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Quantum chaos in the Lorenz equations with symmetry breaking

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The role of phase diffusion for quantum chaos in the quantum-mechanical model of the laser in the Haken limit is discussed. Fractal properties of the support of the asymptotic attracting probability distribution for the system are studied.

I. INTRODUCTION

We will use the term "quantum chaos" to denote the behavior of a system which at a classical level shows chaos. Our interest will be in quantum dissipative systems. It is usual to describe the evolution of such systems in terms of density matrices and master equations.[1–3] Moreover, it is expected that in these systems the density matrix evolves to a time-independent steady state.[4] The support of the asymptotic distribution may be fractal,[5,6] and the short-time response chaotic.

The Lorenz system[7] has played a key role in the study of classical chaos. It can be derived[8] as the "semiclassical" limit of a fully quantum-mechanical theory of the laser. In this limit fluctuations in the electromagnetic field are ignored. Consequently we can return to the quantum-mechanical laser theory and allow for fluctuations in the electromagnetic field. The natural generalization of the Schrödinger equation to the situation where there is dissipation is the master equation. For a single-mode homogeneously broadened laser the density matrix ρ satisfies in the interaction picture a master equation,

$$\dot{\rho}=L\rho=\frac{1}{i\hbar}[H,\rho]+\kappa([a\rho,a^\dagger]+[a,\rho a^\dagger])+\Lambda_A\rho \ , \qquad (1)$$

where the Hamiltonian H is given by

$$H=ig\hbar\sum_{j=1}^{N}(e^{-i\mathbf{k}\cdot\mathbf{x}_j}a^\dagger r_j^- - e^{i\mathbf{k}\cdot\mathbf{x}_j}a r_j^+)$$

and

$$\Lambda_A\rho=\tfrac{1}{2}\sum_{j=1}^{N}\{\gamma_\uparrow([r_j^+,\rho r_j^-]+[r_j^+\rho,r_j^-]) + \gamma_\downarrow([r_j^-,\rho r_j^+]+[r_j^-\rho,r_j^+]) + \gamma_0([r_j^3,\rho r_j^3]+[r_j^3\rho,r_j^3])\} \ .$$

The terms not involving H in Eq. (1) give rise to damping which follow from the standard theory of quantum damping.[2] Here N is the number of atoms in the laser cavity. The atoms are taken as usual to be two-level, i.e., to have just a ground and excited state. $\gamma_\downarrow$ is the Einstein A coefficient, $\gamma_\uparrow$ is the pumping rate, γ_0 is the phase decay rate due to collisions, κ is the cavity damping rate, $\mathbf{k}$ is the wave vector of the field mode, $\mathbf{x}_j$ is the position of the jth atom, and g is the atom-field coupling constant. $(r_j^\pm,r_j^3)$ are the Pauli matrices associated with the jth two-level atom and a is the field mode destruction operator. The quantum states of the system lie in an infinite dimensional Hilbert space spanned by states of the form

$$|n\rangle\prod_{j=1}^{N}|\alpha_j\rangle \ ,$$

where α_j is $(+1,-1)$ and n is a non-negative integer. Moreover,

$$\begin{aligned} a^\dagger|n\rangle\prod_{j=1}^{N}|\alpha_j\rangle &= \sqrt{n+1}\,|n+1\rangle\prod_{j=1}^{N}|\alpha_j\rangle \ , \\ a\,|n\rangle\prod_{j=1}^{N}|\alpha_j\rangle &= \sqrt{n}\,|n-1\rangle\prod_{j=1}^{N}|\alpha_j\rangle \ , \\ r_l^3|n\rangle\prod_{j=1}^{N}|\alpha_j\rangle &= \alpha_l|n\rangle\prod_{j=1}^{N}|\alpha_j\rangle, \quad l=1,2,\ldots,N \\ r_l^\pm|n\rangle\prod_{j=1}^{N}|\alpha_j\rangle &= |n\rangle\,|\alpha_l\pm 2\rangle\Big[\prod_{j\,(\neq l)}|\alpha_j\rangle\Big] \times(\delta_{\alpha_l+2,1}+\delta_{\alpha_l-2,-1}), \\ & \qquad l=1,2,\ldots,N \ . \end{aligned} \qquad (2)$$

Haken[8] has shown that in the semiclassical limit the Lorenz equations can be deduced from Eq. (1). If we evaluate the expectation values of arbitrary operators for the system described by the density matrix satisfying Eq. (1), then such quantities give the true quantum analogue of the Lorenz equations.

