Narrow-band radiation from long-pulse free-electron lasers

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A perturbative theoretical analysis of the operation of a long-pulse free-electron laser (FEL) is presented. This analysis was prompted by the remarkable property of the FEL at the University of California at Santa Barbara of generating radiation with a very small fractional bandwidth of the order of 10^{-8}. To third order in the radiation field, the transverse current driving the FEL contains large crossed-saturation terms between resonator modes. Through the strong crossed saturation the dominant mode depresses the effective gain of other modes to the point of extinction, resulting in single-mode operation. These conclusions hold as long as the long-pulse FEL is operating in a low-gain, homogeneously broadened regime, outside the deep saturation region. The perturbation expansion, nonlinear single-mode, and two-mode problems are discussed in detail.

I. INTRODUCTION

The radiation generated in free-electron lasers (FEL's) driven by short-electron-pulse accelerators shows fractional bandwidths close to the maximum of $(\Delta\omega/\omega)\propto 1/N$,[1] where N is the number of periods of the undulator. On the other hand, FEL's using long electron pulses, such as the one[2] operating at the University of California at Santa Barbara (UCSB), are capable of generating extremely-narrow-band radiation. A theoretical understanding of how this comes about is the main concern of the present paper.

After presenting the basic field equations (undulator and wave) in Sec. II, a Fourier decomposition for the transverse current driving the FEL is outlined in Sec. III. The current Fourier components can be expanded in a perturbation series in powers of the radiation field. Such a perturbation expansion is obtained in Sec. IV using the evolution operator method familiar from quantum mechanics. The well-known linear theory results obtained from the first-order perturbation are given, for completeness, in Sec. V.

We treat the nonlinear theory starting with Sec. VI, where the self- and crossed-saturation terms arising from third-order perturbation, are presented. To third order, the single-mode problem discussed in Sec. VII leads to a limiting intensity (or power density) $I(\infty)$. In Fig. 1, a comparison is made, between the effective gain obtained from the third-order theory, and the result from a computer simulation. The agreement is very good except for the deep saturation region (tail of the gain curve).

The equations describing the evolution of the amplitudes in the two-mode problem are analyzed in Sec. VIII using two different methods. Of crucial importance is the parameter u equal to the ratio of crossed saturation to self-saturation. A FEL with long electron pulses operating in the low-gain regime is a clear case of strong coupling with crossed saturation being twice as large as self-saturation ($u=2$). It is shown that for systems with strong coupling, only single-mode states can be stable solutions of the dynamical equations. As the dominant mode approaches saturation, its power density depresses the effective gain of other modes. When their effective gain becomes negative, these other modes disappear. Thus the theoretical conclusion corroborates the experimental finding that a long-pulse low-gain FEL operates within a very narrow frequency bandwidth.

II. WAVE EQUATION

The magnetic field of an undulator with constant period $\lambda_0=2\pi/k_0$ and uniform amplitude B_0 can be derived from the vector potential

$$\mathbf{A}_u=-i\mathbf{n}B_0e^{-ik_0z}/2k_0+\text{c.c.}\ , \tag{1}$$

where $\mathbf{n}$ is the polarization vector ($\mathbf{n}=\hat{\mathbf{x}}$ for linear and $\mathbf{n}=(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}})$ for circular polarization).

Our major goal in this analysis is to study the longitudinal mode structure of a long-pulse FEL oscillator. However, conceptual simplicity is gained if the oscillator is thought of as a series of amplifier stages, each stage corresponding to a single pass. Moreover, for the purpose of analyzing the basic features of mode interaction, it is sufficient to consider a one-dimensional problem with the radiation field written in terms of a discrete superposition of vector potential plane-wave modes of the form

$$\mathbf{A}=-i\frac{\mathbf{n}mc}{2e}\sum_q a_q(z)e^{i[k_qz-\omega_qt+\phi_q(z)]}+\text{c.c.}\ , \tag{2}$$

where $a_q(z)$ is the adimensional vector potential for mode q and $k_q=\omega_q/c$ is its corresponding wave vector. In the present work we assume that $\omega_q=q\omega_f$, where ω_f is a fundamental angular frequency which in the case of an oscillator corresponds to the spacing between longitudinal modes $c\pi/L_0$.

In the present paper we will consider sufficiently low electron densities so that space-charge effects could be neglected. Using the slowly varying amplitude and phase approximation, the wave equation acting on field (2) leads to

$$\sum_q \omega_q e^{i(k_q z-\omega_q t+\phi_q)}[\partial_z a_q(z)+ia_q(\partial_z\phi_q+\alpha/2L)] +(2-p)(\text{c.c.})=-(\mu_0 e/mp)\mathbf{n}^*\cdot\mathbf{J}\ , \quad (3)$$

where L is the length of the undulator, α is the mode loss per pass, and $p=|\mathbf{n}|^2$ is 1 for linear and 2 for circular polarization. $\mathbf{J}$, the transverse electron current driving the FEL, will be discussed in detail in Sec. III.

