Analytical study of the equation for the longitudinal motion of particles in a radio-frequency-quadrupole accelerator

M. Leo, R. A. Leo, and G. Soliani
Dipartimento di Fisica dell'Università degli Studi di Lecce, I-73100 Lecce, Italy and Gruppo di Lecce, Sezione di Bari, Istituto Nazionale di Fisica Nucleare, I-73100 Lecce, Italy

M. Puglisi and C. Rossi
Dipartimento di Fisica Nucleare e Teorica dell'Università degli Studi di Pavia, I-27100 Pavia, Italy and Sezione di Pavia, Istituto Nazionale di Fisica Nucleare, I-27100 Pavia, Italy

G. Torelli
Dipartimento di Fisica dell'Università degli Studi di Pisa, I-56100 Pisa, Italy and Laboratorio di San Piero, Sezione di Pisa, Istituto Nazionale di Fisica Nucleare, I-56100 Pisa, Italy
(Received 16 June 1986)

We develop a procedure based on the averaging method of Bogoliubov, Krylov, and Mitropolsky to obtain analytical approximate solutions of the equation for the longitudinal motion of a particle in a radio-frequency-quadrupole (RFQ) accelerator under the Kapchinskii-Teplyakov assumption. Our analytical results, which fairly agree (within a few per thousand) with those coming from the numerical integration of the equation under consideration, are exploited in the case of an RFQ device with $N=150$ cells in which the accelerated particles are protons and for values of the efficiency A from 0.05 to 0.5. Starting from an injection energy of 10 keV, a final energy of about 1.5 MeV is achieved. Our procedure may be applied to other fundamental problems arising in the project of an RFQ accelerator, such as the study of the equation for transversal oscillations.

I. INTRODUCTION

One of the crucial points in the treatment of a linear accelerator structure is the problem of conciliating longitudinal and radial stability of the particle beam.[1,2] An important idea in this direction is due to Kapchinskii and Teplyakov (KT), who proposed in 1970 a radio frequency quadrupole (RFQ) device where the rf field can be used both for acceleration and transverse focusing.[3,4] Among the notable functions provided by an RFQ, we mention the possibility of accepting a low-velocity beam and accelerating it to an energy suitable for injection into a drift-tube linac.[1,5] A schematic drawing of an RFQ four vanes resonator is shown in Fig. 1.

As far as we know, so far the equation of motion governing the beam dynamics of a particle in the KT framework has been investigated by means of numerical techniques only. In this work we resort to an analytical procedure to obtain an explicit approximate solution of the longitudinal motion coming from the KT expansion. Our calculation is carried out in absence of space charge contributions. Our approach, which is based on the method of averaging developed by Bogoliubov, Krylov, and Mitropolsky[6,7] (BKM), may also be applied, in principle, to the study of the transversal oscillations of the particles.

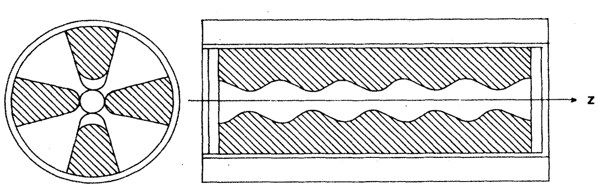

FIG. 1. Schematic (front and side) view of an RFQ resonator.

In Sec. II we write the equation for the longitudinal motion of a nonrelativistic particle starting from the quasistationary KT expansion at the lowest order. In Sec. III we exploit the averaging method to find an approximate solution of the equation reported in Sec. II. Section IV contains a discussion on the comparison between analytical and numerical results and some concluding remarks. Finally, in the Appendix details of calculation are presented.

II. EQUATION FOR THE LONGITUDINAL MOTION

Let us consider the scheme of Fig. 1 and assume a cylindrical coordinate frame. Then in the quasistationary approximation the electric potential for a four vanes resonator with quadrupole symmetry is[3,4]

$$U(r,\psi,z,t)=U_0(r,\psi,z)\sin(\omega t+\phi)\ , \tag{2.1}$$

where $\omega=2\pi/T$, T is the period of the rf and ϕ is the phase of the injected particle.

The assumption underlying (2.1) is verified whenever the free-space wavelength λ of the rf is large in comparison with the radial dimension of the beam.

