PHYSICAL REVIEW A VOLUME 35, NUMBER 1 JANUARY 1, 1987

$1/f$ noise, log-normal distribution, and cascade process in electrical networks

Shoichi Kai, Seiji Higaki, and Masahide Imasaki
Department of Electrical Engineering, Kyushu Institute of Technology, Tobata, Kitakyushu 804, Japan

Hiroshi Furukawa
Institute of Physical Sciences, Faculty of Education, Yamaguchi University, Yamaguchi 753, Japan

(Received 23 June 1986)

The relation between the log-normal distribution and the $1/f$ noise is experimentally studied by use of cascade electrical networks. Results show that the $1/f$ spectrum can be obtained only in networks consisting of resistances and inductances, whereas the log-normal distribution for bifurcated currents is always accompanied for cascade networks in general. The log-normal distribution is observed even in the case where parts of branches of cascade networks are coupled to each other.

I. INTRODUCTION

The so-called $1/f$ noise is observed universally in nature. A noise in a semiconductor such as a flicker noise and a fluctuation in a traffic flow on a highway are examples.[1–4] Mechanisms of these phenomena still remain unsolved. A simple realistic model may help us to understand the mechanisms of these unsolved phenomena. Recently one of us (HF) proposed a cascade model for a crystal growth, which can be also applied to motions of electrons in conductor.[5,6] In the latter case, a $1/f$ noise spectrum is obtained. It is also well known that the self-similar cascade process of bifurcation of physical quantities is closely related to the log-normal distribution.[7] A typical and well-known example is the cascade-energy transfer of eddies in a turbulence, the so called log-normal model by Kolmogorov.[8,9] Intermittent chaos in nonlinear dynamical systems also shows $1/f^{\nu}$ spectra.[10–13] These problems are subjects in different fields of science and engineering, that is, the fluid mechanics[8,11] and the chaotic dynamics,[10] the statistical mechanics,[12,13] the crystal growth kinetics,[14] the electronics, and the communication engineering.[15,16] The aim of this article is to study the cascade electric networks experimentally in order to understand these problems from a unified viewpoint. We expect to find a universal relationship among the cascade process, the log-normal distribution, and the $1/f$ noise.

Several kinds of electrical networks generating $1/f$ noise have been studied. Most of them are infinite networks with self-similar structure, such as an infinite ladder, an infinite lattice circuit consisting of resistances (R) and inductances (L) or capacitances (C).[15,16] One cannot, however, examine the log-normal distribution by use of those networks. We now realize the bifurcating cascade networks which consist of R and L. This network corresponds to the one used for a simulation of crystal growth proposed by Furukawa.[5,6]

II. THEORETICAL AND EXPERIMENTAL

A. Theory

Networks used in the present study are shown in Fig. 1. Figures 1(a), 1(b), and 1(c), respectively, show an R-R cascade of seven bifurcating steps ($n=7$) and an R-L cascade of eight steps ($n=8$) without and with coupling among bifurcating branches. In Fig. 1(c), the strength of the coupling is effectively controlled by the magnitude of R_L. The bifurcating ratio of the current amplitude $|I|$ is $\xi:1-\xi$ where ξ is Z_B/Z_A (inverse ratio of the impedance). Denoting $|I_0|$ as an initial current at the zeroth step of the cascade process, the current $|I_1|$ after the first bifurcation of the process can be given by $\xi|I_0|$ and $(1-\xi)|I_0|$ (see Fig. 2), respectively. Repeating this procedure until jth step of the cascade, the current $|I_j|$ at a branch of jth step can be expressed as

$$|I_j|=\xi|I_{j-1}|=\xi(1-\xi)|I_{j-2}| = \cdots = \xi^m(1-\xi)^{j-m}|I_0| . \quad (1)$$

Here $|I_0|$ is constant and m $(1\le m\le j)$ a variable depending on the path through the current flow. The logarithm of I is

$$\ln|I_j|=m\ln\xi+(j-m)\ln(1-\xi)+\ln|I_0| . \quad (2)$$

The number of branches at the jth step of the cascade is 2^j. Since the variable m must normally distribute, the $\ln|I_j|$ also normally does and therefore I_j distributes log normally:

$$N(|I_j|)=(N_0/|I_j|)\exp(-\{[\ln(|I_j|/|\bar{I}_j|)]^2/2\sigma_j^2\}) . \quad (3)$$

Here N_0, $|\bar{I}_j|$, and σ_j are the normalization constant, the mean value of $|I_j|$ $[=(1/2^j)\sum_i |I_j|_i]$, and the variance at jth step cascade, respectively.

