Stability of a rotating relativistic electron beam in a waveguide near the cyclotron resonance

S. J. Han
Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545
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The stability of a thin electron beam in a uniform magnetic field is studied. It is found that if the conducting wall is close to the beam, the cyclotron-maser action occurs in a fast wave structure. For given geometrical parameters, there is a critical current density below which the beam is stable to perturbations.

The concept of a cyclotron maser was introduced in 1959 by Schneider[1] and first realized experimentally by Hirshfield and Wachtel.[2] Because of its simplicity and its often surprising experimental demonstrations,[3] the cyclotron maser has become one of the most widely studied and best understood devices for intense microwave generation, both in linear[4] and nonlinear[5] regimes. However, many questions in the rigorous analysis of the cyclotron-maser theory remain unanswered and our aim is to address one of them.

The cyclotron-maser action arises from a relativistic effect (i.e., $\omega_c = eB_0/\gamma mc$) and is described as follows: we start with relativistic electrons gyrating in a uniform magnetic field with random initial phases and zero drift velocity in the direction of the magnetic field. Those electrons which are accelerated by the rf wave become heavier and rotate more slowly; similarly those deaccelerated by the rf wave rotate more rapidly. The net result is phase-space bunching which brings about the coherent generation of radiation.[6] It is generally accepted[4] that the azimuthal phase-space bunching is the principal mechanism for the coherent radiation emission by electrons in a magnetic field. This has been demonstrated for a beam so tenuous that space charge can be neglected.[6]

We propose here to investigate, as a viable model for the instability of a rotating electron beam in a uniform magnetic field, the thin electron beam with emphasis on the collective instability of the beam. Although the present study is applicable to cyclotron-maser theory, it can be used for similar problems associated with microwave generation and, thus, is of general interest.

The model we study assumes the electron beam to be a compressible fluid, so that charge bunching can be incorporated in the analysis. Moreover, from the fluid equation, it is possible to determine the precise expression for the density variation with respect to perturbations. We have chosen to study a thin electron beam model because, by employing a Lagrangian description of the perturbations, boundary conditions at the moving interface can be treated exactly. A similar model in the Lagrangian variables was studied by Sprangle;[7] however, his model was characterized by the absence of the fluid equation. Our result complements much recent work[8] on cyclotron-maser theory based on the relativistic Vlasov equation.

The fundamental physical process by which the cyclotron maser operates is surprisingly easy to comprehend in this model; indeed, the essential physics will be described by only two equations which are presented in this paper.

For simplicity, we consider the beam surrounded by perfectly conducting cylinders of radii a_+ and a_- which are aligned with the external magnetic field $\mathbf{B}_{\text{ext}} = B_0\hat{\mathbf{e}}_z$. In a conventional cyclotron-maser device the inner conducting wall is absent; however, the effect of this wall on the beam stability is negligible.

The basic equations we study are

$$\frac{d}{dt}(\gamma\mathbf{v}) = -\frac{e}{m}\left[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right], \tag{1}$$

$$\frac{d}{dt}(\gamma) = -\frac{e}{mc^2}(\mathbf{E}\cdot\mathbf{v}), \tag{2}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{3}$$

where the notations are standard, together with Maxwell's equations.

If we now introduce perturbations in Lagrangian variables in the form $\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\xi}(\mathbf{r}_0, t)$, where $\mathbf{r}_0$ describes the unperturbed trajectory of the fluid element at time t and ξ is its displacement, then the perturbed density to first order[9] becomes $\rho(\mathbf{r}_0+\boldsymbol{\xi}) = \rho(\mathbf{r}_0)(1-\nabla\cdot\boldsymbol{\xi})$. For a thin beam, this can be integrated along the radial direction to give

$$\sigma_0(r_b)(1-\nabla\cdot\boldsymbol{\xi}) = \sigma_0 + \sigma_1, \tag{4}$$

where the ξ_j are assumed to be independent of r. Here r_b is the average radius of the beam.

To obtain the equations of motion and boundary conditions, the following additional relations are necessary:[9]

$$\nabla = \nabla_0 - \nabla_0\boldsymbol{\xi}\cdot\nabla_0, \tag{5}$$

$$\mathbf{v}(\mathbf{r}_0+\boldsymbol{\xi}) = \mathbf{v}_0(\mathbf{r}_0) + \mathbf{v}_0\cdot\nabla_0\boldsymbol{\xi} + \frac{\partial\boldsymbol{\xi}}{\partial t}, \tag{6}$$

$$\mathbf{n}(\mathbf{r}_0+\boldsymbol{\xi}) = \mathbf{n}(\mathbf{r}_0) + \mathbf{n}\mathbf{n}\cdot\nabla_0\boldsymbol{\xi}\cdot\mathbf{n} - \nabla_0\boldsymbol{\xi}\cdot\mathbf{n}, \tag{7}$$

$$\gamma = \gamma_0 + \gamma_1, \tag{8}$$

where $\gamma = (1-\beta^2)^{-1/2}$, $\gamma_1 \ll \gamma_0$, and ∇ and ∇_0 are the differential operators with respect to $\mathbf{r}$ and $\mathbf{r}_0$.