It has been recognized for some time that the operator equation (1) can be written in terms of c-number stochastic equations[3] when the number N of atoms is large. In particular, this can be accomplished as follows. We introduce a generalized c-number Wigner distribution P by

$$\mathrm{Tr}(\chi\rho)=\int d\bar{\xi}_x \int d\bar{\xi}_y \int d\bar{\xi}_x \int d\bar{\xi}_y \int d\bar{\eta}\, P(\bar{\xi},\bar{\xi}^*,\bar{\eta},\bar{\zeta},\bar{\zeta}^*)\exp[i(\bar{\zeta}\zeta+\bar{\zeta}^*\zeta^*+\bar{\xi}^*\xi^*+\bar{\xi}\xi+\bar{\eta}\eta)]\ , \tag{3}$$

where χ is the "characteristic operator"

$$\chi(\xi,\xi^*,\eta,\zeta,\zeta^*)=\exp\left[\sum_{\mu=1}^{N} i(\xi^* r_\mu^+ e^{i\mathbf{k}\cdot\mathbf{x}_\mu}+\eta r_\mu^3+\xi r_\mu^- e^{-i\mathbf{k}\cdot\mathbf{x}_\mu})\right]\exp[i(\zeta^* a^\dagger+\zeta a)]\ . \tag{4}$$

In the limit[9] of N large, P, in terms of the scaled variables

$$\bar{m}=\frac{2}{N}\bar{\eta}\ ,$$
$$\bar{v}=\bar{v}_1+i\bar{v}_2=-\frac{2}{N}\left[\frac{\frac{1}{2}\gamma_\downarrow+2\gamma_0}{\gamma_\downarrow}\right]^{1/2}\bar{\xi}\ ,$$
$$\bar{x}=\bar{x}_1+i\bar{x}_2=n_0^{-1/2}\bar{\zeta}\ ,$$
$$C=g^2N/\kappa\gamma_\downarrow\ ,$$
$$n_0=\gamma_\downarrow N/8\kappa C\ ,$$
$$\bar{\alpha}=n_0^{-1/2}\alpha\ ,$$

satisfies the partial differential equation

$$\begin{aligned}\frac{\partial P}{\partial t}=\Bigg\{&\gamma_\perp\frac{\partial}{\partial\bar{v}_1}(\bar{v}_1+\bar{m}x_1)+\gamma_\perp\frac{\partial}{\partial\bar{v}_2}(\bar{v}_2+\bar{m}x_2)+\kappa\left[\frac{\partial}{\partial\bar{x}_1}(\bar{x}_1-\bar{\alpha}+2C\bar{v}_1)+\frac{\partial}{\partial\bar{x}_2}(\bar{x}_2+2C\bar{v}_2)\right]\\&+\gamma_{\|}\frac{\partial}{\partial\bar{m}}(\bar{m}-\hat{\sigma}-\bar{v}_1\bar{x}_1-\bar{v}_2\bar{x}_2)\\&+\gamma_\perp^2(2C\kappa n_0)^{-1}\\&\times\left[\frac{1}{4}\left[\frac{\partial^2}{\partial\bar{v}_1^2}+\frac{\partial^2}{\partial\bar{v}_2^2}\right]+f^2\frac{\partial^2}{\partial\bar{m}^2}(1-\sigma\bar{m})-\hat{\sigma}f^2\frac{\partial}{\partial\bar{m}}\left[\frac{\partial}{\partial\bar{v}_1}\bar{v}_1+\frac{\partial}{\partial\bar{v}_2}\bar{v}_2\right]\right]+\frac{\kappa}{4n_0}\left[\frac{\partial^2}{\partial\bar{x}_1^2}+\frac{\partial^2}{\partial\bar{x}_2^2}\right]\Bigg\}P\ ,\end{aligned} \tag{5}$$

where

$$\gamma=2\gamma_0+\tfrac{1}{2}\gamma_\downarrow\ ,$$
$$\hat{\sigma}=(\gamma_\uparrow-\gamma_\downarrow)/(\gamma_\uparrow+\gamma_\downarrow)\ ,$$
$$\gamma_{\|}=\gamma_\uparrow+\gamma_\downarrow\ ,$$
$$f=\frac{\gamma_{\|}}{2\gamma_\perp}\ .$$

From Eq. (3) it is clear that the c-number moments of P are the *quantum-mechanical expectation values* associated with the density matrix. The final link in the logic which allows us to treat the quantum theory in terms of stochastic differential equations will now be supplied. Equation (5) is a Fokker-Planck equation (FPE) for P and when the diffusion matrix is effectively positive definite it is well known that the moments of such a P can be obtained from the solution of Ito-Langevin equations which can be uniquely associated with P. These equations are

$$d\bar{x}=-\kappa(\bar{x}-\bar{\alpha}+2C\bar{v})dt+\left[\frac{\kappa}{2n_0}\right]^{1/2}dW_x\ ,$$

$$d\bar{v}=-\gamma_\perp(\bar{v}+\bar{m}\,\bar{x})dt+\frac{\gamma_\perp}{2(\kappa Cn_0)^{1/2}}dW_y\ , \tag{6}$$

$$\begin{aligned}d\bar{m}=&-\gamma_{\|}[\bar{m}-\hat{\sigma}-\tfrac{1}{2}(\bar{v}^*\bar{x}+\bar{v}\,\bar{x}^*)]dt\\&+\frac{\gamma_\perp f}{(\kappa Cn_0)^{1/2}}\\&\times\left[-\frac{\hat{\sigma}f}{2}(\bar{v}^*dW_y+\bar{v}dW_{y^*})\right.\\&\left.+(1-\hat{\sigma}\bar{m}-\hat{\sigma}^2f^2\bar{v}\,\bar{v}^*)^{1/2}dW_z\right]\ ,\end{aligned}$$