The static part of the potential (2.1) obeys the Laplace equation and is given by[5]

$$U_0(r,\psi,z)=\sum_s A_s r^{2(2s+1)}\cos[2(2s+1)\psi] + \sum_n \left[\sum_s B_{ns} I_{2s}(nkr)\cos(2s\psi)\right]\cos(nkz) , \quad (2.2)$$

where A_s,B_{ns} are constants related to the shape of the machine, I_{2s} is the modified Bessel function of order $2s$, $k=2\pi/\beta\lambda$ is the wave number, where λ and βc are, respectively, the rf wavelength and the speed of the particle.

Substituting (2.2) in (2.1) we obtain the so-called KT expansion which reads

$$U=[A_0 r^2\cos(2\psi)+B_{10}I_0(kr)\cos(kz)]\sin(\omega t+\phi) , \quad (2.3)$$

at lowest order. A more convenient notation for (2.3) is

$$U=(V/2)[X(r/a)^2\cos(2\psi) + AI_0(kr)\cos(kz)]\sin(\omega t+\phi) , \quad (2.4)$$

where V is the difference potential between adjacent pole tips,

$$X=1-AI_0(ka) , \quad (2.5)$$

$$A=(\mu^2-1)/[\mu^2 I_0(ka)+I_0(\mu ka)] , \quad (2.6)$$

and a and μ are the parameters characterizing each cell (see Fig. 2).

The longitudinal component of the electric field is given by

$$E=-\frac{\partial U}{\partial z}=(kAV/2)I_0(kr)\sin(kz)\sin(\omega t+\phi) , \quad (2.7)$$

where the quantity AV is the potential difference between the beginning and the end of a unit cell.

In the case of nonrelativistic particles, from (2.7) the equation of motion

$$\frac{d^2z}{dt^2}=C\sin(kz)\sin(\omega t+\phi) \quad (2.8)$$

arises, where

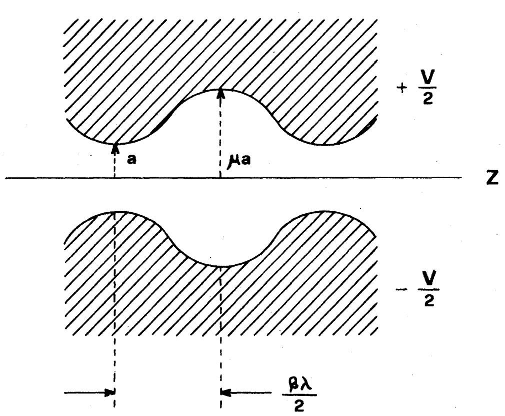

FIG. 2. Geometrical picture of a unit cell. The role of the parameters a and μ is shown, together with the voltage configuration at a given time t_0.

$$C=(qVAk)/(2M) , \quad (2.9)$$

and q,M are the charge and the rest mass of the particle, respectively.

We observe that the wave number k is connected with the velocity of the particle and is a characteristic of each cell. Furthermore, from the synchronism condition between the particle and the rf field, it follows that the integration of Eq. (2.8) for each value of k yields the cell length z in such a way that $kz=\pi$, which is just the half period of the rf field itself.

III. APPROXIMATE SOLUTIONS

Equation (2.8) is a nonlinear second-order ordinary differential equation with a periodic coefficient, which can be suitably analyzed in our context within the averaging method of BKM (Refs. 6 and 7). This procedure can be applied to build up approximate solutions of equations in the standard form

$$\frac{dx_k}{dt}=\epsilon X_k(t,x_1,x_2,\ldots,x_n) \quad (k=1,2,\ldots,n) , \quad (3.1)$$

in which ϵ is a small parameter and X_k can be expressed by

$$X_k(t,x_1,x_2,\ldots,x_n)=\sum_\nu e^{i\nu t}X_{k\nu}(x_1,x_2,\ldots,x_n) , \quad (3.2)$$

where ν are constant frequencies.

A brief formulation of the principle of averaging is reported in the Appendix for the reader's convenience.

In order to put Eq. (2.8) in the form (3.1), let us perform the change of variables

$$\tau=\omega t, \quad \eta=kz-\tau . \quad (3.3)$$

Then Eq. (2.8) becomes

$$\frac{d^2\eta}{d\tau^2}=\frac{\epsilon^2}{2}[\cos(\eta-\phi)-\cos(\eta+2\tau+\phi)] , \quad (3.4)$$

where the (adimensional) parameter ϵ is defined by

$$\epsilon=(kC/\omega^2)^{1/2} , \quad (3.5)$$

which is required to be less than one in order that the BKM method may be applied.

