Renormalization of period doubling in symmetric four-dimensional volume-preserving maps

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We have determined *three* maps (truncated at quadratic terms) that are fixed under the renormalization operator of pitchfork period doubling in *symmetric* four-dimensional volume-preserving maps. Each of these contains the previously known two-dimensional area-preserving map that is fixed under the period-doubling operator. One of these three fixed maps consists of two *uncoupled* two-dimensional (nonlinear) area-preserving fixed maps. The other two contain also the two-dimensional area-preserving fixed map coupled (in general) with a *linear* two-dimensional map. The renormalization calculation recovers all numerical results for the pitchfork period doubling in the symmetric four-dimensional volume-preserving maps, reported by Mao and Helleman [Phys. Rev. A 35, 1847 (1987)]. For a large class of *nonsymmetric* four-dimensional volume-preserving maps, we found that the fixed maps are the same as those for the symmetric maps.

I. INTRODUCTION

Since the discovery of self-similarity and universality in period doubling, the renormalization group has played a central role in the study of transition to chaos. The fixed map of the period-doubling renormalization operator for one-dimensional dissipative maps,[1] and for two-dimensional (2D) invertible area-preserving maps[2] have been found. However, the fixed map for symmetric *four-dimensional* (4D) volume-preserving maps has not been determined yet, even though a numerical period-doubling study for such maps, reported in Ref. 3, suggests that there exist three such maps.

In this paper, we find (up to quadratic terms) *three* fixed maps for that class of 4D maps. The E map consists of two *uncoupled* 2D area-preserving maps. The L map consists of two coupled (in general; uncoupled only if the map is "symplectic") 2D maps, one of which is the (nonlinear) area-preserving fixed map, and the other being a *linear* map. Finally, the U map is the same as the L map but with different coefficients in the linear part. All scaling properties (such as Feigenbaum constants δ's and α's,β's) numerically found[3] for that class of 4D map are recovered here by these three fixed-point maps within 1% error or so.

We have also extended the renormalization scheme to nonsymmetric DeVogelaerelike 4D maps. These maps contain two nonsymmetrically coupled 2D DeVogelaere maps. For these maps, we found the fixed maps are the same as those for the symmetric maps mentioned above, i.e., the E, L, and U maps.

II. SYMMETRIC FOUR-DIMENSIONAL VOLUME-PRESERVING MAPS

Consider the following two symmetrically coupled 2D area-preserving maps:

$$\begin{aligned} x' &= -a_1 y + 2f(x,u) , \\ y' &= \frac{1}{a_1} x , \\ u' &= -a_2 v + 2f(u,x) , \\ v' &= \frac{1}{a_2} u ; \end{aligned} \tag{2.1}$$

$$f(x,u) = b + cx + dx^2 + eu + fu^2 + gxu .$$

This can be regarded as a truncation from the full infinite-dimensional space of maps to an eight-dimensional space. The parameters a_1, a_2, b, c, d, e, f, and g, are the coordinates of this space. The truncation of the map at quadratic terms in x and u, and terms in y and v, provides a space of maps that is large enough to provide a quite good approximation to the true fixed maps of the period-doubling operator. At the same time, this space is small enough to be tractable.

We search for maps that satisfy the renormalization equation

$$T = \underline{R}(T), \quad \underline{R}(T) = \underline{B} T^2 \underline{B}^{-1} , \tag{2.2}$$

where T and $\underline{R}(T)$ are maps within the truncated space. Thus $\underline{R}$ is the period-doubling operator that is composed of squaring (T^2, i.e., composing with itself and truncating back to the chosen space) and rescaling ($\underline{B}$) operators. The truncation does not commute with all choices of $\underline{B}$, so that thought must be given to the desired definition of $\underline{R}$.

These maps have a number of symmetries. They can be factored into the two involutions

$$\begin{aligned} T &= (TS_1)S_1 , \\ S_1^2 &= 1, \quad (TS_1)^2 = 1 , \end{aligned} \tag{2.3}$$

where S_1 is given by

$$x'=a_1 y,\quad u'=a_2 v\ ,$$
$$y'=\frac{1}{a_1}x,\quad v'=\frac{1}{a_2}u\ , \tag{2.4}$$

and also into the involutions

$$T=(TS_2)S_2\ ,$$
$$S_2^2=1,\quad (TS_2)^2=1\ , \tag{2.5}$$

where S_2 is given by

$$x'=a_2 v,\quad u'=a_1 y\ ,$$
$$y'=\frac{1}{a_1}u,\quad v'=\frac{1}{a_2}x\ . \tag{2.6}$$

This second symmetry depends on the change of order of the arguments of the function f in the first and third of Eqs. (2.1). Now S_1 and S_2 commute, and their product $C=S_1S_2$ commutes with T. The latter is given by

$$x'=u,\quad u'=x\ ,$$
$$y=\frac{a_2}{a_1}v,\quad v'=\frac{a_1}{a_2}y\ . \tag{2.7}$$

There is a two-dimensional plane, $x-u=0$ and $a_1y-a_2v=0$, that is fixed under C. If an orbit starts on this plane, it remains there forever. These orbits are called "in phase."