where $W_x=W_1+iW_2$, $W_y=W_3+iW_4$, and $W_z=W_5$. $W_1,W_2,\ldots,W_5$ are independent Wiener processes. Closer correspondence to Lorenz-type systems can be obtained by changing variables to

$$x=-\left[\frac{b}{2}\right]^{1/2}\bar{x}\ ,$$
$$y=(2b)^{1/2}C\bar{v}\ ,$$
$$z=2C\bar{m}\ , \tag{7}$$
$$\tau=\gamma_\perp t\ ,$$

and the parameters b, σ, C, and r are defined by

$$b=\frac{\gamma_{\|}}{\gamma_{\perp}},$$

$$\sigma=\frac{\kappa}{\gamma_{\perp}},$$

$$C=\frac{r}{2}\left[\frac{\gamma_{\downarrow}+\gamma_{\uparrow}}{\gamma_{\uparrow}-\gamma_{\downarrow}}\right],$$

$$r=2C\hat{\sigma}.$$

The equations (6) then have the form

$$dx=\sigma(y-x)dt-\left[\frac{2}{C}\right]^{1/2}\sigma\epsilon\, dW_x\,,$$
$$dy=(xz-y)dt+2\epsilon\, dW_y\,, \tag{8}$$
$$dz=-[b(z-r)+xy^*+x^*y]dt-\frac{br\epsilon}{4C^2}(y^*\,dW_y+y\,dW_{y^*})+(8b)^{1/2}\epsilon\left[1-\frac{rz}{4C^2}-\frac{r^2b}{32C^4}yy^*\right]^{1/2}dW_z\,,$$

where

$$\epsilon=C\left[\frac{\gamma_{\uparrow}+\gamma_{\downarrow}}{\gamma_{\downarrow}N}\right]^{1/2}.$$

The expectation values of products of x, y, and z are proportional to the quantum-mechanical expectation values of the corresponding symmetrized product of the electric field, atomic polarization, and inversion operators.

In our use of the term quantum chaos we have required that the parameter values of the system are chosen such that in the classical approximation the system is chaotic. The development summarized in Eq. (8) requires that N is large. For consistency it is necessary to check that this requirement is compatible with the constraints on the parameters necessary to obtain classical chaos. The Lorenz-Haken[7,8] criterion for chaos is

$$r>\frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}\,. \tag{9}$$

It is easy to show that

$$N=\frac{r^2}{4\epsilon^2}\frac{(\gamma_{\uparrow}+\gamma_{\downarrow})^3}{\gamma_{\downarrow}(\gamma_{\uparrow}-\gamma_{\downarrow})^2}\,. \tag{10}$$

For a given set r, σ, b, and ϵ the minimum N ($N_{\min}$) is given by

$$N_{\min}=\frac{27}{8}\frac{r^2}{\epsilon^2} \tag{11}$$

since $\gamma_{\uparrow}=5\gamma_{\downarrow}$ for the minimum. For $\sigma=5$, $b=1$, $r=15.1$, and ϵ as large as 0.1, $N_{\min}$ is 76 954. Equations (5) and (8) are valid for such N. Even for $\epsilon=1$, $N_{\min}$ is 770 and the FPE is again expected to be valid. The content of a quantum theory can be expressed in terms of expectation values of operators and the use of Eq. (8) allows us to calculate these expectation values for Eq. (1). Hence the stochastic differential equation description that we are using should give an acceptable description of the quantum-mechanical behavior of a system which, at the classical level, gives the Lorenz system well known to have chaotic solutions. Moreover, for physically sensible values of b ($0<b\le 2$, in the context of lasers) the diffusion matrix is effectively positive definite. By our criterion for quantum chaos we have a description of it for a nontrivial dissipative system.

When we are dealing with fluctuations it is important to realize that Eq. (1) has a built-in special symmetry,

$$a\to ae^{i\phi}\,,$$
$$r_j^-\to r_j^- e^{i\phi}\,, \tag{12}$$
$$r_j^3\to r_j^3\,,$$

and similarly the conjugate transformation for the adjoint operators. This symmetry is, of course, reflected in Eq. (8) ($x\to xe^{i\phi}$, $y\to ye^{i\phi}$). It will turn out to be a crucial symmetry for the quantum theory. In order to understand the nature of this symmetry it is important to incorporate a symmetry-breaking interaction. In the quantum-mechanical problem this is achieved by having a classical external field α (with a definite phase) coupling to the resonant field mode in the laser cavity. The master equation (1) is then modified by the addition of a term to L of the form

$$-\kappa(\alpha[a,\rho]+\alpha^*[\rho,a^{\dagger}])\,. \tag{13}$$

This change leads to a corresponding modification of Eq. (8). Specifically the equation for x becomes

$$dx=\sigma(-x+y+\Omega)dt-\left[\frac{2}{C}\right]^{1/2}\sigma\epsilon\, dW_x\,, \tag{14}$$

where

$$\Omega=-\left[\frac{b}{2}\right]^{1/2}\frac{\alpha}{\sqrt{n_0}}$$

but the y and z equations remain unchanged.