Using hereafter a vectorial notation, Eq. (3.4) takes the form

$$\frac{dX}{d\tau}=\epsilon F(\tau,X) , \quad (3.6)$$

where X and F are the vectors

$$X=(X_1,X_2)^T , \quad (3.7)$$

and

$$F(\tau,X)=(X_2,\tfrac{1}{2}[\cos(X_1-\phi)-\cos(X_1+2\tau+\phi)])^T , \quad (3.8)$$

with

$$X_1=\eta, \quad X_2=\frac{1}{\epsilon}\frac{dX_1}{d\tau} , \quad (3.9)$$

where T denotes the transposed operation. As explicit function of τ, F is of period π.

According to the BKM method, now we make the averaging with respect to the variable τ over the interval $(0,\pi)$. We denote by

$$\xi=(\xi_1,\xi_2)^T \tag{3.10}$$

the solution of the resulting equation, namely

$$\frac{d\xi}{d\tau}=\epsilon F_0(\xi) , \tag{3.11}$$

where

$$F_0(\xi)=(\xi_2,\tfrac{1}{2}\cos(\xi_1-\phi))^T . \tag{3.12}$$

Starting from the initial conditions

$$X_1(0)=\xi_1(0), \quad X_2(0)=\xi_2(0) , \tag{3.13}$$

the function ξ can be considered as the first approximation solution of Eq. (3.6). Solving (3.11), we obtain

$$\xi_1=2\arcsin\left[m\ \mathrm{sn}\left[\frac{\epsilon}{\sqrt{2}}\tau+\alpha;m^2\right]\right]+\phi+\frac{\pi}{2} , \tag{3.14}$$

where sn stands for a Jacobi elliptic function of parameter m, and

$$m^2=\sin^2\frac{1}{2}\left[\phi+\frac{\pi}{2}\right]+\frac{1}{2\epsilon^2}\left[k\frac{v_0}{\omega}-1\right]^2 , \tag{3.15}$$

$$\alpha=-\mathrm{sn}^{-1}\left[\frac{1}{m}\sin\frac{1}{2}\left[\phi+\frac{\pi}{2}\right];m^2\right] , \tag{3.16}$$

being v_0 the initial velocity of the particle.

An improved approximate solution of Eq. (3.6) can be written as

$$X=\xi+\sum_n \xi^n \mathcal{X}_n(\tau,\xi) , \tag{3.17}$$

where the vectors $\mathcal{X}_n$ have to be determined in such a way that Eq. (3.6) is satisfied at any order in the parameter ϵ, with the initial conditions $\mathcal{X}_n(0,\xi)=0$.

As shown in the Appendix, we derived the following higher-order contributions:

$$\mathcal{X}_1=(0,-\tfrac{1}{4}\sin(\xi_1+2\tau+\phi)+\tfrac{1}{4}\sin(\xi_1+\phi))^T , \tag{3.18}$$

$$\mathcal{X}_2=(\tfrac{1}{8}\cos(\xi_1+2\tau+\phi)-\tfrac{1}{8}\cos(\xi_1+\phi)+\tfrac{1}{4}\tau\sin(\xi_1+\phi) ,(\xi_2/4)[\tfrac{1}{2}\sin(\xi_1+2\tau+\phi)-\tfrac{1}{2}\sin(\xi_1+\phi)-\tau\cos(\xi_1+\phi)])^T , \tag{3.19}$$

$$\begin{aligned}\mathcal{X}_3=(&(\xi_2/8)[-\cos(\xi_1+2\tau+\phi)+\cos(\xi_1+\phi)-2\tau\sin(\xi_1+\phi)-2\tau^2\cos(\xi_1+\phi)] ,\\ &\tfrac{1}{8}\{\tfrac{1}{16}\cos[2(\xi_1+\phi)]-\tfrac{1}{16}\cos[2(\xi_1+2\tau+\phi)]+\tfrac{1}{4}\cos[2(\xi_1+\tau)]-\tfrac{1}{4}\cos(2\xi_1)\\ &+\tfrac{1}{4}\cos(\xi_1+\phi)\cos(\xi_1+2\tau+\phi)-\tfrac{1}{4}\cos^2(\xi_1+\phi)+(\tau/2)\sin(2\xi_1)\\ &+\sin(\xi_1+\phi)\cdot[-(\tau/2)\cos(\xi_1+2\tau+\phi)+\tfrac{1}{4}\sin(\xi_1+2\tau+\phi)-\tfrac{1}{4}\sin(\xi_1+\phi)]\\ &+(\tau^2/2)\cos(2\xi_1)-\xi_2^2[-\tfrac{1}{2}\sin(\xi_1+\phi)+\tfrac{1}{2}\sin(\xi_1+2\tau+\phi)-\tau\cos(\xi_1+\phi)+\tau^2\sin(\xi_1+\phi)]\})^T .\end{aligned} \tag{3.20}$$