III. TRANSVERSE CURRENT

From now on, for convenience, we will use the adimensional coordinate $\tau=z/L$. In keeping with our one-dimensional treatment let us consider an electron beam with an initial uniform density $\rho=\rho_L\rho_T$, i.e., ρ_L planes per unit length with ρ_T electrons per unit area in each plane. The transverse current density driving the FEL is

$$\mathbf{J}(t,x)=-e\mathbf{v}_T(\tau)\sum_j \delta^3[\mathbf{x}-\mathbf{x}_j(t)] \Longrightarrow -e\rho_T\mathbf{v}_T\sum_j \delta[L\tau-z_j(t)]\ , \quad (4)$$

where $\mathbf{v}_T$, the transverse velocity acquired by the electrons in the undulator field (1), is given by

$$\mathbf{v}_T=i\mathbf{n}(cK/2\gamma)e^{-ik_0L\tau}+\text{c.c.},\quad K=eB_0/mck_0\ . \quad (5)$$

The $\Longrightarrow$ in Eq. (4) indicates the transition from the general three-dimensional situation to the simplified one-dimensional problem.

For $K^2 \ll 1$ one can disregard harmonic radiation. In such a case the current can also be decomposed in the same way as the electric field (2), namely,

$$\mathbf{J}=-\mathbf{n}(m/e\mu_0L)\sum_q \omega_q J_q e^{i(k_qL\tau-\omega_q t+\phi_q)}+\text{c.c.}\ . \quad (6)$$

With this definition, J_q are the adimensional sources for the amplitudes and phases as in

$$\partial_\tau a_q+ia_q(\partial_\tau\phi_q-\alpha/2)=J_q\ . \quad (7)$$

As Fourier coefficients in the expansion (6), the current amplitudes J_q are obtained from Eq. (4) as

$$J_q=-(ec\mu_0L/2pmL_0\omega_q)\int dt\, e^{i[\omega_q t-(k_q+k_0)L\tau-\phi_q]}\mathbf{n}^*\cdot\mathbf{J} =i(e^2\mu_0c\rho_TKL/4m\gamma\omega_qL_0)\sum_j e^{i[\omega_q t_j(\tau)-(k_q+k_0)L\tau]}\ , \quad (8)$$

where the integration is over a fundamental period $T_f=2\pi/\omega_f$ and the sum is over planes of particles with injection times uniformly distributed over this period.

The electron that enters the undulator at time t_{j0} will reach the position $z=L\tau$ at the time

$$t_j(\tau)=t_{j0}+(L\tau/v_0)-\xi_j(\tau)L/c\ , \quad (9)$$

in which v_0 is the initial electron velocity[3] and the small time shift $\xi_j(t)$, due to the bunching, is obtained from the pendulum equation[4]

$$d^2\xi_j/d\tau^2=(L/c^2)d^2z/dt^2 =i\Lambda\sum_q(\omega_q/\omega_R)a_q\psi_{jq}^*e^{iR_{jq}}+\text{c.c.}\ , \quad (10)$$

with

$$\begin{aligned}
&\omega_R=2\gamma^2ck_0/(1+pK^2/2)\ ,\\
&\mu_q=L[k_q+k_0)-\omega_q/v_0]\ ,\\
&\Lambda=\pi pKN/\gamma^2\ ,\\
&R_{jq}=\mu_q\tau-\omega_q t_{j0}+\phi_q\ ,\\
&\psi_{jq}=\exp(-iL\omega_q\xi_j/c)\ .
\end{aligned} \quad (11)$$

The parameter μ_q measures the detuning from the resonant frequency ω_R.

Using Eqs. (9) and (11) we can write the current amplitudes of Eq. (8) as

$$J_q=\sum_i J_q^{(i)}iJ_0(\omega_R/\omega_q\rho_LL_0)\sum_j \psi_{jq}(\tau)e^{-iR_{jq}(\tau)}\ ,\qquad J_0=\frac{e^2c\mu_0L\rho K}{4m\gamma\omega_R}\ , \quad (12)$$

where we have indicated an expansion in powers i of the radiation field contained in the $\xi_j(\tau)$.