It is clear from inspection that the symmetry represented in Eq. (12) no longer holds. Without loss of generality from now on we will choose our phase convention so that Ω is real. In the absence of fluctuations (represented by the Wiener processes) it is possible to consider the fixed-point structure of Eq. (14). The fixed points satisfy

$$x=|x|e^{i\theta},\quad y=|y|e^{i\phi}\,,$$

with

$$br|x|=(|x|\mp\Omega)(b+2|x|^2)\,, \tag{15}$$
$$\theta=\tfrac{1}{2}(\pi\mp\pi)\,,$$
$$|y|e^{i\phi}=|x|e^{i\theta}-\Omega\,, \tag{16}$$
$$\phi=\theta\,,$$

and

$$z=\frac{br}{b+2|x|^2}\,. \tag{17}$$

The 5×5 stability matrix in terms of a basis ordering $\{|x|,|y|,z,\theta,\phi\}$ is

$$\begin{bmatrix} -\sigma & \sigma & 0 & 0 & 0 \\ z & -1 & |x| & 0 & 0 \\ -2|y| & -2|x| & -b & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sigma}{|x|}(|y|+\Omega) & \frac{\sigma|y|}{|x|} \\ 0 & 0 & 0 & \frac{|x|z}{|y|} & \frac{-|x|z}{y} \end{bmatrix} . \quad (18)$$

We find the phase and amplitude parts of the matrix decouple. The eigenvalues corresponding to the lower 2×2 submatrix and to the stability in the phase directions of x and y are

$$\lambda=\frac{1}{2}\left\{-\left[\frac{|x|z}{|y|}+\frac{\sigma}{|x|}(|y|\pm\Omega)\right] + \left[\left[\frac{|x|z}{|y|}+\frac{\sigma}{|x|}(|y|\pm\Omega)\right]^2 \mp \frac{4\sigma z\Omega}{|y|}\right]^{1/2}\right\} \quad (19)$$

and

$$\lambda=-\frac{1}{2}\left\{\left[\frac{|x|z}{|y|}+\frac{\sigma}{|x|}(|y|\pm\Omega)\right] + \left[\left[\frac{|x|z}{|y|}+\frac{\sigma}{|x|}(|y|\pm\Omega)\right]^2 \mp \frac{4\sigma z\Omega}{|y|}\right]^{1/2}\right\} . \quad (20)$$

The $(\pm)$ and $(\mp)$ signs distinguish between the $\theta=0$ case (the upper sign) and $\theta=\pi$ (the lower sign). When $\Omega=0$ the eigenvalue of Eq. (19) becomes zero and the other eigenvalue negative. Moreover, there is no longer just two fixed points but a ring of equivalent fixed points with $|x|=|y|$, $\phi=\theta$, and $z=1$. The zero eigenvalue corresponds to motion around the ring and the negative eigenvalue corresponds to the relaxation of the difference of phases of x and y. In the quantum theory of the laser[3] fluctuations induce a random walk along the ring, and this is termed phase diffusion.

The upper 3×3 submatrix of Eq. (18) gives the stability criterion of the real Lorenz equations, and Eq. (9) gives the criterion for chaos. Just as phase diffusion is important in the region of stable fixed points when fluctuations are present it is expected that they will play a major role in quantum chaos. Our intention is to thoroughly examine the role of phase diffusion. Preliminary work[6] on Eq. (8) has given strong circumstantial evidence that quantum fluctuations manifest themselves mainly in the form of phase diffusion. Indeed the probability distributions for $|x|$ and $|y|$ were found to be less affected by variations of ϵ (the strength of fluctuations) than $\mathrm{Re}(x)$ and $\mathrm{Re}(y)$. Moreover, a fractal dimension of the attractor jumped by about one as ϵ was made nonzero. The value of that fractal dimension was independent of ϵ except for large ϵ (e.g., $\epsilon\sim1.0$). This suggests that the increase in dimension is due to the phase degree of freedom. Moreover, for $\epsilon=1$, the probability distributions of $\mathrm{Re}x$ and $|x|$ were shown[6] to have a radically different structure from that for small ϵ and so quantum chaos has a smoothening effect on the attractor. From Eqs. (10) and (11) we see that this smoothening occurs at parameter values which are consistent with the large-N approximation. In this situation it is important to have a parameter explicitly in the system which can control the phase diffusion. The term in Eq. (13) just provides this control.