As already described by Furukawa,[5,6] the total impedance Z_∞ is given by

$$Z_\infty=(Z_AZ_B)^{1/2} \quad (4)$$

for infinite cascade ($n=\infty$) (Refs. 5 and 6). Therefore, when we choose $Z_A=R$ and $Z_B=i\omega L$, the current power $P(\omega)$ is expressed as

$$P(\omega)\propto|Y(\omega)^2|=1/\omega LR . \quad (5)$$

Here $\omega=2\pi f$. Namely, the power spectrum of the current through the infinite cascade is proportional to

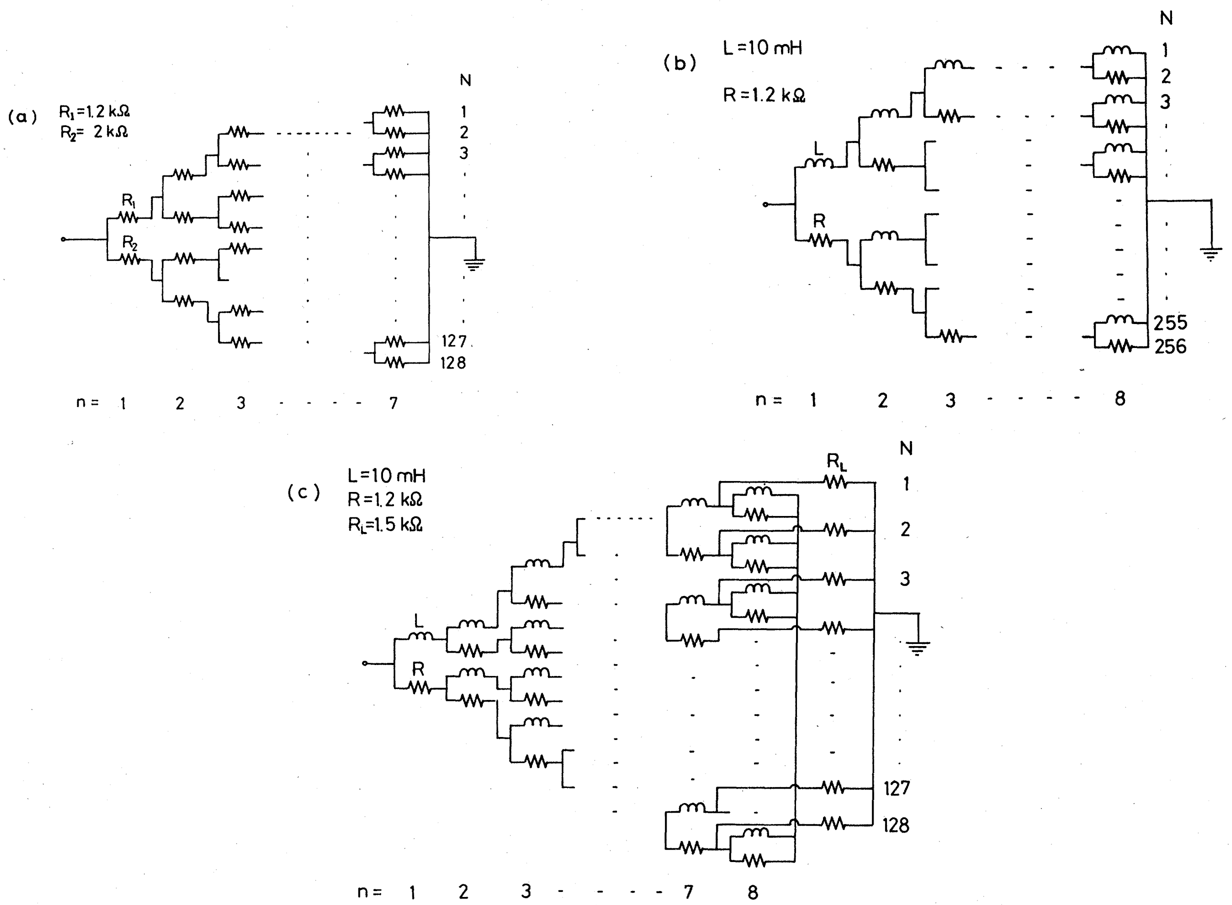

FIG. 1. Cascade networks used in the present study. (a) R-R cascade networks with seven steps of bifurcation ($R_1=1.2$ kΩ and $R_2=2$ kΩ). (b) R-L cascade networks with eight steps of bifurcation ($L=10$ mH, $r_L=21$ Ω, and $R=1.2$ kΩ). (c) R-L cascade networks with coupling (R_L) among branches ($L=10$ mH, $r_L=21$ Ω, $R=1.2$ kΩ, and $R_L=1.5$ kΩ).

f^{-1}. For a finite step ($n<\infty$), the $1/f$ spectrum must be observed in the range

$$R/4n^2L<\omega<4n^2R/L \ , \tag{6}$$

and therefore the band width is given by

$$\Delta \ln\omega \sim 2\ln 4+4\ln n \ . \tag{7}$$

Also the variance of the distribution can be given by

$$\sigma^2=(n/16)[\ln(\omega L/R)]^2 \ . \tag{8}$$

For the R-L cascade networks, we used the inductance $L=10$ mH with the internal resistance $r_L=21$ Ω and the resistance $R=1.2$ kΩ.

To keep Eqs. (6) and (7) valid, ω must be much higher than $(r_L/L)=2\times10^3$ sec^{-1}, that is, the low cutoff frequency is about 330 Hz which makes the band width of $1/f$ spectrum narrower. The current distribution is not sensitively influenced by this narrowing of the band width because it is essentially related to the short time correlation. Contrary to this, the power spectrum is strongly influenced.

B. Experiments and results