IV. PERTURBATION

For the purpose of performing a perturbation expansion, it is convenient to introduce new operators defined by

$$\begin{aligned}
&\psi_{jq}(\tau)=U_{jq}(\tau,\tau_0)\psi_{jq}(\tau_0)\ ,\\
&i\partial_\tau U_{jq}(\tau,\tau_0)=H_{jq}U_{jq}(\tau,\tau_0),\quad H_{jq}=(L/c)\omega_q\xi_j'\ ,
\end{aligned} \quad (13)$$

where $\xi_j'=d\xi_j/d\tau$. U_{jq} and H_{jq} are related in the same way as the evolution and Hamiltonian operators in the interaction representation of quantum mechanics, the evolution variable being τ. Thus, $\psi_{jq}(\tau)=U_{jq}(\tau,0)$ is given by the familiar expansion[5]

$$\psi_{jq}(\tau)=1-i\int_0^\tau dt_1H_{jq}(\tau_1)-\int_0^\tau d\tau_1\int_0^{\tau_1}d\tau_2H_{jq}(\tau_1)H_{jq}(\tau_2) +i\int_0^\tau d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3H_{jq}(\tau_1)H_{jq}(\tau_2)H_{jq}(\tau_3)+\cdots\ . \quad (14)$$

With the use of Eq. (10) the H's can be written as

$$H_{jq}(\tau)=(L/c)\omega_q\xi_j'=-\frac{L}{c}\omega_q\Lambda\sum_b(\omega_b/\omega_R)\int_0^\tau d\tau_1[ie^{-iR_{jb}(\tau_1)}\psi_{jb}(\tau_1)a_b+\text{c.c.}]\ . \quad (15)$$

From Eqs. (14) and (15) it is straightforward to obtain expansions in powers of the radiation field. It is convenient to start with ψ:

$$\psi_{jq}=1+\sum_{i=1}\psi_{jq}^{(i)},\quad \psi_{jq}^{(1)}=2i(L\omega_q\Lambda/c)\sum_b(\omega_b/\omega_R)\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2 a_b\sin R_{jb}(\tau_2)\ , \tag{16}$$

$$\psi_{jq}^{(2)}=4(L\omega_q\Lambda/c)^2\omega_q\sum_{b,c}(\omega_b\omega_c/\omega_R^2)\left[i(\omega_b/\omega_q)\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2 a_b\cos R_{jb}(\tau_2)\int_0^{\tau_2}d\tau_3\int_0^{\tau_3}d\tau_4 a_c\sin R_{jc}(\tau_4)\right.$$
$$\left.-\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3 a_b\sin R_{jb}(\tau_3)\int_0^{\tau_3}d\tau_4 a_c\sin R_{jc}(\tau_4)\right]\ , \tag{17}$$

$$\psi_{jq}^{(3)}=8\left[\frac{L\Lambda\omega_q}{c}\right]^3$$
$$\times\sum_{b,c,d}\frac{\omega_b\omega_c\omega_d}{\omega_R^3}\left[i\frac{\omega_b\omega_c}{\omega_q^2}\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2 a_b\cos R_{jb}(\tau_2)\right.$$
$$\times\int_0^{\tau_2}d\tau_3\int_0^{\tau_3}d\tau_4 a_c\cos R_{jc}(\tau_4)\int_0^{\tau_4}d\tau_5\int_0^{\tau_5}d\tau_6 a_d\sin R_{jd}(\tau_6)$$
$$+\frac{\omega_c}{\omega_q}\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_1}d\tau_3 a_b\sin R_{jb}(\tau_3)$$
$$\times\int_0^{\tau_2}d\tau_4 a_c\cos R_{jc}(\tau_4)\int_0^{\tau_4}d\tau_5\int_0^{\tau_5}d\tau_6 a_d\sin R_{jd}(\tau_6)$$
$$+\frac{\omega_b}{\omega_q}\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_1}d\tau_3 a_b\cos R_{jb}(\tau_3)$$
$$\times\int_0^{\tau_3}d\tau_4\int_0^{\tau_4}d\tau_5 a_c\sin R_{jc}(\tau_5)\int_0^{\tau_2}d\tau_6 a_d\sin R_{jd}(\tau_6)$$
$$+i\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3\int_0^{\tau_1}d\tau_4 a_b\sin R_{jb}(\tau_4)$$
$$\left.\times\int_0^{\tau_2}d\tau_5 a_c\sin R_{jc}(\tau_5)\int_0^{\tau_3}d\tau_6 a_d\sin R_{jd}(\tau_6)\right]\ , \tag{18}$$

where $\omega_q=qc\pi/L_0$, $\omega_b=bc\pi/L_0$, etc., $b,c,\ldots$ are numbers of the same order as q. Using Eqs. (16)–(18) one can obtain the different terms in the power expansion of J_q of Eq. (12). The first-order term will be treated in Sec. V while the remainder of the paper will be devoted to the nonlinear effects.