II. THE QUANTUM ATTRACTOR

As we move from the realm of classical to quantum chaos we can no longer refer to a deterministic strange attractor. The expectation values of operators take the place of values of phase-space variables. For the case of interest to us these expectation values are derivable from moments of probability distributions. The concept of a strange attractor is replaced by a stationary probability distribution with a support which may have a complicated structure except at the finest scales. A good indicator of chaos in classical systems is given by Lyapunov exponents[10] which characterize the exponential fast separation with time of two points on the attractor which are initially close to each other. Our main aim is to demonstrate phase diffusion in our example of dissipative quantum chaos and it will turn out to be useful to develop an analogue of the Lyapunov exponent for the quantum attractor. Instead of two points initially close to each other, we take two very sharply peaked distributions with their peaks very close to each other and observe the rate of separation of the peaks and this rate will give the analogue of the Lyapunov exponent. In a standard way it is possible to define a Lyapunov dimension[10] in terms of the Lyapunov exponents.[10,11] Fractal dimensions of the support of the attractor in the presence of symmetry breaking are important because in the symmetric situation it was just such a dimension which strongly suggested that an additional degree of freedom was coming into play (even when the fluctuation strength ϵ was small). In addition to the Lyapunov dimension we will calculate a fractal dimension D_F, proposed[12] recently, which measures the clustering properties of points on an attracting set.

There are many forms of stochastic differential equations (SDE), and the one that arises naturally in our problem is due to Ito.[2] A one-dimensional SDE has the form

$$dx(t)=a(x(t),t)dt+b(x(t),t)dW(t) . \quad (21)$$

In discretized form the Ito SDE is

TABLE I. Lyapunov-exponent calculations for $r=20$ and $\epsilon=0$.

$\sigma\Omega$	λ_1	λ_2	λ_3	λ_4	λ_5	D_L
0.0	0.469	0.0	0.0	−6.00	−7.47	3.078
1.2	0.459	0.0	−0.01	−5.99	−7.46	3.075
2.4	0.441	0.0	−0.06	−5.94	−7.44	3.064
3.3	0.372	0.0	−0.12	−5.89	−7.36	3.043

$$x(t+\Delta t)-x(t)=a(x(t),t)\Delta t+b(x(t),t)\times[W(t+\Delta t)-W(t)]\ . \tag{22}$$

In particular

$$\langle x(t+\Delta t)-x(t)\rangle=\langle a(x(t),t)\rangle\Delta t \tag{23}$$

since

$$\begin{aligned}\langle b(x(t),t)[W(t+\Delta t)-W(t)]\rangle &=\langle b(x(t),t)\rangle\langle W(t+\Delta t)-W(t)\rangle\\ &=0\ .\end{aligned} \tag{24}$$

Equation (24) follows since $x(t)$ does not depend on the Wiener increment $[W(t+\Delta t)-W(t)]$ and $\langle W(t+\Delta t)-W(t)\rangle$ is zero. The property of Eq. (23) does not hold for non-Ito SDE's. The SDE's of Eqs. (8) and (14) have multiplicative noise. The solutions of these SDE's are based on approximating them by the discretized form of Eq. (22). The Wiener terms are generated through a standard pseudorandom-number generator (of a multiplicative-congruential type). Our work will concentrate on the parameter values $\sigma=5$ and $b=1$ which are possible for a laser. The threshold for chaos in the deterministic Lorenz system is then $r=15$. We study the cases $r=16$ and $r=20$. By continuity the classical equations for sufficiently small Ω must show chaos at these parameter values. However, as Ω is increased there is a threshold above which there is a stable fixed point coexisting with a chaotic attractor. For $r=16$ and 20 the threshold values of Ω are given by $\Omega=0.143$ and 0.695, respectively. At still larger Ω the chaotic solutions are metastable and eventually decay on to the fixed point. Numerical experiments indicate that for $r=20$ and $\Omega=0.72$ the decay time is in excess of 2.5×10^4 time units.

An important though minimal aspect of quantum fluctuations is that they force the variables x and y to be complex. The complex Lorenz equations (with no stochastic terms) has phase symmetry as we have already discussed. The Lyapunov exponents reflect the presence of this symmetry. With full symmetry there are two zero Lypunov exponents, one due to the behavior of trajectories which are infinitesimal time translates of the other, and the other due to lack of "stiffness" in the phase rotation direction. There is also an additional negative exponent that the standard (real) Lorenz equations do not show. This has value $-(\sigma+1)$ and is associated with the relaxation of phase differences between x and y. As the field is introduced the positive exponent is slowly reduced, one of the zero exponents becomes negative and the other is unchanged. The new negative exponent is associated with the relaxation of the phase of x and y towards the preferred one (that of the input field). Table I summarizes these results which hold for the deterministic situation.

If phase diffusion is the main effect of quantum fluctuations in the chaotic regime we would expect the zero Lyapunov exponent for phase diffusion to be unaffected for nonzero ϵ, but $\Omega=0$, since from Eq. (19) we have already seen that the deterministic restoring force for the phase is zero. We can see this from considering a stochastic differential equation

$$\dot\theta(t)=g(\theta(t))\xi(t)\ , \tag{25}$$

where $g(\theta)$ is some differentiable function and $\xi(t)$ is a Gaussian white-noise process.