The experiment was performed using a signal generator, a computer-controlled fast Fourier transform (FFT) analyzer (Iwatsu SM 2100 A), and a white-noise generator (N.F. WG-722) whose statistical property can be varied, for example, a binomial, a Gaussian white noise, and so on. To obtain the distribution and the power spectrum, we use a single sinusoidal wave and a noise source, respectively. We took the distribution of the current amplitude of the branches at each step from the first to the final of the cascade. The power spectrum was obtained, on the other hand, from current fluctuations through the load resistance R_0 which connected all branches at the final step and was varied from 50 Ω to 1.5 kΩ. The variance σ was obtained as

$$\sigma^2=(1/N)\sum_{i=1}^{N}(|\bar{I}_j|-|I_j|_i)^2/|\bar{I}_j^2| \tag{9}$$

experimentally. Figure 3 shows the step dependence of the total impedance of the R-R cascade networks where Z_∞ (=620 Ω) is the value theoretically expected for the

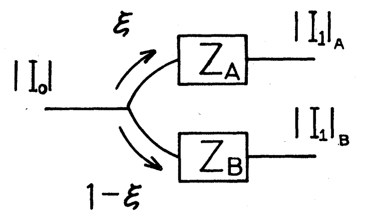

FIG. 2. Current flow at a bifurcating branch.

infinite cascade [Eq. (4)]. The value ($Z_5=605\ \Omega$) for $n=5$ already reaches that of the infinite cascade more than 97%. Therefore the networks with $n>5$ may be used to study a self-similar cascade.

The result obtained from the R-R cascade of seven steps of Fig. 1(a) is shown in Fig. 4. Figure 4(a) indicates the current amplitude distribution in branches at the sixth step ($j=6$), with a single sinusoidal wave current source. This is well fitted to the log-normal distribution. When the R-R bifurcation ratio of the cascade is close to 1:1, the distribution becomes δ-function-like. Applying white-noise current source I_0 to obtain the power spectrum of the current through the cascade networks, it is insensitive to the bifurcation ratio and shows the white spectrum independently of the bifurcating ratio [see Fig. 4(b)].