Let us close this section with a computational detail. Keeping in mind that all frequencies are multiples of the fundamental $\omega_f=c\pi/L_0$, the sum over electrons can be performed in the continuum approximation using

$$\omega_f t_{j0}\to\theta,\quad \xi_j\to\xi(\theta),\quad R_{jq}(\tau)\to R_q(\theta,\tau),\quad \sum_j\to(\rho_L L_0/\pi)\int_{-\pi}^{\pi}d\theta\ . \tag{19}$$

The results reported below were obtained using this transformation.

V. LINEAR THEORY

The current to first order in the radiation field is obtained from Eqs. (12) and (16). It reads

$$J_q^{(1)}=(J_0/\rho_L L_0)\sum_j e^{-iR_{jq}(\tau)}\sum_b(\omega_b/\omega_R)\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2 a_b(\tau_2)\sin R_{jb}(\tau_2)\ . \tag{20}$$

After the replacements of Eq. (19) are made and the τ_1, τ_2, and θ integrations are performed, the first-order current amplitude is obtained. When this first order J_q is inserted into Eq. (8), the real and imaginary parts yield the following evolution equations for the amplitude and phase:

$$da_q(\tau)/d\tau=\tfrac{1}{2}(\Gamma_R-\alpha)a_q(\tau)\ ,$$
$$d\phi_q(\tau)/d\tau=\tfrac{1}{2}\Gamma_I\ , \tag{21}$$

where $\Gamma=\Gamma_R+i\Gamma_I$ and

$$\left.\begin{matrix}\Gamma_R\\ \Gamma_I\end{matrix}\right\} = G_0/\mu^2 \begin{cases}[\sin(\mu\tau)-\mu\tau\cos(\mu\tau)]\\ [\cos(\mu\tau)-1+\mu\tau\sin(\mu\tau)]\ ,\end{cases}$$

$$G_0 = \pi e^2 \rho p K^2 L^2 N / 2\epsilon_0 m c^2 \gamma^3\ . \quad (22)$$

The integral of Γ_R over the undulator length

$$\tilde{\Gamma} = \int_0^1 d\tau\, \Gamma_R = \frac{e^2}{8\epsilon_0 m c^2}\left[-\frac{d}{d(\mu/2)}\left[\frac{\sin(\mu/2)}{\mu/2}\right]^2\right]\frac{p\rho k_0 K^2 L^3}{\gamma^3}\ , \quad (23)$$

is simply related to the small signal gain per pass by $G_s = \exp\tilde{\Gamma} - 1$, while Γ_I yields the linear frequency pulling.

VI. NONLINEAR THEORY

We will comment on the second-order contribution arising from Eq. (17) after discussing the third-order terms which are obtained after inserting Eq. (18) for ψ_{jq} in Eq. (12). When the transformation (19) is performed, the following type of factor has to be calculated:

$$\cdots \sum_{b,c,d} a_b a_c a_d e^{-i(\mu_q \pm \mu_b \pm \mu_c \pm \mu_d)\tau} e^{-i(\phi_q \pm \phi_b \pm \phi_c \pm \phi_d)} \times \int_0^{2\pi} (d\theta/2\pi) e^{i(q \pm b \pm c \pm d)\theta} \cdots\ . \quad (24)$$

The nonvanishing terms can be classified into two types. In the first type 1, at least one of the modes b, c, or d is equal to q, while in the second type 2, the modes b, c, and d are all different from q. Let us now proceed to a discussion of these two types.

(1) When one of the modes (b, c, or d) is equal to q, the other two have to be equal to each other. Let us look at the signs in the exponents of Eq. (24). One of the signs is plus while the other two have to be minus, giving us three sign possibilities. For each one of these possibilities, there are still two cases: (a) One arrangement in which all modes are equal and (b) two arrangements in which one of the (two) modes with minus sign is equal to q while the other two modes are different from q but equal to each other. This two-to-one ratio in favor of (b) over (a), resulting in a crossed saturation twice as large as self-saturation, will prove to be very important for the operation of a long-pulse FEL.

According to the above discussion (or the more detailed analysis in the Appendix), the cases (a) and (b) of 1, contribute to the current amplitude J_q of Eq. (20) with the third-order term

$$-\tfrac{1}{2} s(\mu_q,\tau) a_q \left[a_q^2 + 2\sum_{b(\neq q)} a_b^2\right]\ , \quad (25)$$

where $s = s_R + i s_I$, simply related to the saturation function $S = S_R + i S_I$ to be introduced later, is given by

$$s_R(\mu,\tau) = \sigma[\,3x + (1/8 - 1.25x^2)\sin x + (1.5 - x^2/8)\sin(2x) - x(3.13 + x^2/24)\cos x - \cos(2x)]/\mu^6\ ,$$

$$S_I = \sigma[\,-3 - 0.75x^2 + (1.5 - 1.25x^2)\cos x - (3.88 + x^2/24)x\sin x + x\sin(2x) + (1.5 - x^2/8)\cos(2x)]/\mu^6\ ,$$

$$x = \mu\tau,\quad \sigma = \frac{e^2 \rho K L}{\epsilon_0 m c \gamma \omega_q}\left[\frac{L\Lambda\omega}{c} q\right]^3\ . \quad (26)$$

