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Traveling-wave solutions of the Maxwell-Bloch equations

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A possible class of traveling-wave solutions to the Maxwell-Bloch equations is presented and discussed. These are given by periodic solutions of the Lorenz equations; in particular, solutions for negative values of the parameter σ give stable traveling-wave solutions with phase velocity less than the speed of light.

It is well known that the amplitude of a homogeneously broadened single-mode laser can become unstable only in the bad-cavity limit where the linewidth κ of the singly excited resonant cavity mode exceeds the sum of the relaxation rates of the polarization ($\gamma_\perp$) and population ($\gamma_\parallel$).[1] This criterion is removed for multimode laser as first demonstrated by Risken and Nummedal[2] for the case when the initially excited single cavity mode is exactly resonant with the gain medium, and by several other authors[3] for the case when an additional detuning parameter is introduced into the system. For the latter case, there are two types of instability regions, now generally known as regions of phase instability and amplitude instability.[4] The region of amplitude instability can be considered a simple extension of that first studied by Risken and Nummedal. The region of phase instability occurs immediately above threshold in a manner similar to that observed in inhomogeneously broadened laser systems. The stability properties of the initially excited single mode have been extensively reviewed in Refs. 2 and 3, and it is not our intention to discuss these further here. Rather, we consider one possible class of final-state solutions to the laser equations, namely, traveling waves simply related to the set of periodic orbits of a low-dimensional dynamical system: the Lorenz equations.[5]

In appropriately normalized units, the laser equations for a resonant ring cavity system are[2]

$$\begin{aligned} &c\frac{\partial E}{\partial x}+\frac{\partial E}{\partial t}=\kappa(P-E)\,,\\ &\frac{\partial P}{\partial t}=\gamma_\perp ES-\gamma_\perp P\,,\\ &\frac{\partial S}{\partial t}=-\frac{\gamma_\parallel}{2}\lambda(E^*P+EP^*)-\gamma_\parallel(S-\lambda-1)\,. \end{aligned}\tag{1}$$

Here E and P are the slowly varying envelopes of the (complex) electric field and polarization, respectively, S is the (real) population inversion, and λ is a pump parameter ($\lambda=0$ at threshold, and is greater than zero above threshold). The cavity mirrors impose the periodicity conditions

$$F(x+L,t)=F(x,t)\,,\tag{2}$$

where F is either E, P, or S. Equations (1) have a constant-amplitude solution whose stability properties have been discussed elsewhere.[2-4] This solution corresponds to continuous laser operation in a single mode. We consider traveling-wave solutions of the form

$$\begin{aligned} &E(x,t)=\mathcal{E}(t-x/v)\,,\\ &P(x,t)=\mathcal{P}(t-x/v)\,,\\ &S(x,t)=\mathcal{S}(t-x/v)\,, \end{aligned}$$

in which case Eqs. (1) are replaced by

$$\begin{aligned} &\frac{dE}{dt}=\kappa'(\mathcal{P}-\mathcal{E})\,,\\ &\frac{dP}{dt}=\gamma_\perp\mathcal{E}\mathcal{S}-\gamma_\perp\mathcal{P}\,,\\ &\frac{dS}{dt}=-\frac{\gamma_\parallel}{2}\lambda(\mathcal{E}^*\mathcal{P}+\mathcal{E}\mathcal{P}^*)-\gamma_\parallel(\mathcal{S}-\lambda-1)\,, \end{aligned}\tag{1$'$}$$

where $\kappa'=\kappa/(1-c/v)$. Notice that these are the single mode equations, apart from the replacement of κ' for κ. Note also that even if κ is very much less than $\gamma_\perp+\gamma_\parallel$ there is no such restriction on κ'; for values of the phase velocity v less than c, κ' can even be negative.

The boundary conditions (2), when applied to the traveling-wave forms assumed in deriving Eqs. (1$'$), mean that

$$\begin{aligned} &\mathcal{E}(\tau)=\mathcal{E}(\tau+T)\,,\\ &\mathcal{P}(\tau)=\mathcal{P}(\tau+T)\,,\\ &\mathcal{S}(\tau)=\mathcal{S}(\tau+T)\,, \end{aligned}\tag{2$'$}$$

with

$$\tau=t-x/v\,.$$

In other words, the traveling-wave-type solutions of (1) are periodic orbits of Eqs. (1$'$) with basic period $T=L/v$ (v unknown).

