# Euclidean quantum mechanics

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Euclidean quantum mechanics is not limited to an analytical continuation in time from the Schrödinger equation to the heat equation. It is a new classical statistical theory founded on a new probabilistic interpretation of the heat equation and constitutes the closest classical analogy of quantum mechanics.

## I. INTRODUCTION

The Euclidean approach of quantum field theory has its origin in the works of Schwinger,[1] Nakano,[2] Symanzik,[3] and Nelson.[4] Initially, it was proposed as a technical tool, namely, a way to introduce the well-defined probabilistic methods of statistical mechanics in the Feynman formal (but physically "true") approach of field theory,[5-7] yet, gradually, from the seventies, it has been presented as a complete basis for the rigorous (but still physically inconclusive) study of this theory.[8,9]

In nonrelativistic quantum mechanics, the Euclidean theory is attributed to Kac.[10] It is summarized in the Feynman-Kac formula for the integral kernel of the semigroup $T_t = e^{-tH}$, namely, the analytic continuation which replaces $\tau$ by $-it$ in the quantum unitary group of evolution used by Feynman, $U_\tau = e^{-i\tau H}$, for the Hamiltonian $H = -\frac{1}{2}\Delta + V$.

In spite of the power of this Euclidean approach, due to the extensive use of the integral representation of the semigroup $T_t$, the physical meaning of such an analytical continuation is quite obscure. This explains the traditional resistance of theoretical physicists to any interpretation of quantum phenomena involving the classical diffusion processes usually associated with the heat equation (or Schrödinger equation "in imaginary time")

$$-\frac{\partial \theta^*}{\partial t} = H\theta^* \ . \tag{1.1}$$

This resistance is quite justified, because the kind of physical phenomena described by the heat (or diffusion) equation (1.1) is as different as possible from the quantum-type phenomena. In fact, all the qualitative features of any known classical statistical theory founded on Eq. (1.1) are absent from quantum theory. And the first one is that Eq. (1.1) is supposed to describe only irreversible phenomena.

So, from the point of view of theoretical physics, the Feynman-Kac formula is a formula looking for a theory. Nothing structurally analogous to quantum theory is associated with it. In this puzzling context, the method advocated by Fényes and Nelson[11-13] for associating classical diffusion processes (in real time) to the solutions of the Schrödinger equation itself,

$$i\frac{\partial \psi}{\partial \tau} = H\psi \ , \tag{1.2}$$

has a singular position. Its mathematical consistency did not wear down, in general, the above-mentioned resistance of the physicists. The resulting stochastic processes are apparently too different from the ones usually investigated in statistical mechanics. And first of all they are, of course, time symmetric (or reversible). Nevertheless, too many physicists still do not realize that the basic (formal) Feynman path-integral formulation of quantum mechanics has been profoundly inspired, indeed, by the analogy with classical diffusion processes and cannot be appreciated without having this reference in mind.

In any case, the discussion about the respective advantages of the imaginary-time or the real-time probabilistic approaches of nonrelativistic quantum mechanics may have important consequences for the rigorous construction of physical quantum fields, because the frame founded on the Feynman-Kac formula, in spite of its evident physical obscurities, is often considered as the only possible Euclidean version of quantum mechanics, and therefore the only possible starting point for a Euclidean program. The aim of the present work is to prove that this last assertion is basically incorrect. The genuine Euclidean quantum mechanics is far from being limited to an analytical continuation in time from the Schrödinger equation (1.2) to the heat equation (1.1). It is associated with a radically new probabilistic interpretation of the diffusion equation, whose dynamical structure is much closer to quantum theory than any other "classical" analogy. I say new, but its key kinematical idea has been discovered by Schrödinger in 1931 (Refs. 14 and 15) and forgotten by the theoretical physicists since then. It has been presented elsewhere, under the name of (Schrödinger's) "stochastic variational dynamics."[16,17] Here, it will be shown in particular that all the basic concepts introduced by Feynman for his path-integral formulation of quantum mechanics have their Euclidean analogs involving real, and well-defined, "average."

The interest of Euclidean quantum mechanics is twofold. First it may be a useful conceptual laboratory for constructive quantum field theory, suggesting in the most elementary case the nature of the difficulties one meets when constructing the involved stochastic process. Also, as the closest classical analogy of quantum mechanics it may help to clarify the physical foundations of this theory in an experimental context much easier, in principle, to control. The organization of this paper is the following.