The orientation of the rescaling that goes with period doubling is closely associated with the symmetries of the map. Thus the rescaling operator $\underline{B}$ can be simplified upon choosing coordinates for the maps in which the symmetries are simple. Thus we introduce the new coordinates

$$\overline{X}=\tfrac{1}{2}(x+u),\quad \overline{U}=\tfrac{1}{2}(x-u)\ ,$$
$$\overline{Y}=\frac{1}{2A_1}(a_1y+a_2v),\quad \overline{V}=\frac{1}{2A_2}(a_1y-a_2v)\ , \tag{2.8}$$

in terms of which the map is given by

$$\overline{X}'=-A_1\overline{Y}+2F(\overline{X},\overline{U})\ ,$$
$$\overline{Y}'=\frac{1}{A_1}\overline{X}\ ,$$
$$\overline{U}'=-A_2\overline{V}+2G(\overline{X},\overline{U})\ ,$$
$$\overline{V}'=\frac{1}{A_2}\overline{U}\ . \tag{2.9}$$

Here a symmetry of the map of Eq. (2.1) takes the form that $F(\overline{X},\overline{U})$ is even in $\overline{U}$ and $G(\overline{X},\overline{U})$ is odd in $\overline{U}$. These functions are truncated at quadratic terms, yielding a new set of parameters:

$$F(X,U)=B+CX+DX^2+FU^2\ ,$$
$$G(X,U)=EU+GXU\ . \tag{2.10}$$

These coordinates have the effect of making the fixed plane of the commutator C of Eq. (2.7) a coordinate surface, $\overline{U}=\overline{V}=0$. A further transformation,

$$X=\overline{X},\quad U=\overline{U}\ ,$$
$$Y=\overline{Y}-\frac{1}{A_1}F(\overline{X},\overline{U}),\quad V=\overline{V}-\frac{1}{A_2}G(\overline{X},\overline{U})\ , \tag{2.11}$$

which is a DeVogelaere transformation,[4] makes the fixed plane of TS_1 of Eq. (2.3) and (2.4) above a coordinate surface, $Y=0$. Thus our map becomes

$$X'=-A_1Y+F(X,U)\ ,$$
$$Y'=\frac{1}{A_1}[X-F(X',U')]\ ,$$
$$U'=-A_2V+G(X,U)\ ,$$
$$V'=\frac{1}{A_2}[U-G(X',U')]\ , \tag{2.12}$$

where $F(X,U)$ and $G(X,U)$ are expressed in Eq. (2.10). In this coordinates the dominant period doubling occurs along the X axis, and the rescaling operator $\underline{B}$ is diagonal.

This map is volume preserving, but additional conditions are required for it to be symplectic,[5] that is, for the map to be a canonical transformation. Specifically, when the canonical pairs of variables are chosen to be (x,y) and (u,v) the map is symplectic when $2A_2F=A_1G$. There are many other possibilities for writing maps of this class as canonical transformations, each with its own condition.

III. RENORMALIZATION CALCULATION

Iterating the map (2.12) twice, truncating the expressions for X'' and U'' at terms of order X^2,U^2,Y,V, and rescaling X,Y,U,V by $\alpha_1,\beta_1,\alpha_2,\beta_2$, respectively, we have for the new map $\underline{R}(T)$:

$$X''=-Y\left[2\frac{\alpha_1}{\beta_1}A_1(C+2B)\right]+[2\alpha_1B(1+C+B)]+X[-1+2C(C+2B)]+X^2\left[\frac{2}{\alpha_1}(C+2B+C^2)\right]+U^2\left[2\frac{\alpha_1}{\alpha_2}F(C+2B+E^2)\right]\ ,$$
$$U''=-V\left[2\frac{\alpha_2}{\beta_2}A_2(E+GB)\right]+U[-1+2E(E+GB)]+XU\left[\frac{2}{\alpha_1}G(E+GB+CE)\right]\ . \tag{3.1}$$