Given two δ-function peaked distributions with the peaks infinitesimally close to each other [the separation being $\delta\theta(t)$] we need to average the local rate of separation of the positions of the peaks over the probabilistic attractor. This rate of separation is given by

$$\frac{d}{dt}\delta\theta(t)=g'(\theta(t))\delta\theta(t)\langle\xi(t)\rangle=0 \tag{26}$$

and so the Lyapunov exponent is zero. There will be another zero Lyapunov exponent in general for our system of stochastic differential equations. If we take an initial δ-function distribution and allow it to evolve for an infinitesimal time interval and take this as the second distribution (whose peak is close to the first one) then the average rate of separation of the peaks is zero. This can be illustrated by the following example:

$$\dot x(t)=f(x(t))+g(x(t))\xi(t)\ , \tag{27}$$

where f is twice differentiable

$$x(t)=x(0)+\int_0^t f(x(t'))dt'+\int_0^t g(x(t'))dW(t')\ , \tag{28}$$

TABLE II. Lyapunov-exponent calculations for $r=20$ and $\Omega=0$.

ϵ	λ_1	λ_2	λ_3	λ_4	λ_5	D_L
0.0	0.469	0.0	0.0	−6.00	−7.47	3.078
10^{-3}	0.456	0.0	0.0	−6.00	−7.46	3.076
3×10^{-3}	0.461	0.0	0.0	−6.00	−7.46	3.077

TABLE III. Lyapunov-exponent calculations for $r=20$ and $\sigma\Omega=1.2$.

ϵ	λ_1	λ_2	λ_3	λ_4	λ_5	D_L
0.0	0.459	0.0	−0.01	−5.99	−7.46	3.075
10^{-3}	0.444	0.0	−0.02	−5.99	−7.44	3.071
3×10^{-3}	0.452	0.0	−0.03	−5.99	−7.45	3.071

$$\langle x(t+\Delta t)-x(t)\rangle=\left\langle \int_t^{t+\Delta t} f(x(t'))dt' \right\rangle \sim \langle f(x(t))\rangle \Delta t \, , \tag{29}$$

for infinitesimal Δt. Consequently,

$$\frac{d}{dt}\langle x(t+\Delta t)-x(t)\rangle=\langle \{f'(x(t))\dot{x}(t)+\tfrac{1}{2}[g(x(t))]^2 \times f''(x(t))\}\rangle \Delta t \tag{30}$$

since for Ito stochastic processes[2]

$$df(x(t))=f'(x(t))dx(t)+\tfrac{1}{2}[g(x(t))]^2 f''(x(t))dt \, .$$

Since Δt is infinitesimal Eq. (30) shows that $d\langle[x(t+\Delta t)-x(t)]\rangle/dt$ is infinitesimal. Hence the Lyapunov exponent vanishes. The general method that we have used to calculate these exponents numerically is given in the Appendix. Our Lyapunov-exponent calculations are summarized in Tables I to IV. For Eq. (8) we find that the two zero Lyapunov exponents do indeed survive the presence of the noise which is in general multiplicative. [The multiplicative nature of the noise is enhanced for small values of C. From Eq. (7) we note that the minimum value of C is $r/2$ and the typical values of y and z on the chaotic attractor are of the order of $\pm r/2$ and so the magnitude of the noise terms may vary by a factor of 3.] By taking $f(x)$ in Eq. (27) to be $-kx$ it is easy to show that the phase diffusion Lyapunov exponent (which vanishes when $k=0$) becomes $-k$. Our numerical calculations with Eq. (14) show that with nonzero Ω and the resultant suppression of phase diffusion one of the zero Lyapunov exponents becomes increasingly negative with increasing Ω (see Tables III and IV). This body of numerical calculations strongly supports the importance of phase diffusion in the presence of quantum fluctuations.

The increase of fractal dimension by about one mentioned at the beginning of this section is thought again to be a signal of the additional phase degree of freedom. We will now report further calculations involving fractal dimensions. It is possible to introduce the Lyapunov dimension[10] D_L

$$D_L=n_L+\frac{\sum_{i=1}^{n_L}\lambda_i}{|\lambda_{n_L+1}|} \, , \tag{31}$$

where the Lyapunov exponents λ_i satisfy

$$\lambda_i \geq \lambda_{i+1} \, , \tag{32}$$

$$\sum_{i=1}^{n_L}\lambda_i \geq 0 \, , \tag{33}$$

and

$$\sum_{i=1}^{n_L+1}\lambda_i < 0 \, . \tag{34}$$

n_L gives the integer part of D_L. For large classes of deterministic systems Pesin's identity[10] gives

$$D_L=D_F \, . \tag{35}$$

However, for our quantum system this does not hold. Quantum effects force us to have x and y complex. Even when $\epsilon=0$ since the Lyapunov exponents "know" the presence of the phase degree of freedom D_L exceeds D_F by one. D_L varies very little with noise. In particular, very small scale effects where the attractor should be dominated by noise do not contribute to D_L. This can make D_L a good indicator of quantum chaos in some systems. D_L is well defined in the presence of nonzero Ω since the Lyapunov exponents are.