Figure 5 shows the result for the R-L cascade without coupling among branches. In this case, we have a log-normal distribution and a $1/f$ spectrum as expected from Eq. (5). Since $Z_B=\omega L$, the bifurcation ratio can be controlled by change of the frequency. The solid line in Fig. 5 shows the theoretical curve of the log-normal distribution. Figures 5(a), 5(b), and 5(c) show distributions of current flow amplitude at each branch in the seventh step, respectively, for $\omega<\omega_c$, $\omega=\omega_c$, and $\omega>\omega_c$. Here $f_c=19$ kHz at which $Z_A=Z_B$ and the distribution will be expected to be δ-function-like, but the observed one is slightly different from the theoretical expectation because of the scatters of elements (R and L). One can find that no significant change in the current distribution occurs when frequency changes. That is, the log normality of the distribution function is maintained (see Fig. 5). Figure 6 shows the frequency dependence of the variance σ. The

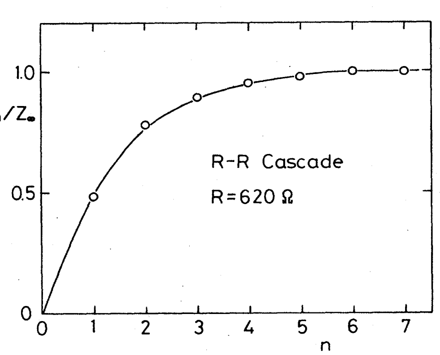

FIG. 3. Cascade-step dependence of total impedance (resistance). Here $R_1=R_2=620\ \Omega$, that is, $Z_\infty=620\ \Omega$ for infinite cascade.

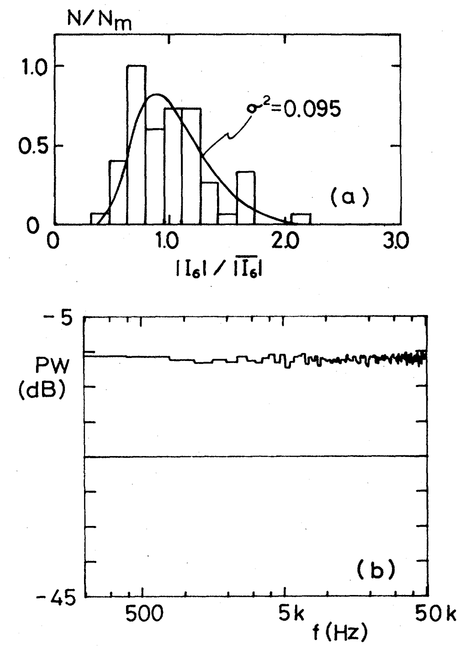

FIG. 4. Current amplitude distribution (a) and the power spectrum (b) of R-R cascade as shown in Fig. 1(a). The solid line shows a log-normal distribution with $\sigma^2=0.095$, $|\bar{I}_6|=7.72\ \mu$A.

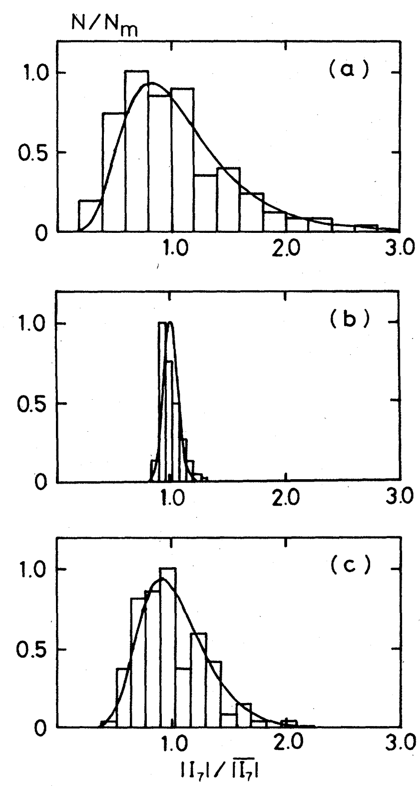

FIG. 5. Current amplitude distribution at the seventh step for the R-L cascade shown in Fig. 1(b). (a) $\omega<\omega_c$ ($f=10$ kHz). $\sigma^2=0.187$ (theoretical value), $|\bar{I}_7|=7.72\ \mu$A. (b) $\omega=\omega_c$ ($f=19$ kHz). $\sigma^2=0.004$ (best fitted value by eye), $|\bar{I}_7|=1.77\ \mu$A. (c) $\omega>\omega_c$ ($f=30$ kHz). $\sigma^2=0.081$ (theoretical value), $|\bar{I}_7|=1.44\ \mu$A. The solid lines are drawn with σ^2 given above.