Truncation for the expressions for Y'' and V'' are chosen to preserve the form of Eq. (2.12), and are thus not independent. This new map $\underline{R}(T)$ should have the form of map (2.12). Thus $\underline{R}$ is a map from

$$\mathbf{P}=(A_1,A_2,B,C,D,E,F,G)$$

to

$$\mathbf{P}'=(A_1',A_2',B',C',D',E',F',G')\ ,$$

and from Eq. (3.1), it is given by

$$\begin{aligned}
&Y:\ A_1'=2\frac{\alpha_1}{\beta_1}A_1(C+2BD)\,,\\
&1:\ B'=2\alpha_1 B(1+C+BD)\,,\\
&X:\ C'=-1+2C(C+2BD)\,,\\
&X^2:\ D'=\frac{2}{\alpha_1}D(C+2BD+C^2)\,,\\
&U^2:\ F'=2\frac{\alpha_1}{\alpha_2^2}F(C+2BD+E^2)\,,\\
&V:\ A_2'=2\frac{\alpha_2}{\beta^2}A_2(E+GB)\,,\\
&U:\ E'=-1+2E(E+GB)\,,\\
&XU:\ G'=\frac{2}{\alpha_1}G(E+GB+CE)\,.
\end{aligned}\tag{3.2}$$

The accumulation value of parameter (denoted by $\mathbf{P}_\infty$) can be determined by setting $\mathbf{P}'=\mathbf{P}=\mathbf{P}_\infty$ in Eq. (3.2) and solving it for $\mathbf{P}_\infty$. Then A_1,A_2,D,F sets the scales of Y,V,X,U, and thus are arbitrary. Hence these equations also determine α_1, β_1, α_2, and β_2. We found, see Appendix, 16 solutions for $\mathbf{P}_\infty$ (and α's,β's) listed in Table I. The map (2.12) or its equivalents, the map (2.1) and (2.9) with the 16 different $\mathbf{P}_\infty$'s are the fixed maps of the period-doubling renormalization operator. But, as will be discussed in Sec. IV, only *three* of them are fixed maps of the pitchfork period-doubling renormalization operator.

For all these 16 fixed maps, one can determine the eigenvalues of the following (parameter) perturbation about the fixed maps:

$$\underline{D}\equiv\left.\frac{\partial\mathbf{P}'}{\partial\mathbf{P}}\right|_{\mathbf{P}_\infty}.\tag{3.3}$$

This 8×8 matrix has eight eigenvalues, λ_i, $i=1,2,\ldots,8$. Four of them are 1, corresponding to

$$\left.\frac{\partial A_1'}{\partial A_1}\right|_{\mathbf{P}_\infty}=\left.\frac{\partial A_2'}{\partial A_2}\right|_{\mathbf{P}_\infty}=\left.\frac{\partial D'}{\partial D}\right|_{\mathbf{P}_\infty}=\left.\frac{\partial F'}{\partial F}\right|_{\mathbf{P}_\infty}\equiv1\,.\tag{3.4}$$

The remaining four eigenvalues are eigenvalues of the following two 2×2 matrices:

$$\underline{M}_1=\left.\frac{\partial(B',C')}{\partial(B,C)}\right|_{\mathbf{P}_\infty}=\left.\begin{bmatrix}1+2\alpha_1B+\frac{4}{\alpha_1}B & 2\alpha_1B+\frac{2}{\alpha_1}B(2C+1)\\ 4C & 4(B+C)\end{bmatrix}\right|_{\mathbf{P}_\infty},\tag{3.5}$$

$$\underline{M}_2=\left.\frac{\partial(E',G')}{\partial E,G)}\right|_{\mathbf{P}_\infty}=\left.\begin{bmatrix}4E+2BG & 2BE\\ \frac{2}{\alpha_1}G(C+1) & 1+\frac{2}{\alpha_1}BG\end{bmatrix}\right|_{\mathbf{P}_\infty}.\tag{3.6}$$

TABLE I. Sixteen solutions of Eq. (3.2) when $\mathbf{P}'=\mathbf{P}$, i.e., $A_1'=A_1$, $B'=B$, $C'=C,\ldots$. We have set the scales in X, Y, U, and V (i.e., D, A_1, F, and A_2) being 1. Note that the solutions 1,2,3 (denoted by E,L,U map) are the fixed maps for the pitchfork period-doubling bifurcation without degeneracy.