 

Section II introduces Euclidean quantum mechanics (EQM) in its different aspects. First the experimental context, namely, the original Schrödinger *Gedankenexperiment.* Then a brief summary of the properties of the new class of diffusion processes involved in this construction (the Bernstein processes). Section II B refers to results recently published[16,17] and contains the necessary tools for the path-integral approach of EQM. Section II C introduces the Hilbert space $\bar{v}^*$ which is the Euclidean analogue of the usual quantum $L^2$ space. Section II D is devoted to the detailed presentation of the path-integral formulation of EQM, starting from a regularized version of classical (Euclidean) action. The laws of motion of the Bernstein diffusion process are found and a new path-integral representation for the involved solutions of the heat equation (1.1) is discussed. At the end of Sec. II D the correspondence between the key concepts introduced by Feynman in his path-integral approach and the concepts of EQM is indicated. Section II E describes the Hilbert-space formulation of EQM starting from the classical Euclidean version of the Poisson bracket and showing how to define the basic operators of the theory. The relation between the Hilbert-space expectations and their probabilistic interpretations is also given.

Section III investigates the interpretation of the theory, first through the role of probability (Sec. III A), and then discussing its physical content. It is argued that a naively realistic interpretation of the results of EQM is hardly tenable.

Section IV justifies the title of the present paper. One shows that it is possible to realize some processes of EQM starting from solutions of the Schrödinger equations (1.2) and, reciprocally, that from such Bernstein processes one gets other Markovian diffusions, in real time, which are nothing but the processes associated by Nelson to the Schrödinger equation.

We conclude, in Sec. V, with some perspectives on the generalization of EQM in field theory.

This paper is mainly focused on the physical structure of EQM. The mathematically inclined reader should consult Ref. 18 for a really precise description.

## II. EUCLIDEAN QUANTUM MECHANICS

### A. Schrödinger's *Gedankenexperiment*

Let us consider some *Gedankenexperiment* involving a system of *classical* particles of unit mass diffusing in a medium with (positive) diffusion constant $\lambda$, under the effect of a force field $F=\nabla V$. No deterministic theory is able to describe the outcome of such an experiment. However, ample experimental evidences suggest that these classical phenomena are related to an initial-value problem for the diffusion equation

$$-\lambda\frac{\partial\theta^*}{\partial t}=-\frac{\lambda^2}{2}\Delta\theta^*+V\theta^*\equiv H\theta^* , \qquad (2.1)$$

where $\theta^*=\theta_t^*$ is a real scalar field on $\mathscr{M}\times I=\mathbb{R}^3\times[-T/2,T/2]$ (for example) and $V:\mathbb{R}^3\to\mathbb{R}$ is the scalar potential of the force field $F$. The diffusion constant $\lambda$ and the Planck constant $\hbar$ have a very different theoretical status since the first one can be made, in principle, arbitrarily small, in contrast to $\hbar$ regarded as the lower bound for the inevitable perturbation due to a quantum measurement. Nevertheless, the structure of EQM will be independent of the value of $\lambda$. So, for greater convenience we set, from now on, $\lambda\equiv\hbar$. This means that we interpret Eq. (2.1) according to the (terrible) Euclidean terminology, as a "Schrödinger equation in imaginary time." It is well known that, for a nonzero force field $F$, $\theta_t^*$ *cannot* be interpreted as the probability that a diffusive particle is found in the volume element $d^3x$ about $x$ at the time $t$. Mathematically, this is reflected in the fact that the evolution equation (2.1) on $I=[-T/2,T/2]$ is described by a contraction semigroup, formally

$$\theta_t^*=e^{-(t+T/2)H}\theta_{-T/2}^*,\quad t\in I \qquad (2.2)$$

under some technical assumptions on the Hamiltonian operator $H$. If we consider Eq. (2.2) on $L^2(\mathbb{R}^3)$, for example, with an (essentially) self-adjoint $H$, the word "contraction" means that the norm of $\theta_t^*$ is less than one. So, in probabilistic terms, we "lose" some probability during the evolution, therefore this one is not reversible in time (the semigroup has no inverse). One of the key points of EQM is to prove that positive $\theta^*$ are nothing but the Euclidean version of the quantum probability amplitude. From the point of view of the physical experiment of classical diffusion, in EQM, we change nothing. What is changed is the way to use the statistical data resulting from the experiment. Our *Gedankenexperiment* takes place during the time interval $I=[-T/2,T/2]$, between a controllable source of classical diffusive particles and a screen on which a final probability of presence is observed. The experimental arrangement itself (for example, a screen with two holes somewhere in between the source $S$ and the screen 0) may be considered as a black box. The only known information about what is going on in the box is the given force field $F$ (for example, a free evolution between $S$ and the holes and another free evolution between the holes and the screen 0). Cf. Fig. 1. We consider the following theoretical problem. Given the initial density of probability , $p_{-T/2}(x)dx$, and *any* final probability $p_{T/2}(y)dy$, is it possible to describe a unique probabilistic evolution in between?