Equation (18) is rather complicated and a large number of integrations have to be performed in deriving Eqs. (25) and (26). In order to simplify things, s in these equations was obtained under the assumption that all μ's are approximately equal. This assumption is consistent with our purpose of analyzing, in this paper, the interaction among near modes. Equation (25) represents a "true" third-order interaction to be discussed later.

(2) Equation (18) also contributes to the current with terms in which all three frequencies are different as in

$$\sum_l \left[\sum_m C_{qlm} a_{q+m} a_{q-m-l}\right] a_{q+l}\ .$$

Summed over the amplitudes $a_{q+m} a_{q-m-l}$, this part of the current functions as an effective mixing between the modes q and $q+l$. The second-order current also produces a two-mode mixing. When all such small interactions are added, we end up with an effective coupling of the form

$$\sum_l g_{q,q+l} a_{q+l}(\tau)\ , \quad (27)$$

in the right-hand side of Eq. (7).

The general multimode problem is complicated by the fact that, besides the saturation terms (24), we have to deal with couplings like (27) which are responsible for mode mixing. In the remainder of this paper we will restrict our treatment to one and two modes, in which case that type of mixing does not occur. Such a simplification is useful since the mechanism responsible for the narrow frequency bandwidth can already be discovered in the two-mode problem.

VII. NONLINEAR SINGLE-MODE PROBLEM

When only a single mode is present, the second term inside the large parentheses in Eq. (25) is, of course, absent. Adding the remaining term to Eqs. (21), one ends up with the single-mode evolution equations which, up to third order in the radiation field, read

$$da_q(\tau)/d\tau = \tfrac{1}{2}[\Gamma_R(\mu_q,\tau) - \alpha] a_q(\tau) - \tfrac{1}{2} s_R(\mu_q,\tau) a_q^3\ ,$$
$$d\phi_q(\tau)/d\tau = \tfrac{1}{2}[\Gamma_I(\mu_q,\tau) - s_I(\mu_q,\tau) a_q^2]\ . \quad (28)$$

Since in this section we are dealing with a single mode, we will drop the subindex q. The intensity or power density I is, of course, simply related to the amplitude by

$$I = C a^2,\quad C = \tfrac{1}{2} c\epsilon_0 (mc/e)^2 \omega^2\ . \quad (29)$$

From the first of Eqs. (28) and (29) we see that the intensity evolves during a pass in accordance with

$$dI(\tau)/d\tau = I(\tau)[\Gamma_R(\tau) - \alpha - S_R(\tau)I(\tau)] \, , \quad S = S_R + S_I = (s_R + s_I)/C \, , \tag{30}$$

with C as in Eq. (29), and s_R and s_I as in Eq. (26). The solution of Eq. (30) relates $I(\eta+1)$ at the end of pass $\eta+1$ with $I(\eta)$ at the beginning by

$$I(\eta+1) = I(\eta)e^{\Gamma_c}/[1 + I(\eta)S_c] \, . \tag{31}$$

In this equation Γ_c is

$$\Gamma_c = \tilde{\Gamma} - \alpha = G_0 f(\mu)/\mu^3 - \alpha, \quad f(\mu) = 2(1 - \cos\mu) - \mu \sin\mu \, , \tag{32}$$

and

$$S_c = \int_0^1 d\tau S_R(\tau) \exp \int_0^\tau d\tau' [\Gamma_R(\tau') - \alpha] \approx \frac{2G_0}{\epsilon_0} \left[\frac{eL\Lambda}{mc} \right]^2 \frac{F(\mu)}{\mu^7} \, , \tag{33a}$$

$$F(\mu) = 6.53 + \mu^2/2 - (5.50 - 1.13\mu^2)\cos\mu - (5.38 + \mu^2/24)\mu \sin\mu + (\mu^2/16 - 1.03)\cos(2\mu) - 0.56\sin(2\mu) \, . \tag{33b}$$

The expression after $\approx$ in Eq. (33a) is valid for small Γ_c.