	B	C	α_1	β_1	δ_1	δ_1'	E	G	α_2	β_2	δ_2	δ_2'
1 E map							C	2	α_1	β_1	-4.4510	1.8762
2 L map							1	0	±2.9116	$2\alpha_2$	4	1
	-0.9282	-0.1959	-4.0280	16.5338	8.9474	-4.0443						
3 U map							-0.5	0	±3.8104	$-\alpha_2$	-2	1
4							$1/(2C^2)$	13.4565	Imaginary	Imaginary	32.3107	2.0263
5							C	2	α_1	β_1	-4.2333	1.9246
6							1	0	±5.0256	$2\alpha_2$	4	1
	1.2192	2.1689	9.9476	5.3612	7.1150	14.8328						
7							-0.5	0	±3.2148	$-\alpha_2$	-2	1
8							$1/(2C^2)$	4.1814	2.3655	24.6005	10.5915	2.0547
9							C	2	α_1	β_1	2.2569	-1.7529
10							1	0	±1.2605	$2\alpha_2$	4	1
	0.3179	-0.7782	0.9265	-0.2640	2.8233	-1.7032						
11							-0.5	0	0.4463	$-\alpha_2$	-2	1
12							$1/(2C^2)$	0.8815	0.9994	2.2100	3.9565	1.5108
13							1		±4	$2\alpha_2$	4	1
	0	1	4	8	4	1						
14							-0.5	0	$\pm\sqrt{10}$	$-\alpha_2$	-2	1
15							1	0	Imaginary	Imaginary	4	1
	0	0.5	0.5	0.5	-2	1						
16							-0.5		$\pm\frac{1}{2}$	$-\alpha_2$	-2	1

The two eigenvalues of M_i $(i=1,2)$ are called δ_i and δ_i', and listed also in Table I.

IV. THREE FIXED MAPS OF PITCHFORK PERIOD-DOUBLING RENORMALIZATION OPERATOR

Among the 16 fixed maps listed in Table I, the first three recover the numerical results[3] of period doubling (following so-called E, L, and U bifurcation "path") in the symmetric 4D volume-preserving maps.

The map (2.12) [or its equivalent maps (2.1) and (2.9)] under scale changes in Y,V,U and under a translation $X \to X+X^*$ (X^* independent of X,Y,U,V) is still a fixed map of the renormalization operator $\underline{R}$ with the rescaling operator $\underline{B}$ replaced by $\underline{B}^*(X)=[\alpha_1(X-X^*)]$. In order to obtain a compact form of the fixed map, we thus set $A_1=A_2=D=F=1$ (i.e., make scale changes $Y\to Y/A_1$, $V\to V/A_2$, $X\to X/\sqrt{D}$, $U\to U/\sqrt{F}$), and let the constant term in $F(X,U)$ be zero (i.e., translate X by $X^*=\frac{1}{2}\{(1-C)-[(1-C)^2-4B]^{1/2}\}$. The fixed map (2.9) becomes

$$\begin{aligned} X' &= -Y+2(C_1X+X^2+U^2)\ , \quad C_1=1-(1-C^2-4B)^{1/2}\ , \\ Y' &= X, \\ U' &= -V+2(C_2U+GXU), \quad C_2=E+1/2G(C_1-C)\ , \\ V' &= U\ . \end{aligned} \tag{4.1}$$

Note this compact form has only three parameters (C_1, C_2, and G). Also note that the rescaling $\underline{B}^*$ and truncation do not commute, and that if the order is reversed the renormalization form[6] of Helleman is recovered.

For the solution 1 in Table I, $C_1=C_2=-1.2678,\ldots$, $G=2$, and therefore the map (4.1) can be transformed into

$$\begin{aligned} x' &= -y+2(C_1x+x^2)\ , \\ y' &= x\ , \\ u' &= -v+2(C_1u+u^2)\ , \\ v' &= u \end{aligned} \tag{4.2}$$

by $x=X+U$, $y=Y+V$, $u=X-U$, $v=Y-V$. Obviously, this map consists of two uncoupled 2D area-preserving maps in the normal form.[6] This map is equivalent to the map (2.1) with $e=f=g=0$ (i.e., no coupling term). Hence, this fixed map corresponds to the E route in Ref. 3, and so is called E map. Around this E map the $\underline{R}$ has relevant eigenvalues $\delta_1=8,9\ldots$ and $\delta_2=-4.45\ldots$, which are quite close to the numerical results of the Feigenbaum constant $\delta_1=8.721\ldots$ and $\delta_2=-4.404\ldots$ for the E route. The eigenvalue δ_1' is the eigenvalue α_1 associated with a shift of the X coordinate, but it is not clear whether δ_2' is a further relevant eigenvalue.