We may now linearize the equation of motion. The unperturbed motion is given as

$$(\mathbf{v}_0\cdot\nabla_0)\mathbf{v}_0=-\frac{e}{\gamma_0 m}\left[\mathbf{E}_0^{\rm ext}+\frac{\mathbf{v}_0}{c}\times\mathbf{B}_0^{\rm ext}\right], \tag{9}$$

and the first-order equation reads

$$\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}+2\mathbf{v}_0\cdot\nabla_0\frac{\partial\boldsymbol{\xi}}{\partial t}+(\mathbf{v}_0\cdot\nabla_0)(\mathbf{v}_0\cdot\nabla_0)\boldsymbol{\xi}$$

$$=-\frac{e}{\gamma_0 m}\left[\mathbf{E}+\frac{\mathbf{v}_0}{c}\times\mathbf{B}+\frac{1}{c}\left[\mathbf{v}_0\cdot\nabla_0\boldsymbol{\xi}+\frac{\partial\boldsymbol{\xi}}{\partial t}\right]\times\mathbf{B}_0^{\rm ext}\right]$$

$$+\frac{e}{m\gamma_0^2}\left[\mathbf{E}_0^{\rm ext}+\frac{\mathbf{v}_0}{c}\times\mathbf{B}_0^{\rm ext}\right]\gamma_1\ . \tag{10}$$

where $\gamma_0=(1-\beta_0^2)^{-1/2}$ and $\beta_0=v_0/c$. Here $\mathbf{E}$ and $\mathbf{B}$ are the first-order electromagnetic fields. Notice that we have neglected the self-fields, which can be justified for a thin beam.

Next, if we set $\mathbf{E}_0^{\rm ext}=0$, then Eq. (9) gives $\mathbf{v}_0=r_b\omega_c\hat{e}_\vartheta$, where r_b is the average beam radius and $\omega_c=eB_0/\gamma_0 mc$. The small perturbations can all be taken as $f_j e^{i(\omega t-l\vartheta-kz)}$ with ω to be determined and $\boldsymbol{\xi}=(\xi_r,\xi_\vartheta,\xi_z)$.

If we now consider the TE mode, which couples strongly with the beam, Eq. (2) gives the first-order relativistic correction term as

$$\gamma_1=i\left[\frac{e}{mc^2}\right]\frac{v_0E_\theta}{\omega-l\omega_c}\ . \tag{11}$$

Equations (4) and (11) together explain the essence of the cyclotron-maser action.

Substitution of the assumed form for the perturbed terms yields the results

$$(\omega-l\omega_c)^2\xi_r+i\omega_c(\omega-l\omega_c)\xi_\vartheta$$

$$=\frac{e}{\gamma_0 m}(E_r+\beta_0B_z)-i\frac{e\omega_c}{\gamma_0 m}\beta_0^2\frac{E_\vartheta}{\omega-l\omega_c}\ , \tag{12}$$

$$-i\omega_c(\omega-l\omega_c)\xi_r+(\omega-l\omega_c)^2\xi_\vartheta=\frac{e}{\gamma_0^3 m}E_\vartheta\ , \tag{13}$$

$$(\omega-l\omega_c)^2\xi_z=\frac{e}{\gamma_0 m}(E_z-\beta_0B_r)\ , \tag{14}$$

where the rf fields are all in the first order.

The boundary conditions that we adopt at the interface[9,10] are

$$\mathbf{n}\times(\mathbf{B}^+-\mathbf{B}^-)+(\mathbf{n}\cdot\boldsymbol{\beta})(\mathbf{D}^+-\mathbf{D}^-)=\frac{4\pi}{c}\mathbf{K}\ ,$$

$$\mathbf{n}\cdot(\mathbf{B}^+-\mathbf{B}^-)=0 \tag{15}$$

$$\mathbf{n}\times(\mathbf{E}^+-\mathbf{E}^-)=0,\ \ \mathbf{n}\cdot(\mathbf{E}^+-\mathbf{E}^-)=4\pi\sigma \tag{16}$$

where $\mathbf{K}=\sigma_0(1-\nabla_0\cdot\boldsymbol{\xi})\mathbf{v}(\mathbf{r}_0+\boldsymbol{\xi})$.