In this paper we take $\mathcal{E}$ and $\mathcal{P}$ to be real; in a future publication[6] we show that the general case with complex fields and with the presence of detuning between the atomic resonance and the cavity modes can be treated in a similar way, and present the corresponding relationship with quasiperiodic solutions to the complex Lorenz equations.[7] It is then more convenient to use the simple transformations given in Ref. 1 to rewrite Eqs. (1$'$) in the "standard" form normally considered in dynamical systems, first quoted by Lorenz:[5]

$$\begin{aligned} &\dot X=\sigma(Y-X)\,,\\ &\dot Y=(r-Z)X-Y\,,\\ &\dot Z=XY-bZ\,. \end{aligned}\tag{3}$$

Here $\sigma=\kappa'/\gamma_\perp=\kappa/(1-c/v)\gamma_\perp$, $b=\gamma_\parallel/\gamma_\perp$, $r=\lambda-1$, $X=\sqrt{b}E$, $Y=\sqrt{b}P$, $Z=\lambda-1-S$, and $\tau=(t-x/v)/\gamma_\perp$. Differentiation is with respect to τ.

Consider the superluminal case first ($v>c$), in which case σ is positive. It is known[8] that when $r>1$, Eqs. (3) have two symmetric fixed-point solutions, which are stable in the region $1<r<r_c$,

$$r_c=\frac{\sigma(\sigma+b+3)}{\sigma-b-1} . \tag{4}$$

Several authors[7,8] have studied the Lorenz equations in (r,b) parameter space for a fixed value of σ ($\sigma=10$). Here we fix b, κ (but not κ'), and L and study the Lorenz equations with σ as an eigenvalue (i.e., σ must be such that the periodicity condition is satisfied) and have r as the variable control parameter. Equation (4) is quadratic in σ; the fixed points are then unstable in a region $\sigma_1<\sigma<\sigma_2$, and it is known that these become unstable by a subcritical Hopf bifurcation. It is not difficult to show that the homoclinic explosion that produces the limit cycles involved in the Hopf bifurcation occurs at two values of $\sigma(\sigma_{hc1}<\sigma_1, \sigma_{hc2}>\sigma_2)$.

Fixing r and b we can plot the period T as a function of σ for all the periodic orbits of Eqs. (3); using the classification scheme described in the book by Sparrow[8] we show such plots for a few of the simpler orbits in Fig. 1. Any such periodic orbit is an acceptable solution to the laser equations (1) *provided* that the period of the orbit is equal to L/v, as required by (2′). This criterion is most usefully represented graphically by noticing that

$$T_v=L/v=L/c\left(\frac{\sigma-\kappa/\gamma_\perp}{\sigma}\right) . \tag{5}$$

A plot of T_v vs σ is also shown in Fig. 1. Intersections of T_v with the curves for periodic orbits give acceptable solutions, which satisfy Eqs. (1) and (2).

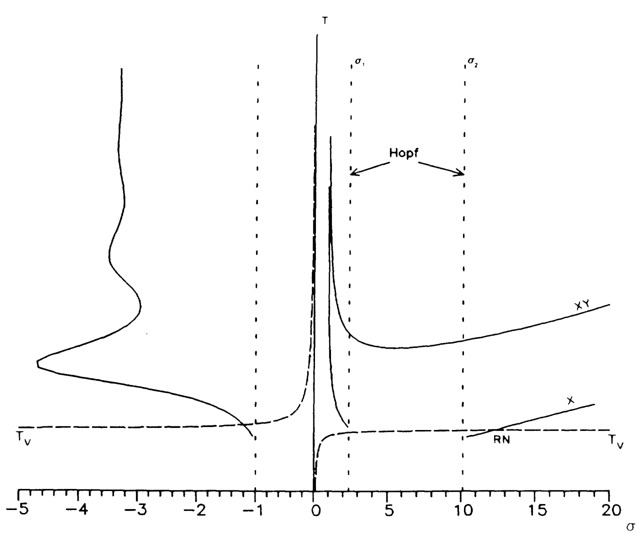

FIG. 1. Period of three simple orbits in the Lorenz equations plotted against σ (for $b=0.5$ and $r=16$). Here we plot the period of the X orbit, XY symmetric orbit, and the symmetric principal periodic orbit for negative σ. T_v [Eq. (5)] is also shown in this figure for both positive and negative σ (broken curves). There is a Hopf bifurcation of the origin at $\sigma=-1$, and two Hopf bifurcations of the fixed points C_1 and C_2 at σ_1,σ_2. The first homoclinic orbit of the origin occurs at $\sigma\simeq0.9217$, there is a second one at $\sigma\simeq44$ (not shown). A heteroclinic orbit connects C_1 and C_2 at $\sigma\simeq-3.31$ (notice the different horizontal scales for negative and positive σ).