For the solution 2 in Table I, the map (4.1) has parameters $C_1=-1.2678\ldots$, $C_2=1$, and $G=0$, which means the 2D UV map is *linear*. The relevant eigenvalues $\delta_1=8.9\ldots$ and $\delta_2=4$ are very close to the numerical values ($\delta_1=8.721\ldots$, $\delta_2=4$) for the L route (see Ref. 3). Thus we call this fixed map L map. Similar discussions can be made for the third solution (called U map) in Table I. The L and U map contain the 2D (nonlinear) area-preserving fixed map coupled with the linear 2D map with the linear coefficient $2C_2=2$ and -1, respectively.

The solutions 4, 9—12, 15, and 16 have at least one of the orbital scaling factors $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ either imaginary or less than one. They are thus not fixed maps for the pitchfork period-doubling bifurcation.

The solutions 5—7 have $C_1=C_2=-1.2727\ldots$, and $G=2,0,0$, respectively. They are same (in error of 1% or so) as the solution 1—3. We therefore consider them as degeneracies of the E, L, and U map. The solution 8 corresponds to the map (4.1) with $C_1=-1.2727\ldots$, $C_2=1.9800\ldots$, and $G=4.1814$. Suggesting these values of parameters (C_1, C_2, and G) being the accumulation values of parameters in the period doubling, we numerically searched the period-doubling sequences and found $\delta_1=8.721$, $\delta_2=4$, which means the solution 8 also corresponds to L route.

The solutions 13 and 14 correspond to the tangent bifurcation because two of eigenvalues of their Jacobian matrix at their fixed point ($X_0=Y_0=U_0=V_0=0$ when $B=0$) are 1.

Note that for the linear 2D map, eigenvalues of its Jacobian matrix are $e^{\pm i\theta}$, where $\cos\theta=C_2$.[6] Thus $C_2=1,-\frac{1}{2}$ correspond to $\theta=0,2\pi/3$, for which the eigenvalues $e^{\pm i\theta}$ interchange when the period doubles.

In sum, among the 16 solutions in Table I, the first

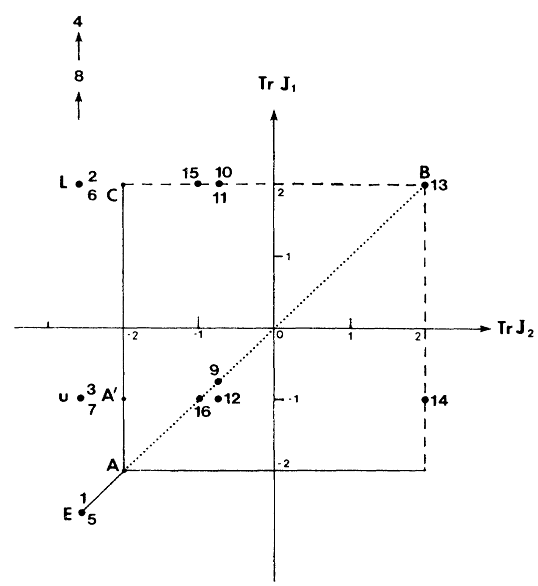

FIG. 1. The 16 fixed maps in Table I. They are marked by the heavy dots, and denumbered by the bold Arabic numbers. TrJ_1 and TrJ_2 are traces of the two matrices J_1 and J_2 into which the Jacobian matrix decomposes, see Eq. (4.3). The square is stable region. The solid, dashed, and dotted lines denote respectively the pitchfork period-doubling bifurcation line, the tangent bifurcation line, and the complex bifurcation line.

three are the fixed map of the pitchfork period-doubling renormalization operator without degeneracy. They recover the numerical results in error of 1% or so.

If the map (2.12) [or its equivalent maps (2.1), (2.9)] is symplectic (i.e., $2A_2F=A_1G$, see Sec. II), then $F=1,0,0$, respectively for the E, L, and U map. Thus all three fixed maps (not only E map) consist of two *uncoupled* 2D maps. This is the only difference between the fixed maps for the class of symmetric *volume-preserving* maps and those for its subclass, the symmetric *symplectic* maps.