It is to be noted that the surface current $\mathbf{K}$ conserves charge, i.e.,

$$\frac{\partial\sigma}{\partial t}+(\nabla_0-\nabla_0\boldsymbol{\xi}\cdot\nabla_0)\cdot\mathbf{K}=0\ , \tag{17}$$

where $\sigma=\sigma_0(1-\nabla_0\cdot\boldsymbol{\xi})$.

In what follows we consider only the long-wavelength limit $k=0$ in the z direction and defer the more general case with $v_\perp/v_\parallel\gg1$ to a future study. The jump conditions for the TE mode then yield

$$\hat{E}_r^--\hat{E}_r^+=-4\pi i\sigma_0k_\theta\xi_\theta,\ \ \hat{E}_\vartheta^--\hat{E}_\vartheta^+=-4\pi i\sigma_0k_\vartheta\xi_r\ ,$$

$$\hat{E}_z^--\hat{E}_z^+=0,\ \ \hat{B}_\vartheta^--\hat{B}_\vartheta^+=-\frac{4\pi i}{c}\sigma_0(\omega-l\omega_c)\xi_z\ , \tag{18}$$

$$\hat{B}_z^--\hat{B}_z^+=\frac{4\pi\sigma_0}{c}(i\omega\xi_\vartheta+\omega_c\xi_r)\ ,$$

where $k_\vartheta=l/r_b$ and the hat in the rf fields implies the first order.

It is to be noted that in the limit $k=0$, Eq. (14) becomes redundant; Eqs. (12) and (13) completely describe the onset of instability.

In order to obtain a dispersion relation, the rf fields in Eqs. (12) and (13) must be expressed in terms of ξ_r and ξ_ϑ. The first step in this procedure is to replace the rf fields with the mean fields $E_r=\frac{1}{2}(\hat{E}_r^++\hat{E}_r^-)$ and $E_\vartheta=\frac{1}{2}(\hat{E}_\vartheta^++\hat{E}_\vartheta^-)$.

If we now introduce the wave admittance $b_\pm=\mp(ik_\vartheta c/\omega)B_z^\pm/E_\vartheta^\pm$ for the TE mode,[11] then the mean fields take the form

$$E_r=-\frac{4\pi\sigma_0k_\vartheta}{b_++b_-}\left[\tfrac{1}{2}(b_+-b_-)\left[\frac{\omega_c}{\omega}\xi_r+i\xi_\vartheta\right]+b_-b_+\xi_r\right], \tag{19}$$

$$E_\theta=\frac{4\pi i\sigma_0k_\vartheta}{b_++b_-}\left[\left[\frac{\omega_c}{\omega}\xi_r+i\xi_\vartheta\right]-\tfrac{1}{2}(b_+-b_-)\xi_r\right]. \tag{20}$$

It is now possible to derive a dispersion relation that determines the onset of instability. First, substitution of Eqs. (19) and (20) into Eqs. (12) and (13), the use of the relation $E_r+\beta_0B_z=[1+\beta_0(\omega/k_\vartheta c)]E_r$, which was derived by using Faraday's law for the free-space waves, leads to the dispersion relation. In practice, the dispersion relation involves higher transcendental functions of ω for a given mode number and other physical quantities. However, one important simplification is possible, for Eq. (11) suggests that the radiation emission by an electron in a magnetic field is at its peak near the cyclotron resonance $\omega=l\omega_c+i\epsilon$, so that the normal modes associated with the eigenvalues computed in some small region around the resonance will display an exponential growth for the wave in resonance. Therefore, the wave admittance functions $b_\pm$ can be approximated by evaluating them at $\omega=l\omega_c$. Second, if this frequency is written as $\omega=l\omega_c+\delta\omega_l$ with $\delta\omega_l$ small, the dispersion relation reduces to a polynomial in $\delta\omega_l$ of order seven, which can be easily seen by inspection of Eqs. (12) and (13).

The detailed expressions of the coefficients of the polynomial are complicated functions of k_ϑ and other physical parameters. Thus the determination of eigenvalues presents a difficult problem that must be treated numerically. It is, however, to be noted that in spite of their complex expressions, the coefficients are all real and hence the roots will be either real or will occur in complex conjugate pairs. This serves to check the numerical accuracy of the solutions to the polynomial.

For given physical quantities (see the legend in Fig. 2), numerical studies show that the polynomial possesses, in addition to three real roots, two pairs of complex conjugate roots depending on the mode number. This conclusion is borne out by a detailed numerical study in which geometrical and other parameters are varied to check the numerical stability. Full details will be presented elsewhere,[12] and we summarize here certain features of the results. First, the instability invariably occurs for long-wavelength perturbations when the conducting walls are close to the beam. In particular, there are two fast growing waves. Each wave has its own domain of the mode number for which the instability occurs. One is stable to long-wavelength perturbations (up to $l=11$) but is unstable to short-wavelength perturbations (see Fig. 1). The other depends, in a complex way, on the wave admittance (see Fig. 2). Second, in addition to the above fast growing waves, there are two slow waves; one of which has a negative phase velocity.