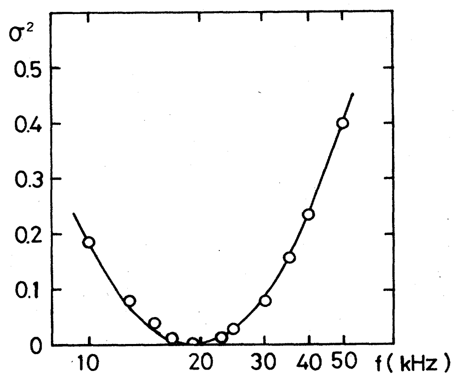

FIG. 6. Frequency dependence of the variance σ^2 for the eight-step R-L cascade. The data are taken from branches of the seventh step of the cascade ($j=7$, $n=8$). ○, experimental results; ——, theoretical results.

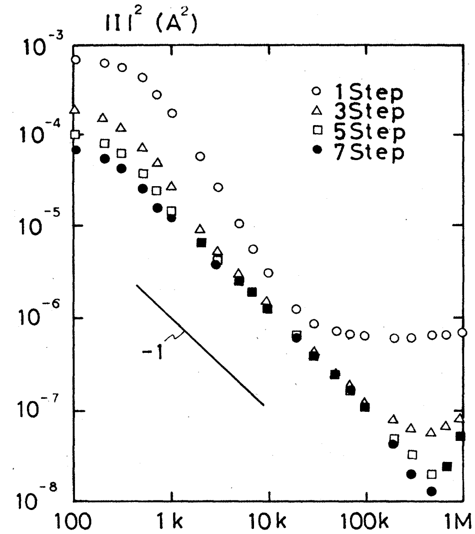

FIG. 7. Frequency dependence of the current power in the R-L cascade networks at a branch of $n=1$–7.

solid line demonstrates a theoretical curve obtained from Eq. (8). As expected from Eq. (8), σ^2 linearly varies with the step number n experimentally, such as $\sigma^2=0.08$, 0.135, and 0.187 at $f=10$ kHz, and $\sigma^2=0.037$, 0.062, and 0.081 at 30 kHz for $n=3$, 5, and 7, respectively. The band width showing the $1/f$ spectrum increases with the increase in the step number of the cascade (see Fig. 7). There occurs, however, narrowing of the band width $\Delta\omega$ for the low-frequency regime ($f<300$ Hz) due to the internal resistance of inductance (L) and for the high-frequency regime ($f>500$ kHz) due to the internal capacitance of L, even for large n. The agreement of $\Delta\omega$ with the theoretical prediction by Eq. (7) is very good up to five-steps cascade, but becomes poor with the increase in n by the reason described above. Figure 8 shows the