To end this section, we plot all 16 fixed maps in Fig. 1. The Jacobian matrix $\underline{J}$ of the map (4.1) at $X_0=Y_0=U_0=V_0=0$ decomposes into two 2×2 matrices $\underline{J}_1$ and $\underline{J}_2$,

$$\underline{J}=\begin{bmatrix}\underline{J}_1 & 0\\ 0 & \underline{J}_2\end{bmatrix} . \tag{4.3}$$

Traces of $\underline{J}_1$ and $\underline{J}_2$ for the 16 fixed maps are calculated and plotted in Fig. 1. The square in the figure is stable region. The left side of the pitchfork bifurcation line AC is the region for pitchfork bifurcation, and thus the E, L, and U maps are located there.

V. RENORMALIZATION IN NONSYMMETRIC VOLUME-PRESERVING FOUR-DIMENSIONAL MAPS

In Secs. II–IV, we dealt with the *symmetric* volume-preserving 4D maps. They could be either Hénon-like, such as the map (2.1), or DeVogelaere-like, such as the map (2.12). Hénon-like maps can be transformed into DeVogelaere maps by the DeVogelaere transformation (2.11). In this section, we give a renormalization scheme for *nonsymmetric* volume-preserving 4D maps, either Hénon-like or DeVogelaere-like.

A nonsymmetric Hénon-like 4D map is

$$\begin{aligned} x' &= -a_1 y + 2F(x,u) ,\\ y' &= \frac{1}{a_1}x ,\\ u' &= -a_2 v + 2G(x,u) ,\\ v' &= \frac{1}{a_2}u , \end{aligned} \tag{5.1}$$

where the nonlinear function $G(x,u)\neq F(u,x)$, so the map is called nonsymmetric [compare with the symmetric map (2.1)]. This map can be put in DeVogelaere form,

$$\begin{aligned} X' &= A_1 Y + F(X,U) ,\\ Y' &= \frac{1}{A_1}[X - F(X',U')] ,\\ U' &= -A_2 V + G(X,U) ,\\ V' &= \frac{1}{A_2}[U - G(X',U')] , \end{aligned} \tag{5.2}$$

where

$$F(X,U)=B_1+C_1X+D_1X^2+E_1U+F_1U_1^2+G_1XU , \tag{5.3}$$

$$G(X,U)=B_2+C_2X+D_2X^2+E_2U+F_2U^2+G_2XU .$$

If $E_1=G_1=B_2=C_2=D_2=F_2=0$ we come back to the symmetric case, see Eq. (2.10).

We perform the same renormalization, as we did for the symmetric map in Secs. II–IV, now for the nonsymmetric map (5.2). We iterate this map twice, truncate at quadratic terms, and rescale X,Y,U,V by the orbital scaling factors $\alpha_1,\beta_1,\alpha_2,\beta_2$, respectively. Then we have expressions for X'' and U'', which are similar to Eq. (3.1). The coefficient of V in the X'' expression should be zero because V does not appear in X' expression of the original map (5.2). Hence we have

$$E_1+2B_2F_1+B_1G_1=0 . \tag{5.4}$$

Similarly, the coefficient of Y in U'' expression should also be zero, i.e.,

$$C_2+2B_1D_2+B_2G_2=0 . \tag{5.5}$$

The other coefficients in X'' and U'' expressions give recursion relation for the parameters:

$$A_1'=2\frac{\alpha_1}{\beta_1}A_1Q_1,\quad Q_1\equiv C_1+2B_1D_1+B_2G_1 , \tag{5.6a}$$

$$B_1'=2\alpha_1[B_1(1-D_1+Q_1)+B_2(E_1+B_2F_1)] , \tag{5.6b}$$

$$C_1'=-1+2Q_1 , \tag{5.6c}$$

$$D_1'=\frac{2}{\alpha_1}[D_1(C_1^2+Q_1)+C_2^2F_1+C_1C_2G_1] , \tag{5.6d}$$

$$E_1'=2\frac{\alpha_1}{\alpha_2}E_1Q_1 , \tag{5.6e}$$

$$F_1'=2\frac{\alpha_1}{\alpha_2^2}[F_1(E_2^2+Q_1)+E_1^2D_1+E_1E_2G_1] , \tag{5.6f}$$

$$G_1'=\frac{2}{\alpha_2}[G_1(C_1E_2+C_2E_1+Q_1)+2(C_1E_1D_1+C_2E_2F_1)] , \tag{5.6g}$$

$$A_2'=2\frac{\alpha_2}{\beta_2}A_2Q_2,\quad Q_2\equiv E_2+2B_2F_2+B_1G_2 , \tag{5.6h}$$