Up to now we have studied the evolution of the mode amplitudes (or intensities) for an amplifier, or within a single pass of an oscillator. It is a simple matter, however, to iterate Eq. (31) over passes and obtain for I at pass η the expression

$$I(\eta) = I_0 e^{\eta\Gamma_c}/[1 + I_0 S_c (e^{\eta\Gamma_c} - 1)/G_L] \, , \quad I_0 \equiv I(\eta=0), \quad G_L = e^{\Gamma_c} - 1 \, . \tag{34}$$

G_L is, of course, the small signal gain minus losses per pass. For low-gain ($\exp\Gamma_c - 1 \approx \Gamma_c$), I as given by Eq. (34) coincides with the solution of the differential equation

$$dI(\eta)/d\eta = I(\eta)[\Gamma_c - S_c I(\eta)] \, . \tag{35}$$

From the foregoing equations it is easy to see that the third-order theory leads to the limiting intensity

$$I_\infty \equiv I(\infty) = G_L/S_c \, . \tag{36}$$

For small Γ_c, this limiting intensity can be approximated by

$$I_\infty \approx \frac{\Gamma_c}{S_c} = \left[\frac{\epsilon_0 c^3}{2\pi^2} \left[\frac{mc}{e} \right]^2 \right] \left[\frac{f(\mu)}{F(\mu)} \right] \mu^4 \left[\frac{\gamma^2}{pKLN} \right]^2 \, . \tag{37}$$

The factor in large square brackets is 3.518×10^7 W and the function inside the large parentheses is 1.55 for $\mu = 2.6$, at the maximum of the small signal gain.

With the general definition of the gain per pass G as the ratio of the intensity change to the initial intensity, from Eqs. (31) and (37) one derives

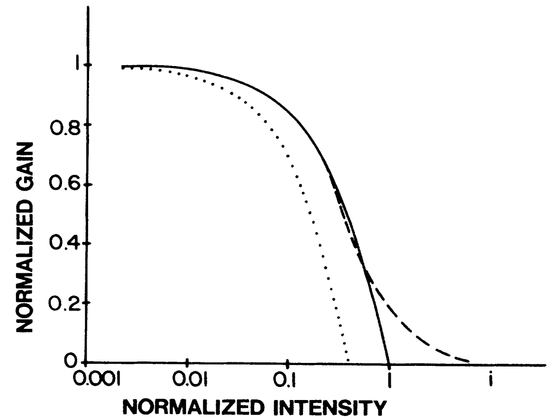

FIG. 1. Normalized gain as a function of normalized intensity. (a) Solid line: Gain for a single mode, or for the dominant mode 1 in the two-mode problem, derived from third-order perturbation [Eq. (38)]. The abscissa is $i = I_1/I_{1\infty}$, where I_1 is the intensity of the single or dominant mode and $I_{1\infty}$ its third-order limiting intensity. (b) Dashed line: One-dimensional computer simulation for the normalized gain of a single mode. $i = I/I_{1\infty}$, with $I_{1\infty}$ the same as in (a). (c) Dotted line: Gain for the weaker mode 2 in the two-mode problem as a function of $i = I_1/I_{1\infty}$.

$$G = \frac{G_L(1-i)}{1 + G_L i}, \quad i = I/I_\infty \tag{38}$$

valid up to $I = I_\infty$. The solid curve in Fig. 1 shows a graphical representation of the normalized gain $G(i)/G(0)$ for $\Gamma_c = 0.5$, obtained from the third-order Eq. (38), while the dashed curve is the result of a one-dimensional computer simulation. The third-order theory is seen to give a good account of the behavior of a low-gain FEL except for the deep saturation region (the tail of the dashed curve). The dotted curve in the figure will be discussed in Sec. VIII.

VIII. TWO-MODE PROBLEM

In a FEL with short electron pulses, the spatial structure of the electron pulse couples (locks) a large number of modes. For long pulses, on the other hand, this locking does not exist and the only cold mode mixing comes from small terms of the type of Eq. (27). Since the g in those equations are small, the loaded resonator modes do not depart much from the cold modes. Moreover, much about the more general multimode situation can be learned from the two-mode problem to which we now devote our attention.

Let us consider a situation with two modes having almost equal detuning μ and gain minus losses Γ_R. Due to the interaction of Eq. (24), the power densities $I_1 = Ca_1^2$ and $I_2 = Ca_2^2$, where C is given in Eq. (29), satisfy the system of coupled equations

$$dI_1/d\tau = I_1[\Gamma_R - S_R(I_1 + uI_2)] \, , \quad dI_2/d\tau = I_2[\Gamma_R - S_R(I_2 + uI_1)] \quad (u=2) \, . \tag{39}$$

In a FEL the above equations hold with $u=2$, but it is worthwhile to continue the discussion with a generic u.

As it was already remarked, the fact that $u > 1$ (crossed saturation larger than self-saturation) is extremely important for the behavior of a long-pulse FEL. The saturation function S_R in Eq. (39) is related, through the second of Eq. (30), to s_R of Eq. (26) which was obtained with the simplification of taking the same detuning for both modes. This is a good approximation for nearby modes.