For the behavior of D_F to be consistent with the interpretation in terms of phase diffusion, it is necessary that, for length scales large compared with l_p, the D_F is that of the deterministic attractor. Here l_p is a typical length scale associated with phase diffusion. This D_F should rise by about one to reflect the additional phase degree of freedom for scales of the order of l_p. The D_F should be relatively flat and then at very small scales should rise to five reflecting the domination of the support of the probabilistic attractor by noise. With the breaking of phase symmetry by introducing nonzero Ω the scale l_p should decrease

TABLE IV. Lyapunov-exponent calculations for $r=20$ and $\sigma\Omega=3.3$.

ϵ	λ_1	λ_2	λ_3	λ_4	λ_5	D_L
0.0	0.372	0.0	−0.12	−5.89	−7.36	3.043
10^{-3}	0.406	0.0	−0.11	−5.91	−7.32	3.050
3×10^{-3}	0.389	0.0	−0.11	−5.91	−7.35	3.049

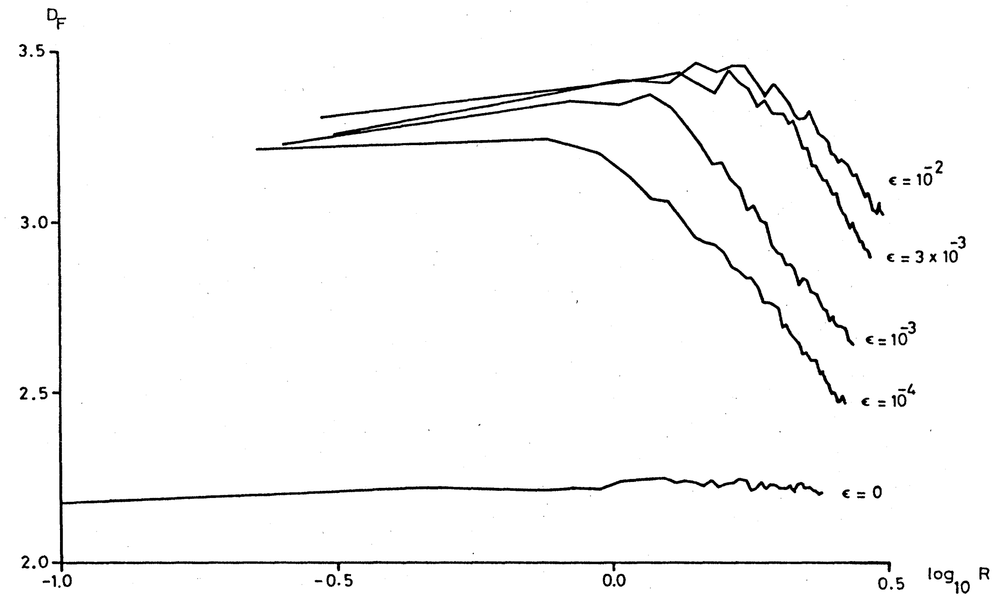

FIG. 1. D_F vs $\log_{10}R$ for $\Omega = 0.48$ with different ϵ.

and hence D_F should increase by about one at smaller r.

Various definitions of fractal dimensions have been proposed. The original capacity dimension[5] is not practical to use for higher-dimensional attractors. An alternative measure of dimension[12] is derived from the radii of spheres needed to cover the attracting set. If a sphere containing N points has radius R_N then

$$\overline{R}_N \sim N^{1/D} ,$$

where $(\overline{R}_N)^d$ is the average volume of a box containing N points and d is the topological dimension. For a self-similar situation there is a single D_F for all N. We adopt a more generalized usage and allow for cases where a single scaling law does not hold for all radii. In the symmetric case[6] D_F was 3.07 ± 0.04 for a large range of scales but started rising steeply at the smallest scales. This was interpreted[6] as a noise-related inner scale. For each determination of D_F we have used a data set of 50 000 and averaged over 1280 center points. We find that in the cases where the noise is zero the dimension is 2.20 and there is no evidence for deviation from power-law behavior. When there are fluctuations the situation is different. In Fig. 1 (for which $\Omega = 0.48$) there is no longer a well-defined D_F. It rises as $\log R$ decreases but then reaches a plateau value at some quite well-defined radius $R(\epsilon,\Omega)$. For fixed Ω, $R(\epsilon,\Omega)$ becomes larger and more sharply defined when ϵ increases (see Fig. 1). We would like to identify $R(\epsilon,\Omega)$ with the typical scale l_p at which phase diffusion effects appear. l_p can be inferred from the values of the mean and variance of Imx (see Table V). (Of course for the deterministic case such variables are strictly zero.) $R(\epsilon,\Omega)$ and l_p are of the same order of magnitude. Again, consistent with the interpretation in terms of the phase diffusion degree of freedom, $R(\epsilon,\Omega)$ decreases as Ω increases (see Fig. 2).