$$B_2'=2\alpha_2[B_2(1-F_2+Q_2)+B_1(C_2+B_1D_2)] , \tag{5.6i}$$

$$C_2'=\frac{2\alpha_2}{\alpha_1}C_2Q_2 , \tag{5.6j}$$

$$D_2'=\frac{2\alpha_2}{\alpha_1^2}[D_2(C_1^2+Q_2)+C_2^2F_2+C_1C_2G_2] , \tag{5.6k}$$

$$E_2'=-1+2E_2Q_2 , \tag{5.6l}$$

$$F_2'=\frac{2}{\alpha_2}[F_2(E_2^2+Q_2)+E_1^2D_2+E_1E_2G_2] , \tag{5.6m}$$

$$G_2'=\frac{2}{\alpha_1}[G_2(C_1E_2+C_2E_1+Q_2)+2(C_1E_1D_2+C_2E_2F_2)] . \tag{5.6n}$$

In order to determine the accumulation values of the parameters $\mathbf{P}=(A_1,B_1,\ldots,G_1,A_2,B_2,\ldots,G_2)$, we set $\mathbf{P}'=\mathbf{P}$, i.e., $A_1'=A_1$, $B_1'=B_1,\ldots$, the same as we did in Secs. II–IV. Among the 14 parameters, four of them give scales in X,Y,U,V coordinates. They are, say, D_1,A_1,F_1,A_2. The remaining ten parameters, together with the four scaling factors $\alpha_1,\beta_1,\alpha_2,\beta_2$, can be determined from the 14 equations. Instead of solving these 14 equations directly, we use a simple way to find the fixed maps. The way is based on the two constraints Eqs. (5.4) and (5.5), and on the fact that a fixed map after translations in X and U directions is still a fixed map. We first make an X- and an U-coordinate change to have

$$E_1=0,\quad B_2=0\ . \tag{5.7}$$

The two constraints, Eqs. (5.4) and (5.5), now are

$$B_1G_1=0\ , \tag{5.8}$$

$$C_2+2B_1D_2=0\ . \tag{5.9}$$

The Eq. (5.8) has two solutions: $B_1\neq 0$ ($G_1=0$) and $B_1=0$. We will discuss these two cases in the following.

When $B_1\neq 0$, $G_1=0$, the Eq. (5.6i), using Eq. (5.9), gives $C_2=0$. Then Eq. (5.9) requires $D_2=0$. Now, Eqs. (5.6a)–(5.6d), (5.6f), (5.6h), (5.6l), (5.6n) are just the same as the Eq. (3.2) of the symmetric case. Of course, we get the same fixed maps as listed in Table I, with $B_1=B$, $C_1=C$, $E_2=E$, $G_2=G$. The only remaining Eq. (5.6m) becomes

$$F_2'=\frac{2}{\alpha_2}F_2(E_2^2+Q_2)\ . \tag{5.10}$$

When $C_1=E_2$, F_2 can be any value because $\alpha_2=2(E_2^2+Q_2)$. But F_2 must be zero when $C_1\neq E_2$. In sum, the $B_1\neq 0$ case has fixed maps (5.2) with

$$\begin{aligned} F(x,u)&=B+CX+X^2+U^2\ ,\\ G(x,u)&=EU+F_2U^2+GXU\ , \end{aligned} \tag{5.11}$$

where the values of B,C,E,G are listed in Table I; $F_2=0$ for solutions 1,5,9, F_2 is arbitrary otherwise. The corresponding $\alpha_1,\beta_1,\alpha_2,\beta_2$ values found from Eqs. (5.6a)–(5.6n) are the same ones as listed in Table I.

The second case due to Eq. (5.8) is $B_1=0$. In this case, using Eqs. (5.7)–(5.9), we have $B_1=B_2=E_1=C_2=0$. Hence the solutions of Eqs. (5.6a)–(5.6n) are the solutions 13–16 listed in Table I.

Linearization around fixed maps yields larger matrices, and therefore more eigenvalues, when there are more parameters in the representation of the map. Thus, while considering an enlarged space of nonsymmetric maps does not yield any more fixed maps of the Feigenbaum operator, it does provide more eigenvalues for the ones obtained in Secs. II–IV.

In the present case, the new eigenvalues that have eigenmaps that break the symmetry are degenerate with the eigenvalues whose eigenmaps do not break the symmetry. Of course they must be counted as additional eigenvalues, because their eigenmaps are different.