Since for a single mode the intrapass Eq. (30) leads to the interpass Eq. (35), it is only natural to assume that corresponding to Eqs. (39), the two-mode interpass evolution is described by

$$\begin{aligned} dI_1/d\eta &= I_1(\eta)[\Gamma_c - S_c(I_1+uI_2)] , \\ dI_2/d\eta &= I_2(\eta)[\Gamma_c - S_c(I_2+uI_1)] . \end{aligned} \tag{40}$$

It is convenient now to transform to a polar system according to

$$\begin{aligned} a_1(\eta) &= r(\eta)\cos\Theta(\eta) , \\ a_2(\eta) &= r(\eta)\sin\Theta(\eta) , \\ R &= r^2 . \end{aligned} \tag{41}$$

With this, the system of Eqs. (39) turns into

$$\begin{aligned} dR/d\eta &= R(\eta)(\Gamma_c - S_cR(\eta)\{1+\tfrac{1}{2}(u-1)\sin^2[2\Theta(\eta)]\}) , \\ d\Theta/d\eta &= -(\tfrac{1}{8})(u-1)S_cR(\eta)\sin[4\Theta(\eta)] . \end{aligned} \tag{42}$$

At (or near) saturation the system is described by stable solutions (with constant Θ) of the Eqs. (42). The boundary between weak and strong coupling is $u=1$ for which any Θ is possible. For other u the only equilibrium points are $\Theta=0$, $\pi/4$, and $\pi/2$. For weak coupling ($u<1$), $\Theta=0$, $\pi/2$ are unstable while $\Theta=\pi/4$ is the stable point where the system saturates with equal amounts of the two modes. On the other hand, for strong coupling ($u>1$, case of the FEL with $u=2$), $\Theta=\pi/4$ is unstable while $\Theta=0$ (pure mode 1) and $\Theta=\pi/2$ (pure mode 2) are the only stable equilibrium points. Thus near saturation a strongly interacting system like the FEL operates at a single mode which evolves according to Eq. (34). We will refer to the analysis based on the transformation (41) as the "polar" method.

It is instructive to corroborate the above results by a different method. Since no analytical solution of Eqs. (40) with $u=2$ is known, it is useful to analyze exactly soluble cases with different values of u and try to extrapolate. For $u=0$ (no coupling) each mode grows independently of each other and the intensities behave as

$$I_i(\eta) = I_{i0}\Gamma_c e^{\Gamma_c\eta}/[\Gamma_c + I_{i0}S_c(e^{\Gamma_c\eta}-1)] \qquad (u=0,\quad i=1,2) . \tag{43}$$

It is easy to see that in this case the system saturates with equal amounts of I_1 and I_2. As we saw from the polar method this is true in general for $u<1$.

Another case in which an exact analytic solution is readily found is for $u=1$. It reads

$$\left.\begin{matrix} I_1(\eta) \\ I_2(\eta) \end{matrix}\right\} = \left\{\begin{matrix} I_{10} \\ I_{20} \end{matrix}\right\} \frac{\Gamma_c e^{\Gamma_c\eta}}{\Gamma_c + (I_{10}+I_{20})S_c[\exp(\Gamma_c\eta)-1]} , \qquad u=1 . \tag{44}$$

In this case, at saturation $I_1/I_2 = I_{10}/I_{20}$, in agreement with the polar method.

Let us for the moment assume that even in the strong coupling ($u>1$) situation the dominant mode has, in analogy with Eq. (43), an intensity given by

$$I_1(\eta) = I_{10}\Gamma_c e^{\Gamma_c\eta}/[\Gamma_c + I_{10}S_c(e^{\Gamma_c\eta}-1)] . \tag{45}$$

Then, the intensity of the other mode is a solution of

$$\frac{dI_2}{d\eta} = I_2\left[\Gamma_c - S_c\left[I_2 + \frac{u\Gamma_c I_{10}e^{\Gamma_c\eta}}{\Gamma_c + I_{10}S_c(\exp\Gamma_c\eta - 1)}\right]\right] . \tag{46}$$

Keeping in mind that the I's are positive definite, let us introduce a new function $F(\eta)$ that satisfies

$$dF(\eta)/d\eta = F(\eta)\left[\frac{\Gamma_c - uS_c\Gamma_c I_{10}e^{\Gamma_c\eta}}{\Gamma_c + S_cI_{10}(e^{\Gamma_c\eta}-1)}\right] , \tag{47}$$

with the initial condition $F(0)=I_{20}\equiv I_2(0)$. It is easy to see that for positive I_2 and F we have

$$I_2(\eta) \le F(\eta) = \frac{I_{20}e^{\Gamma_c\eta}}{[1+S_cI_{10}(e^{\Gamma_c\eta}-1)/\Gamma_c]^u} . \tag{48}$$