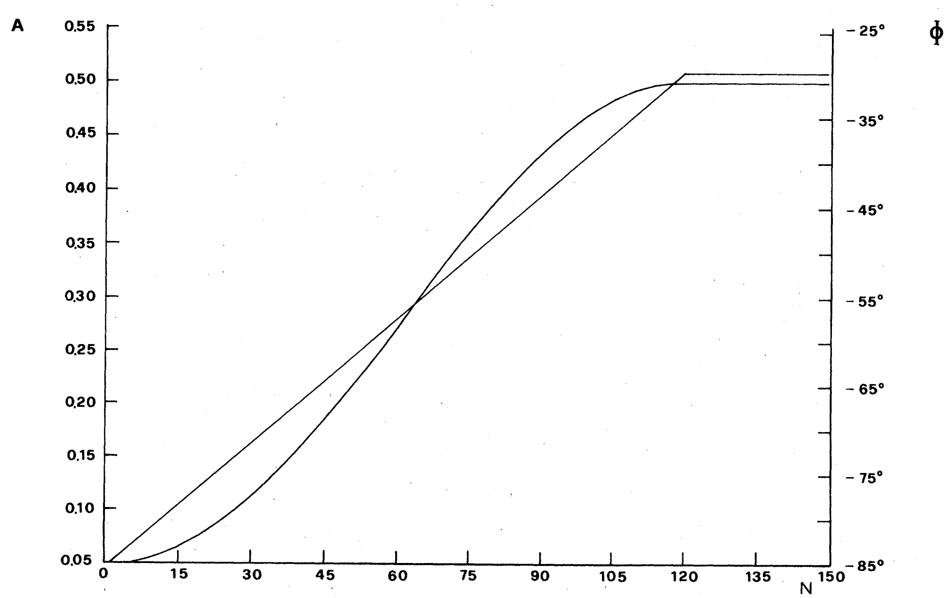

FIG. 3. Behavior of the parameters A and ϕ for the whole machine (corresponding to a number of cells $N=150$).

Inserting (3.18), (3.19), and (3.20) in (3.17), we get a third-order-approximate solution of Eq. (3.6).

IV. DISCUSSION

The third-order analytical approximate solution derived in Sec. III has been tested in the case of an RFQ device with $N=150$ cells in which we have taken protons as particles to be accelerated. The behavior of the efficiency A and that of the phase ϕ have been chosen as follows (see Fig. 3):

$$A=0.05\left\{1+\frac{9}{2}\left[1+\cos\pi\left[1+\frac{N-1}{119}\right]\right]\right\},$$

$$\phi=-85^\circ+(N-1)\frac{55^\circ}{119},$$

for $1\leq N\leq 120$, and

$$A=0.5,\quad \phi=-30^\circ,$$

for $120\leq N\leq 150$.

Starting from an injection energy of 10 KeV, a final energy of about 1.5 MeV is achieved. The length of the whole machine turns out to be of 3.51 m.

As one expects, since the particle velocity (in the synchronism condition) is growing, then the value of the wave number tends to decrease accordingly. This is shown in Figs. 4 and 5, respectively, where the analytical values of the proton energy and the wave number are quoted in terms of the number of cells.

The solution found in Sec. III has been compared with that coming from the numerical integration of Eq. (2.8). The last has been integrated by means of a fourth-order Runge-Kutta predictor-corrector method.[8] We point out that the values of the physical quantities (proton energy, wave number, etc.) calculated analytically agree with the corresponding ones determined via the numerical integration within a few per thousand. Anyway, we observe that in the case in which only the lowest-order term of the expansion (3.17) is taken into account, the analytical results turn out to be rather close to the numerical ones (within 1%). Of course, this fact might be of help to accelerator designers, in order to obtain some useful indication for a preliminary study of an RFQ device.

