifiers is the experimental control parameter μ. A beam splitter samples the laser intensity between the modulator and the fiber: This represents the feedback function of the system. The feedback signal is detected by a photomultiplier which has a much higher bandwidth than the amplifiers or the silicon detector.

The autocorrelation of the feedback signal was determined for various settings of the device through two independent techniques. In all cases, $x_b = \pi/2$ and $R = 0.98$. The straightforward time-domain method is described first. For each setting of the device, a time series consisting of 262 000 data points was obtained using a computerized data-acquisition system. Subtracting off the dc component (which only alters the dc level of the autocorrelation) yields the working time series V_j. The sampling time t_s was 85 ns in most cases, the delay time $\tau_R = 6$ μs, and response time γ^{-1} either 1.2 or 2.5 μs.[13] The unnormalized autocorrelation $\Gamma_f(\tau)$ where $\tau = it_s$, was computed by the summation $\Gamma_f(\tau) = \Sigma V_j V_{j+i}$ over 250 000 j's; the largest value of i was 125, so the maximum τ exceeds 10 μs. The alternative technique (same parameter settings) involves determining the power spectrum of the feedback function and finding its Fourier transform to obtain the autocorrelation, yielding $\Gamma_f(\tau)$, in good agreement with the time-series result.

Figure 3 exhibits the agreement between experimental and theoretical correlation times δ of the feedback. This demonstrates that the hybrid device is described well by Eqs. (1) and (7) and allows us to estimate the Lyapunov dimension of the experimental chaotic attractor by the conjecture $d_L = \tau_R/\delta$. This conjecture should be especially interesting to experimentalists since it is much easier to measure δ than to calculate the correlation dimension from the time series, for dimensions larger than a few units.[8] The validity of $d_L = \tau_R/\delta$ was also confirmed for the Mackey and Glass equation with Farmer's parameters for which d_L increases linearly with τ_R. *Since $d_L = \tau_R/\delta$ is obeyed for both a periodic feedback (hybrid) and short-range force (Mackey and Glass) as well as both periodic and short-range forces in a ring cavity, we conjecture that it should be valid in any delayed-feedback dynamical system which exhibits deterministic chaos.*

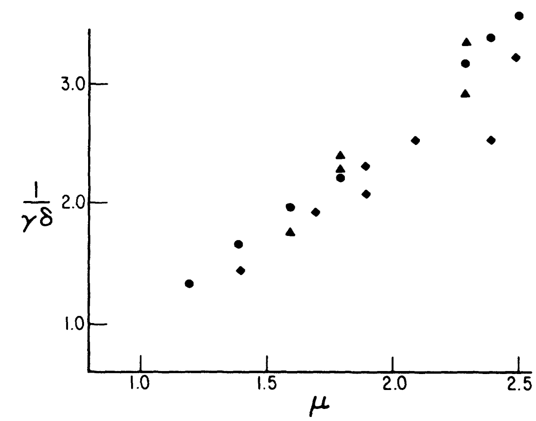

FIG. 3. Linear dependence of $(\gamma\delta)^{-1}$ vs μ, where γ^{-1} is the system response time, δ is the half-width at e^{-1} of the autocorrelation of the feedback function, and μ is the hybrid-device bifurcation parameter. Filled circles represent theoretical results; filled triangles, experimental time domain; filled diamonds, experimental power spectrum transform.

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[8]See P. Allen, J. G. Caputo, B. Malraison, and Y. Gagné, J. Mec. Theor. Appl. **3**, 133 (1984); J. Holzfuss and G. Mayer-Kress, in *Proceedings of the International Workshop on Dimensions and Entropies in Chaotic Systems, Pecos River, New Mexico, 1983*, edited by G. Mayer-Kress (Springer-Verlag, Berlin, 1985), Vol. 32, which show that the high-dimension analysis is very difficult because the errors increase linearly with the embedding dimension. Conversely, d_L which does not require any embedding dimension concept, is very accurate. Indeed j converges fairly quickly, and the error on the fractal part is less than 10%. Further, potential inconsistencies inherent to the calculation of the correlation dimension have recently been pointed out [F. A. Hopf *et al.*, Phys. Rev. Lett. **57**, 1394 (1986)].

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