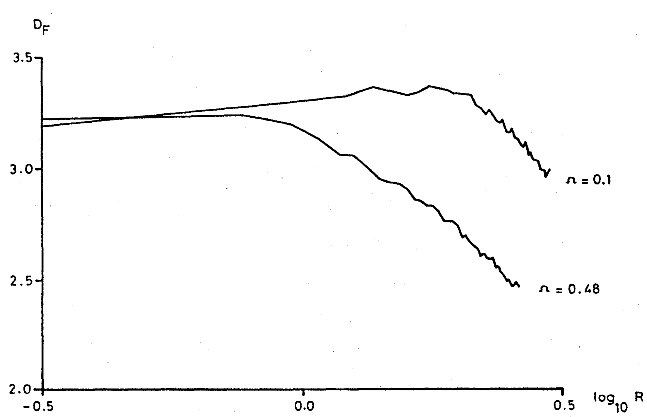

FIG. 2. D_F vs $\log_{10}R$ for $\epsilon = 10^{-4}$ with $\Omega = 0.1$ and 0.48.

In conclusion, by the use of phase-symmetry breaking in conjunction with the calculation of Lyapunov exponents and fractal dimensions we have obtained strong evidence that the same phase diffusion which is the dominant effect of fluctuations for the laser (i.e., fixed-point situation) plays a similar role for the chaotic regime.

TABLE V. $\langle (\mathrm{Im}x)^2 \rangle$ for various values of ϵ.

ϵ	$\langle (\mathrm{Im}x)^2 \rangle$	$\langle (\mathrm{Im}x)^2 \rangle^{1/2}$
0.0	0.0	0.0
10^{-4}	0.35	0.6
10^{-3}	0.62	0.8
3×10^{-3}	0.95	0.97
10^{-2}	1.42	1.19

APPENDIX: CALCULATION OF LYAPUNOV EXPONENTS

We shall follow very closely the method proposed by Eckmann and Ruelle,[10] and consider a set of m-dimensional continuous-time stochastic differential equations,

$$\frac{dx(t)}{dt}=F(x(t))+G(x(t))\xi \ , \tag{A1}$$

where $x(t)$, $F(x(t))\in \mathrm{R}^m$, and $G(x(t))$ is an $m\times m$ matrix. By reasoning similar to that given with Eq. (23) the separation of the peaks $u(t)$ of distributions which were initially infinitesimally close δ-function distributions is given by

$$\frac{d}{dt}u(t)=(D_{x(t)}F)u(t) \ . \tag{A2}$$

$D_{x(t)}F$ is the matrix of partial derivatives of the m components of F. We will refer to this as the tangent space equations, and they enable us to find a time average of the rate of separation (which gives then an average over the attractor for long enough time). It is possible to write a solution of (A1) for a particular realization of the noise as

$$x(t)=f^t(x(0)) \ . \tag{A3}$$

We then have

$$u(t)=(D_{x(0)}f^t)u(0) \tag{A4}$$

and can define

$$T_x^t=D_x f^t \ . \tag{A5}$$

Hence Eq. (A2) implies that

$$\frac{d}{dt}T_{x(0)}^t=(D_{x(t)}F)T_{x(0)}^t \ . \tag{A6}$$

The initial condition $T_{x(0)}^0$ is taken to be the unit matrix and represents the m-independent directions of the initial separation of the peaks. Eckmann and Ruelle proceed by obtaining first an effective discrete time dynamical system. The time step τ of the discrete system has to be chosen with care. The contribution of a Lyapunov exponent λ to $T_{x(0)}$ is of the form $e^{\lambda\tau}$ and for this not to vary too much for λ of different magnitudes it is necessary that τ is small. However, in order to have Lyapunov exponents averaged over the attractor it is necessary to iterate a number (n) of times in the discrete system. n is of the order of τ^{-1}; $T_{x(0)}^n$ is a product of n matrices. For reasons of computation time n should not be too large and correspondingly τ too small. We have found that τ of the order of one is satisfactory and defined the discrete stochastic map f by

$$\tilde{f}(x)=f^\tau(x) \tag{A7}$$

Clearly the characteristic exponents of $\tilde{f}$ are τ times the corresponding ones for f.

Now we have

$$T_x^{n\tau}=T^\tau(f^{n-1}(x))\cdots T^\tau(f(x))T^\tau(x) \ . \tag{A8}$$

For convenience of notation we shall suppress the τ dependence in the above from now on. The matrix $T(x)$ can be "QR decomposed" into a product of an orthogonal matrix (Q_1) and an upper triangular one (R_1) with non-negative diagonal elements. We define

$$T_2'=T(\tilde{f}(x))Q_1 \tag{A9}$$

and QR decompose T_2',

$$T_2'=Q_2R_2 \ . \tag{A10}$$

In this way

$$T_x^n=Q_nR_n\cdots R_1=Q_nR \ . \tag{A11}$$

It can be shown[10] that the diagonal elements of R (which are just the products of the corresponding elements of the R_j) satisfy

$$\lim_{n\to\infty}\left[\frac{1}{n}\ln R_{ii}\right]=\lambda_i \ ,$$

where λ_i is a Lyapunov exponent of $\tilde{f}$. There are readily available routines which can effect a QR decomposition and the method outlined above has been found to be numerically satisfactory.

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