The additional eigenvalues lead to new insights into the nature of the fixed maps. Consider first the E map. This map is composed of two area-preserving fixed maps, one in the (x,y) sector and one in the (u,v) sector. Thus there are two relevant eigenvalues that are equal to the well-known relevant eigenvalue of the area-preserving maps, approximated here as 8.974. In symmetry coordinates, one eigenmap perturbs the two uncoupled maps together, preserving symmetry. The other eigenmap perturbs the two uncoupled maps oppositely, breaking symmetry. Similarly, the two degenerate δ_2 eigenvalues, approximately -4.4510, have eigenmaps that couple the two halves of the E map. Again, the coupling can be symmetric or antisymmetric.

We understand the remaining ten eigenvalues we have found for this map as follows. Eight are associated with coordinate transformations. Four of the eigenvalues of unity are related to independent scaling of the four coordinates of the map. Two more arise because the renormalization is identical, given by $\alpha_1=\alpha_2$, in the x and u coordinates. Thus linear transformations between x and u are indifferent and there are two additional eigenvalues of unity. The last pair of coordinate generated eigenvalues arise from displacements of the origin of the x and u coordinates. The associated eigenvalues are α_1 and α_2, which we identify with δ_1' and its symmetry breaking counterpart.

The remaining two eigenvalues δ_2' and its unsymmetric partner, cannot be identified with anything seen in numerical work. We believe that they are artifacts of the truncation.

Our conclusion is that the E map has four relevant eigenvalues, plus the usual group of coordinate-change generated eigenvalues.

VI. CONCLUSIONS

In this work we have considered period doubling in four-dimensional volume-preserving maps. Period doubling in these maps is controlled by the maps that are fixed under the obvious generalization of the Feigenbaum renormalization operator. In contrast to previous work that has identified the fixed maps numerically, here we have searched for them algebraically.

We have found algebraic approximations to the three fixed maps that have been identified previously. No further fixed maps have been identified.

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APPENDIX

In this appendix we outline the way to find the 16 solutions of Eq. (3.2). The eight equations are for eight unknowns B, C, E, G, α_1, β_1, α_2, and β_2. Note that A_1, A_2, D F are scales in Y, V, X, and U coordinates, and that we can set $A_1=A_2=D=F=1$.

We first observe that the second, third, and fourth equations are for unknowns B,C,α_1 only. We thus find an equation for C from these three equations, for $B\neq 0$,

$$4C^5+10C^4+4C^3+5C^2+6C+1=0\ , \qquad \text{(A1)}$$

which has three real solutions $(-0.1959\ldots, -2.1689\ldots, -0.7782\ldots)$. If $B=0$, we find $C=1$, $-\frac{1}{2}$, from the third equation. All these five C values are listed in the fourth column of Table I. They are the same as 2D area-preserving fixed maps.[2] It is straightforward to determine B,α_1,β_1, in terms of C, from the first four equations.

The fourth and eighth equations give, when $G\neq 0$,

$$(C-E)(2C^2E-1)=0\ . \qquad \text{(A2)}$$

Hence E equals either C or $1/(2C^2)$. If $G=0$, then $E=1$ or $-\frac{1}{2}$ from the seventh equation. That is, for every C (and B,α_1,β_1) the parameter E has four possible values (two possible values if $B=0$), see Table I. The nonzero values of G are then determined by, for $B\neq 0$,

$$G=\begin{cases} 2 & \text{if } E=C \\ \dfrac{1}{B}\left[\dfrac{1}{2}+\dfrac{1}{2E}-E\right] & \text{if } E=\dfrac{1}{2C^2}\ . \end{cases} \qquad \text{(A3)}$$

The α_2,β_2 can be found from the fifth and seventh equations.

[1]M. J. Feigenbaum, J. Stat. Phys. **21**, 669 (1979).
[2]J. Greene, R. S. Mackay, F. Vivaldi, and M. J. Feigenbaum, Physica **3D**, 468 (1981).
[3]J. M. Mao and R. H. G. Helleman, Phys. Rev. A **35**, 1847 (1987).
[4]R. DeVogelaere, Technical Report No. 62-1, Department of Mathematics, University of California, Berkeley, 1962 (unpublished); *Contributions to the Theory of Nonlinear Oscillations,* edited by S. Lefshetz (Princeton University Press, Princeton, 1958), Vol. IV, p. 53.
[5]V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1979).
[6]R. H. G. Helleman, in *Long-Time Predictions in Dynamics,* edited by W. Horton, L. Reichl and V. Szebehely (Wiley, New York, 1982), pp. 95—126.