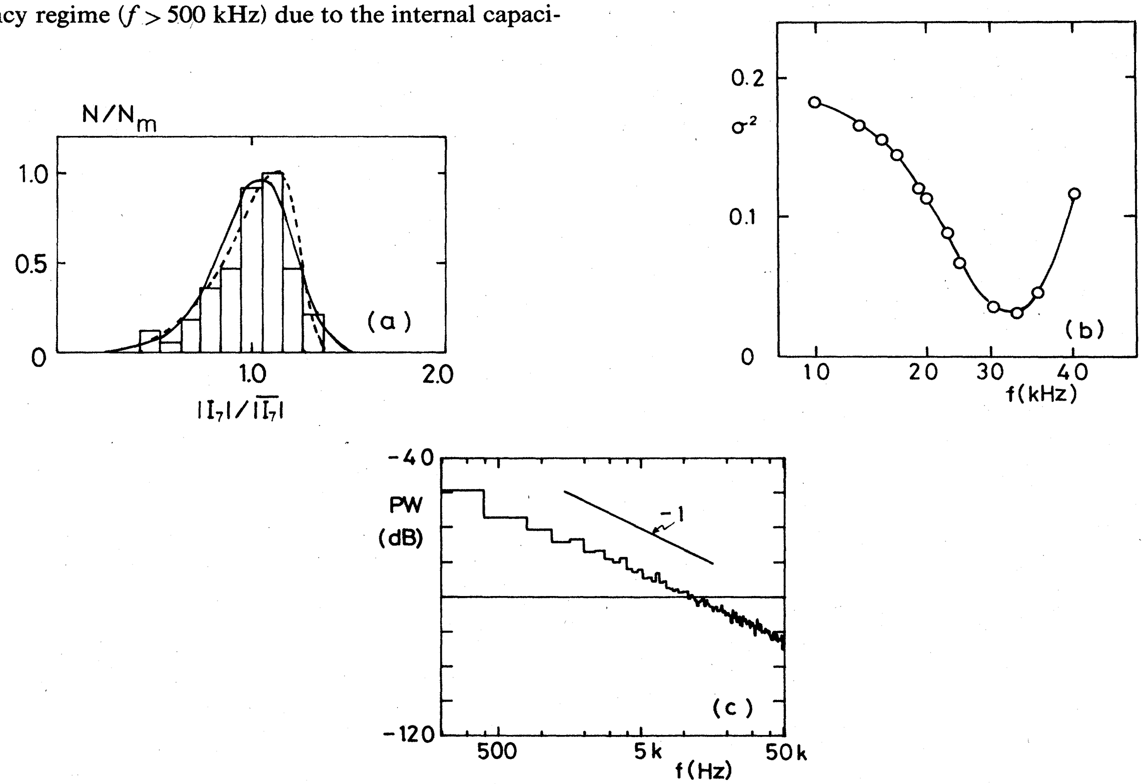

FIG. 8. (a) Current amplitude distribution, (b) frequency dependence of the variance, and (c) the power spectrum of the coupled R-L cascade. (a) The solid line shows the inversely plotted log-normal distribution with $\sigma^2=0.04$ ($f=40$ kHz). $|\bar{I}_7|=0.59$ μA and upper limit $|I_m|=1.21$ μA, and the dotted line shows the distribution expected from Lifshiz-Slyozov-Wagner theory (Ref. 20). (b) ○, the experimental result; ——, the best fitted line by eye. (c) The solid line shows $1/f$.

current distribution, the current variance, and the power spectrum at the cascade with coupling (R_L) among branches. With the increase in R_L from 20 Ω to 1.5 kΩ, the distribution deviates from the log-normal distribution and becomes the oppositely asymmetric distribution (with negative skewness) for all frequencies. For $R_L=1.5$ kΩ, we have tried to fit the distribution with an inversely plotted log-normal distribution where $|I|$ in Eq. (3) is replaced by $|I_m|-|I|$, where $|I_m|$ is the upper limit of the variables.[7] The result is shown in Fig. 8(a) by the solid line which gives good agreement with the inverse log-normal distribution. Frequency dependence of the variance, however, is quite different from those of noncoupling cases and shows more complicated features with the shift of f_c ($=32$ kHz) [Fig. 8(b)]. In addition, the distribution is not so sharp at f_c, which is clearly recognized as non-Gaussian profile. Nevertheless, the power spectrum shows a $1/f$ type profile similar to that of noncoupling cascade [Fig. 8(c)].

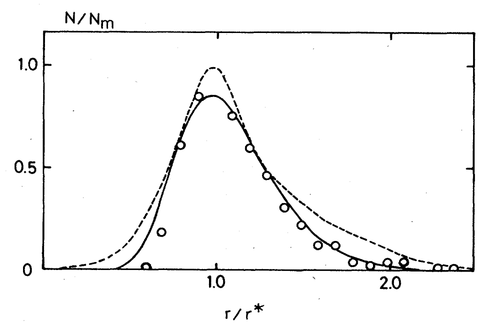

FIG. 9. Crystal size distribution of $NaNO_3$ in the aqueous solution. The fitting with the log-normal distribution (solid line) was done using $\sigma^2=0.073$. The dotted line shows the distribution curve expected from a second-order reaction controlled growth (SORG) proposed by Kahlweit (Ref. 20). N/N_m and $r/r*$ were used as the same values for data at $T=298$ K and $t=67$ h later in the original paper (Ref. 17).