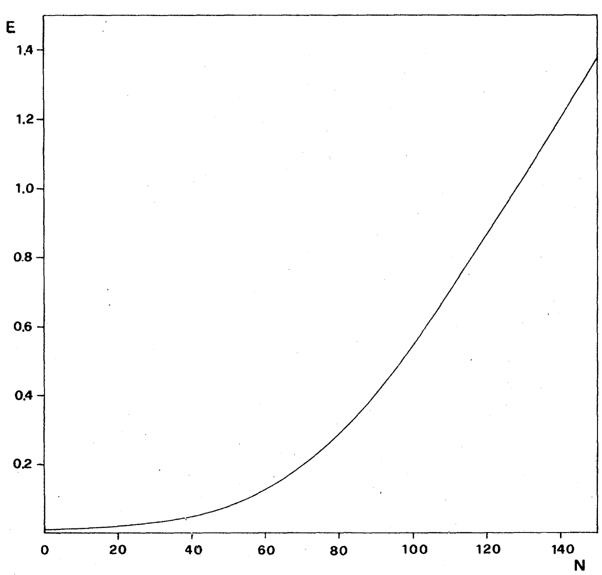

FIG. 4. Energy E in MeV of the particle obtained analytically vs the number of cells N.

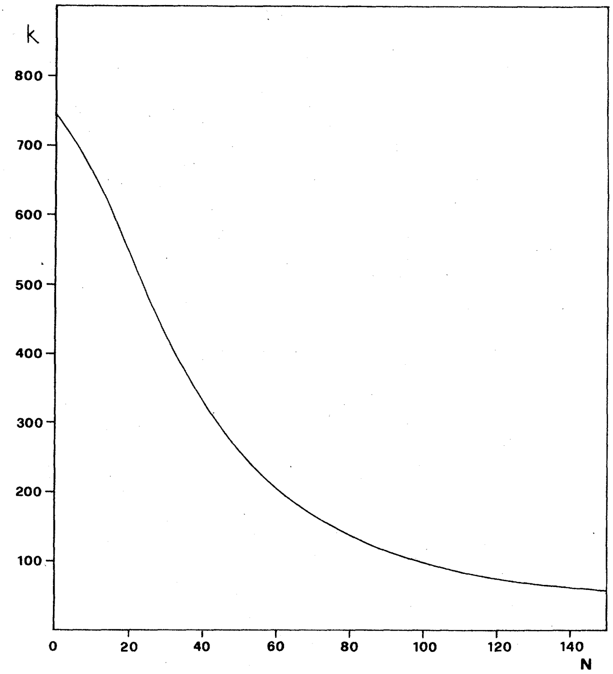

FIG. 5. Plot of the analytical value of the wave number k vs the number of cells N.

It is also important to note that ϵ remains less than one along the whole accelerator. Moreover, since in a nonrelativistic approximation we have $\epsilon=0.5(AV/V_0)^{1/2}$, where V_0 is the injection voltage in a cell (eventually the first), it follows that the accuracy of the solution does not depend on the mass of the accelerated particle.

Both for checking the validity of the analytical solution and its computer formulation, many solutions have been tested with the usual algebraic test where each cell is divided into a suitable number (40) of intervals and the acceleration in each interval is held constant.

Clearly, an analytical solution offers the possibility of optimizing an RFQ structure in a reasonably short computation time. Finally, we remark that the procedure developed in Sec. III may be employed in principle to tackle other fundamental problems arising in the project of an RFQ machine. One of these is concerned with the study of the equation for the transversal oscillations.

APPENDIX

Here we report a brief introduction on the method of averaging (see Ref. 7 for a deep analysis of the mathematical foundations), and sketch our procedure for deriving the approximate solutions (3.18), (3.19), and (3.20).

In doing so, let us consider the system of equations

$$\frac{dx}{dt}=\epsilon X(t,x) , \tag{A1}$$

where x,X are n-component vectors, ϵ is a small parameter, and each component of X is of the type (3.2).

Let us build up the averaged equation

$$\frac{d\xi}{dt}=\epsilon X_0(\xi) , \tag{A2}$$

where $X_0(\xi)$ is expressed by

$$X_0(\xi)=\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T X(t,\xi)dt\right] . \tag{A3}$$

The crucial point of the BKM method is that under very general conditions (see Theorem following), the difference $x(t)-\xi(t)$ may be made as small as desired. In fact, the following Theorem holds (see Ref. 7, Chap. 6, for the proof):

Theorem. Let us suppose that $X(t,x)$ satisfies the conditions:

(i) for some domain D two positive constants, say M and λ, exist such that for all values of $t\geq 0$ and for any points x,x',x'' of D, the inequalities

$$|X(t,x)|\leq M;\quad |X(t,x')-X(t,x'')|\leq\lambda|x'-x''| \tag{A4}$$

are fulfilled;