Thus, in the FEL strong coupling ($u=2$) situation $I_2 \to 0$ for $\eta\to\infty$. This can be understood by looking at the second of Eqs. (40). For I_1 larger than I_2, the effective gain exponent for I_2 is $\Gamma_{\text{eff}} = (\Gamma_c - 2S_cI_1)$ which turns negative as I_1 becomes sufficiently large. Thus, while growing, the dominant mode suppresses the other mode.[6,7] It is also clear that for larger η the intensity I_1 of the dominant mode behaves very much as in Eq. (45), since in the first of Eq. (40), I_2 becomes negligible. Then, the gain for the dominant mode 1 is practically the same as in the single-mode case. As discussed in Sec. VII, in the third-order theory this gain is given by Eq. (38) with $i = I_1/I_{1\infty}$, and represented by the solid curve in Fig. 1.

For a given $I_1 > I_2$ we can solve the second of Eqs. (40) and obtain the gain for the intensity I_2 of the weaker mode 2 as

$$G_2 = \frac{e^{\Gamma_c - 2G_Li}}{1+(I_1/I_2)G_Li} - 1 \quad (i = I_1/I_{1\infty}) . \tag{49}$$

This third-order weak-mode gain is represented, for $\Gamma_c = 0.5$, by the dotted curve in Fig. 1. As can be seen, at the point $i=0.38$ where the dominant-mode gain was decreased to half its maximum by the self-saturation, the crossed saturation has already reduced to zero the gain of the weak mode.

The third-order theory is expected to give a reasonable account of the behavior of a low-gain long-pulse FEL operating outside the deep saturation region (tail of the dashed curve in Fig. 1). With sufficiently high losses the FEL operation can be kept outside that deep saturation region. Under those conditions the following picture holds: The strong crossed saturation among modes (for near modes it is twice the self-saturation) results in an intense competition whereby the dominant mode is able to reduce the effective gain of other nearby modes to the point of extinction. The outcome is operation at a single mode.

For FEL's operating inside the deep saturation region, terms of higher order (than the third) in the radiation field have to be taken into account. These, as well as the effects they produce (sideband instabilities,[8-11] for instance), are outside the scope and intent of the present paper.

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APPENDIX: EXPLICIT COUNT OF THE RELEVANT MODE ARRANGEMENTS IN THE THIRD-ORDER COUPLING

The mode behavior of a long-pulse FEL depends crucially on the structure of Eq. (25). Thus, it might be worthwhile to show explicitly which terms of Eq. (24) actually contribute. The θ integral in Eq. (24) imposes, of course, the equality $q=\pm b\pm c\pm d$. This yields the eight cases

$$q=-b-c-d\ , \tag{A1}$$

$$q=b+c+d\ , \tag{A2}$$

$$q=-b-c+d\ , \tag{A3}$$

$$q=-b+c-d\ , \tag{A4}$$

$$q=-b+c+d\ , \tag{A5}$$

$$q=b-c-d\ , \tag{A6}$$

$$q=b-c+d\ , \tag{A7}$$

$$q=b+c-d\ . \tag{A8}$$

(A1) cannot, of course, be satisfied, while case (A2) with $b=c=d=p=q/3$ will be disregarded on grounds that this gives a p too far away from q and outside the gain bandwidth.

For the type 1(a) terms in which $b=c=d=q$, only cases (A5), (A7), and (A8) are possible giving a factor of 3.

Let us now consider type (b) terms in which among b,c,d there is at least one q and one $p\neq q$. Here again, only cases (A5), (A7), and (A8) are possible. The common feature of these cases is that one of the signs is minus and the other two are plus. Let us consider one of these cases, say (A8). If $d=q\rightarrow b=c=q$, we are back in type (a). For (b) then, $d=p$ leading to $q=b+c-p$ which is satisfied in the following two arrangements:

$$d=p,\quad \begin{cases} b=q, & c=p \\ b=p, & c=q\ . \end{cases} \tag{A9}$$

A similar analysis holds for the other cases (A5) and (A7) which yield the arrangements

$$b=p,\quad \begin{cases} c=q, & d=p \\ c=p, & d=q\ , \end{cases} \tag{A10}$$

$$c=p,\quad \begin{cases} b=q, & d=p \\ b=p, & d=q\ . \end{cases} \tag{A11}$$

The conclusion is that for type 1(b), three cases contribute with two arrangements each for a total factor of 6 [as compared to 3 for type 1(a)]. This is the origin of the 2 to 1 weights in the terms inside the large parentheses of Eq. (25). The common factor (3) was absorbed in the function $s(\mu,\tau)$ of that equation.

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