C. Discussion

Furukawa proposed a cascade growth model of a crystal (grain) growth process and gave the possibility of the log-normal distribution in a grain size.[5,6] Some experimental facts show such a behavior.[17-19] For example, the experimental result, which is a crystal growth from supersaturated aqueous solution of $NaNO_3$ done by Hohman *et al.*[17] fits better with a log-normal distribution than the distribution expected from the second-order reaction kinetic growth (SORG) proposed by Kahlweit.[20] This result is shown in Fig. 9. The distribution as shown in Fig. 8 is also similar to Lifshiz-Slyozov-Wagner (LSW)—type distribution in a diffusion controlled growth process.[20] The coupling among current pieces increases through all branches of the eighth step in our networks when R_L increases. In the case of the crystal growth, it can be said that the increase in the coupling among grains is due to the increase in an interaction among crystals through the bulk. It may be said that the growth kinetics in polydispersed crystals is similar to the cascade process.[5,6] Similar aspects have been observed in formation processes (transient phenomena) of a dissipative structure in an electrohydrodynamics of nematic liquid crystals under application of Gaussian white noise in an electric field externally.[21] Thus it seems that the cascade process is observed in many phenomena and may have universal properties for nonequilibrium systems.

III. SUMMARY

The results obtained in the present study for the cascade process are as follows: (1) it shows a log-normal distribution at the independent cascade which has no coupling among branches of the process, (2) it shows inversely log-normal distribution when it has the coupling, and (3) it shows a $1/f$ spectrum if the bifurcating kinetics contain different properties such as L and R. We would like to stress that the cascade process can introduce not only log-normal but also inversely log-normal (or LSW-like) distributions.

ACKNOWLEDGMENT

This work was partially supported by the Yamada Science Foundation and by a Grant-in-Aid for Special Project Research from the Ministry of Education, Science, and Culture of Japan.

[1]A. van der Ziel, Solid-State Electron. **9**, 1139 (1966).
[2]A. van der Ziel, *Noise in Measurement* (Wiley, New York, 1976); R. D. Black, P. J. Restle, and M. B. Weissman, Phys. Rev. Lett. **51**, 1476 (1983).
[3]T. Musha, Physica (Utrecht) **80A**, 387 (1975).
[4]T. Musha and H. Higuchi, Jpn. J. Appl. Phys. **15**, 1271 (1976).
[5]H. Furukawa, Phys. Lett. **110A**, 316 (1985).
[6]H. Furukawa, Prog. Theor. Phys. **73**, 586 (1985); *ibid.* **73**, 1141 (1985).
[7]J. Aitchison and J. A. C. Brown, *The Lognormal Distribution* (Cambridge University, Cambridge, England, 1957).
[8]A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[9]H. G. E. Hentschel and I. Procaccia, Phys. Rev. A **28**, 417 (1983).
[10]Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980).
[11]U. Frisch and R. Morf, Phys. Rev. A **23**, 2673 (1981).
[12]Y. Aizawa, Prog. Theor. Phys. **72**, 659 (1984).
[13]H. Mori, B. C. So, and S. Kuroki, Physica (Amsterdam) **21D**, 355 (1986).
[14]H. Furukawa, Phys. Rev. A **31**, 4002 (1985).
[15]K. Motoishi and T. Koga, Trans. IECE **J65-A**, 237 (1982).
[16]S. Erfani, M. Ahmadi, V. Ramachandran, and M. O. Ahmad, Proc. ISCAS **85**, 1003 (1985).

[17]H. Hohman and M. Kahleweit, Ber. Bunsenges. Phys. Chem. **76**, 933 (1972).
[18]P. S. Sahni, D. J. Srolovitz, G. S. Grest, M. P. Anderson, and S. A. Safran, Phys. Rev. B **28**, 2705 (1983).
[19]P. Sherman, *Emulsion Science* (Academic, London, 1968), Ch. 3.
[20]For review, M. Kahlweit, Adv. Colloid Interface Sci. **5**, 1 (1975).
[21]S. Kai, S. Wakabayashi, and M. Imasaki, Phys. Rev. A **33**, 2612 (1986).