We may now understand what physical mechanism drives these instabilities. From Eq. (11), it is not difficult to see that the fast wave, which is stable to long-wavelength perturbations, arises from the relativistic effect (i.e., the cyclotron-maser action). This follows from the fact that if a microwave is emitted by the electrons through the cyclotron resonance interaction, the frequency of the wave is determined by $\omega \simeq l\omega_c$. Hence, for a given magnetic field, the interaction at higher cyclotron harmonics is necessary to generate microwaves. This has been tested by varying the external magnetic field strength. We notice that for given geometrical parameters the maximum growth rate is found near the mode number $l=28$ for which $b_+ + b_- \simeq 0$. One can see from Eqs. (19) and (20) that the maximum of the mean rf fields passes through the peak position $l=28$; these rf fields are the prime driving force for the space-charge bunching. The

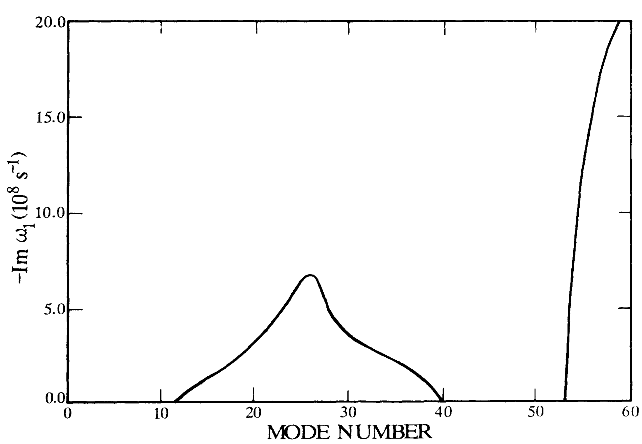

FIG. 1. The growth rate for the cyclotron modes is plotted against the mode number l for given geometrical and other physical parameters (see the legend in Fig. 2).

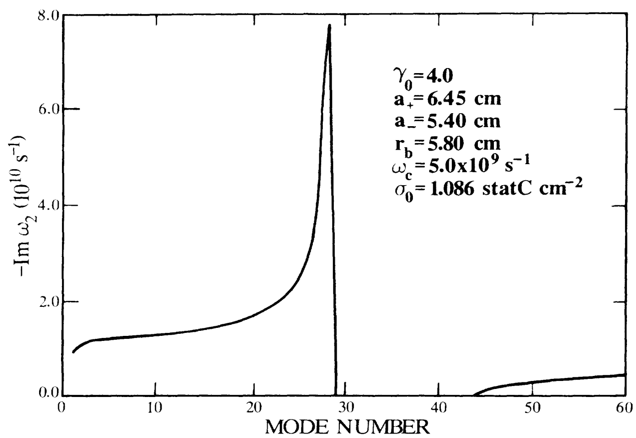

FIG. 2. The growth rate for the fast wave is plotted against the mode number.

other fast growing wave arises from a strong coupling of a fast wave to the slow wave with negative phase velocity, which is well known.[7] The slow wave, which is a space-charge wave, can modify the TE mode supported by the waveguide. This explains why this fast wave depends strongly on the wave admittance. Here again the maximum growth rate is found near the mode number $l=28$. It should be noted that in the short-wavelength limit ($l>40$) the instability can be damped out by viscosity or other dissipative processes. Thus the instability in this regime may not be as important as the one in the long-wavelength perturbations.

One may further ask whether the two waves can be distinguished in experiments. The answer is *no*, because their phase velocities become almost identical in the wavelength regime in which the two instabilities occur simultaneously.

Most interesting is the critical charge density, $\sigma_0 = 1.7\times 10^{-2}$ statC/cm^2, below which the beam is stable. Perhaps, below the critical charge density, the fluid density is too low to support the azimuthal charge bunching, which seems plausible but has not been proved. Finally, we note that if one moves the outer conducting wall to a sufficient distance (12.5 cm), the beam becomes stable to long-wavelength perturbations (up to $l=7$), which is in qualitative agreement with the analysis by Levy.[13]

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$$b_{\pm} = \mp \left(\frac{k_\theta c}{\omega} \right) \frac{N_l'(pa_\pm)J_l(pr) - J_l'(pa_\pm)N_l(pr)}{N_l'(pa_\pm)J_l'(pr) - J_l'(pa_\pm)N_l'(pr)} ,$$

where the prime denotes the differentiation with respect to the argument and $p = l\omega_c/c$.

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