As well as the positive-σ solutions considered above, it is also possible to have $\sigma<0$, corresponding to subluminal traveling waves. The Lorenz equations (3) are now studied for negative σ. The linear stability of the origin is easily carried out and reveals that there is a Hopf bifurcation (supercritical) at $\sigma=-1$ for all values of $r>1$. Moreover, numerical studies indicate that there is a codimension-one heteroclinic orbit of the Sil'nikov type[9] linking the two symmetric fixed points.[10]

Our numerical investigations reveal that the periodic orbit born in the Hopf bifurcation at $\sigma=-1$ acquires extra turns around the unstable manifolds of the fixed points and eventually becomes the (symmetric) heteroclinic connection; that is, the "principal periodic orbit" in the language of Glendinning and Sparrow.[9] This behavior is shown in Fig. 2, where the asymptotic behavior toward the heteroclinic orbit for some specific value of r and b is indicated. The evolution of the period against σ for this orbit is also shown in the left-hand side of Fig. 1. The periodicity condition (2′) must still hold; consequently, we show also the plot of T_v [Eq. (5)] for $\sigma<0$. The intersection of these two curves gives again an acceptable solution which satisfies both Eqs. (1) and (2). In addition to the symmetric principal orbit, there exists an infinite class of asymmetric and period-doubled orbits winding their way up to attain homoclinic status, in agreement with Glendinning and Sparrow.[9] We concentrate on the symmetric principal orbit in this paper.

Having found and classified all these types of traveling-wave solutions to Eqs. (1) it is necessary to consider next their stability. At present, we do this by substituting the appropriate traveling-wave solution into Eqs. (1) and simply observing whether the wave form persists indefinitely. Some periodic solutions for the region $\sigma>0$ are stable, but studies to date indicate that most are unstable. One exception is the simple pulsed solution found by Risken and Nummedal, which corresponds to a simple x-type orbit of the Lorenz equations, and which is labeled RN in Fig. 1. In contrast, the traveling-wave solutions for negative σ would appear to be stable, and have been observed to act as attractors for the system (1). Integrating numerically the partial differential equations (1) and (2) we have observed several instances when an initial wave form from the region $\sigma>0$ becomes unstable, a transient regime ensues for a while before the system settles down to a different type of traveling wave which can be identified as one of the periodic orbits for the $\sigma<0$ case.

We believe certain results recently published by Lugiato *et al.*[11] are related to our periodic orbits for $\sigma<0$ (in their paper, they present a "square-wave" periodic solution found by numerical integration to a set of partial differential equations closely related to our system), even though their model is slightly different from ours in that cavity losses are considered discreet rather than uniformly

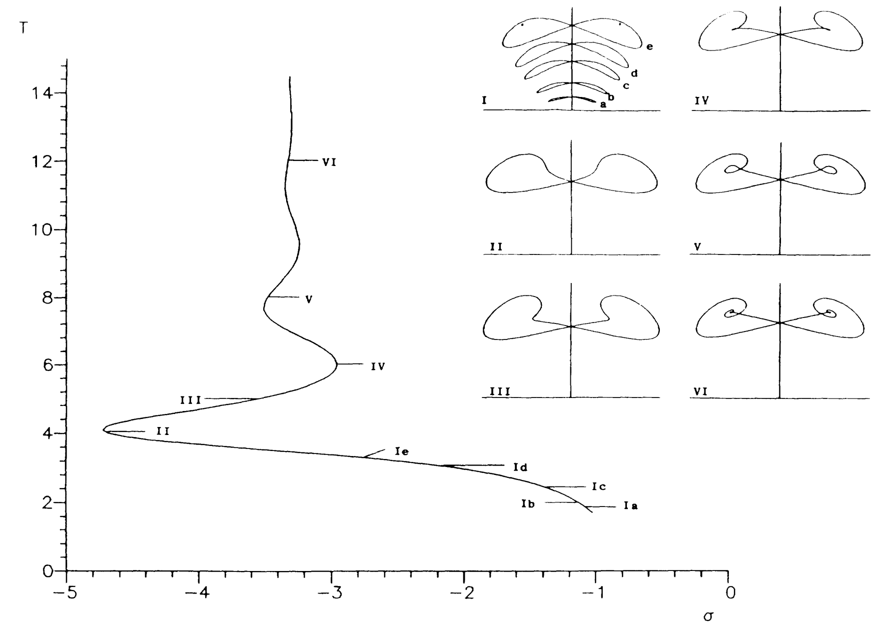

FIG. 2. The symmetric principal periodic orbit for $\sigma < 0$ in the Lorenz equations. This orbit is born in a Hopf bifurcation of the origin ($\sigma = -1$) and eventually becomes the heteroclinic connection between C_1 and C_2 ($t \to \infty$). The Z-X projection is shown for several stages in the orbit evolution, together with a plot of the period against σ. In this figure, $r = 16$ and $b = 0.5$.

distributed as here, and our gain media is assumed to fill the cavity.

The subluminal waves are also likely to be the "slow" solutions reported by Hillman and Koch[12] when looking for bichromatic states in homogeneously broadened lasers. It is interesting to note the presence of multistability at very low pump values above threshold, where the cw solution coexists with the subluminal waves (the symmetric principal orbit exists for r values just above threshold). These results may account for the behavior experimentally observed by Hillman *et al.*[13] In particular, the spontaneous split in frequency (and disappearance of the central resonant peak), multistable operation, and hysteresis loops can be explained by the model. A more detailed study of this particular point is in progress and results will be presented elsewhere.

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