(ii) for all x in D the limit

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T X(t,x)dt\right]=X_0(x) \tag{A5}$$

exists. Thus, corresponding to any positive ρ and η as small as desired and for as large a value of L as wanted, one may find a positive ϵ_0 such that if $\xi=\xi(t)$ is a solution of the equation

$$\frac{d\xi}{dt}=\epsilon X_0(\xi) ,$$

defined in the interval $0<t<\infty$ and lying in the domain D along with its entire ρ neighborhood, then, for $0<\epsilon<\epsilon_0$, in the interval $0<t<L/\epsilon$ the inequality

$$|x(t)-\xi(t)|<\eta \tag{A6}$$

is valid. In (A6), $x=x(t)$ represents the solution of the equation

$$\frac{dx}{dt}=\epsilon X(t,x) , \tag{A7}$$

which coincides with $\xi(t)$ at $t=0$. (A ρ neighborhood of a set A means the set of all points whose distances from A are less than ρ.)

In our case, where Eq. (3.6) is under consideration, $F(\tau,x)$ is a periodic function of period π. We have, therefore,

$$F_0(\xi)=\frac{1}{\pi}\int_0^\pi F(\tau,\xi)d\tau . \tag{A8}$$

On the other hand, the difference $x(\tau)-\xi(\tau)$ is given by (3.17). We have limited ourselves to deal with the expansion

$$x(\tau)=\xi+\epsilon\chi_1(\tau,\xi)+\epsilon^2\chi_2(\tau,\xi)+\epsilon^3\chi_3(\tau,\xi) . \tag{A9}$$

Introducing (A9) in (3.6), expanding the function $F(\tau,x)$ in a Taylor's series in the parameter ϵ and equating the coefficients of the corresponding powers of ϵ in both sides of (3.6), we are led to the following relations:

$$\frac{\partial\chi_1(\tau,\xi)}{\partial\tau}=F(\tau,\xi)-F_0(\xi) , \tag{A10a}$$

$$\frac{\partial\chi_2(\tau,\xi)}{\partial\tau}=\left[\chi_1^\alpha\frac{\partial}{\partial\xi^\alpha}\right]F(\tau,\xi)-\left[F_0^\alpha\frac{\partial}{\partial\xi^\alpha}\right]\chi_1(\tau,\xi) , \tag{A10b}$$

$$\frac{\partial\chi_3}{\partial\tau}=\left[\chi_2^\alpha\frac{\partial}{\partial\xi^\alpha}\right]F(\tau,\xi)-\left[F_0^\alpha\frac{\partial}{\partial\xi^\alpha}\right]\chi_2(\tau,\xi)+\frac{1}{2}\left[\chi_1^\alpha\chi_1^\beta\frac{\partial^2}{\partial\xi^\alpha\partial\xi^\beta}\right]F(\tau,\xi) , \tag{A10c}$$

where $\alpha,\beta=1,2$ and the summation convention over repeated indexes is understood.

We recall that

$$\chi_n(0,\xi)=0 . \tag{A11}$$

As a consequence, $\chi_1^{(1)}=0$; furthermore, since $F(\tau,\xi)$ is linear in the variable ξ_2, the third term in (A10c) is vanishing. Finally, the quantities (3.18), (3.19), and (3.20) are obtained respectively by integrating Eqs. (A10a), (A10b), and (A10c) with the initial conditions (A11).

[1]K. R. Crandall, R. H. Stokes, and T. P. Wangler, 1979 Linac Accelerator Conference (unpublished).

[2]R. H. Stokes, K. R. Crandall, J. E. Stovall, and D. A. Swenson, IEEE Trans. Nucl. Sci. **26**, 3469 (1979).

[3]I. M. Kapchinskii and V. A. Teplyakov, Prib. Tekh. Eksp. **119**, 19 (1970).

[4]I. M. Kapchinskii and N. V. Lazarev, IEEE Trans. Nucl. Sci. **26**, 3462 (1979).

[5]For an introduction to the RFQ device see, for example, J. Le Duff, *Dynamics and Acceleration in Linear Structures,* in Proceedings of the CERN Accelerator School (CERN, Geneva, Switzerland, 1985), p. 144; M. Puglisi, Polarized Proton Technical Note No. 28, Brookhaven National Laboratory, 1983 (unpublished).

[6]N. Krylov and N. N. Bogoliubov, *On Nonlinear Mechanics* (Acad. Sci. USSR, 1945).

[7]N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Nonlinear Oscillations* (Gordon and Breach, New York, 1961).

[8]IBM Scientific Subroutine Package No. H20-0205-3, 1969 (unpublished).