Of course, for an arbitrary $p_{T/2}(y)$, most of the diffusive particles will follow another probabilistic evolution. We do not mind this. Our aim is to describe the relatively

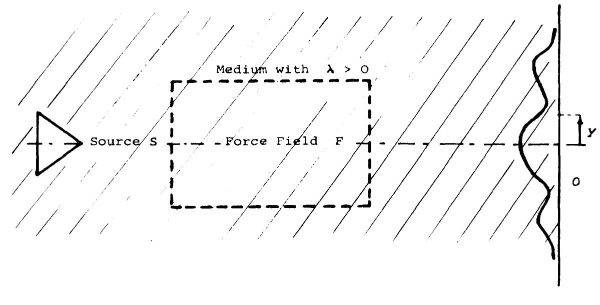


FIG. 1. Schrödinger's *Gedankenexperiment.*

unprobable events compatible with our *pair* of data. This is clearly always possible if we repeat the random experiment long enough. EQM will be the dynamical theory of this particular class of classical diffusions. This very unorthodox way to approach some probabilistic evolutions is due to Schrödinger[14,15] (1931). It has been developed into a physical theory in Refs. 16 and 17, under the name of (Schrödinger's) stochastic variational dynamics.

### B. A new class of diffusion processes

It is indeed always possible to construct stochastic processes $Z_t$, $t \in I$, with value in $\mathscr{M}$, the configuration (or state) space, compatible with our data. But, as it may be guessed, the resulting processes are neither unique nor Markovian. They belong to a strictly larger class of time-symmetric processes, discovered by Bernstein in 1932.[19] A Bernstein process $Z_t$ satisfies, by definition,

$$E[g(Z_t) \mid \mathscr{P}_s \cup \mathscr{F}_u] = E[g(Z_t) \mid Z_s, Z_u] \tag{2.3}$$

for any $-T/2 < s < t < u < T/2$ and Borel measurable $g$. $\mathscr{P}_s$ and $\mathscr{F}_u$ are, respectively, the sigma algebra containing the information on the process $Z_t$ before the time $s$ and after the time $u$. $E[\ \mid \sigma]$ denotes the conditional expectation given $\sigma$. Notice that, in modern terms, (2.3) is a one-dimensional version of the "local Markov property" used, for example, in constructive field theory,[20-22] but Bernstein has, of course, to be credited for the introduction of this concept long before modern needs. In contrast to a Markovian process, a Bernstein process $Z_t$, $t \in I$, is constructed from the data of a three-point analog of the concept of transition probability. This Bernstein transition, and its density, are denoted by

$$H(s,x;t,A;u,y) = \int_A h(s,x;t,\xi;u,y)d\xi, \quad -\frac{T}{2} \le s < t < u \le \frac{T}{2} \tag{2.4}$$

for a Borel set $A$. By definition, this transition is a probability on the Borel sets of $\mathscr{M}$ with respect to the intermediate position $\xi$. Also the Markovian data of an initial probability is replaced here by the data of a joint probability, with density $m$, for the initial and final positions on $I$, $Z_{-T/2}$, and $Z_{T/2}$. Let us recall that a probability space is a triple $(\Omega, \sigma_I, P)$ where $\Omega$ is the sample space of the possible events, in our case the space of all paths $I \rightarrow \mathscr{M}$ for $Z_t$. $\sigma_I$ is the sigma algebra of the observable events, in our case the one generated by $Z_t$ for $t \in I$, i.e., which contains the history of this process from time $-T/2$ to $T/2$. Finally, $P$ is a probability measure on $\sigma_I$.

The following crucial result is due to Jamison:[23] There is a unique probability measure $P_m$ such that with respect to the probability space $(\Omega, \sigma_I, P_m)$, $Z_t$, for $t \in I$, is a Bernstein process and

(a) for $B_S$ and $B_E$ the starting and ending Borel sets

$$P_m(Z_{-T/2} \in B_S, Z_{T/2} \in B_E) = \int_{B_S \times B_E} m(x,y)dx\,dy \ . \tag{2.5}$$

(b) For any $-T/2 \le s \le t \le u \le T/2$ and Borel set $B$,

$$P_m(Z_t \in B \mid Z_s, Z_u) = H(s, Z_s; t, B; u, Z_u) \ . \tag{2.6}$$

In addition, the finite-dimensional distribution density of $Z_t, t \in I$, is given by (for $-T/2 \le t_1 < t_2 < \cdots < t_n \le T/2$)

$$p_m(x_1,t_1,x_2,t_2,\ldots,x_n,t_n) = \int_{\mathscr{M}\times\mathscr{M}} m(x,y) h\left[-\frac{T}{2},x;t_1,x_1;\frac{T}{2},y\right] h\left[t_1,x_1;t_2,x_2;\frac{T}{2},y\right] \cdots h\left[t_{n-1},x_{n-1};t_n,x_n;\frac{T}{2},y\right] dx\,dy \ . \tag{2.7}$$

Let us come back to the starting diffusion equation (2.1). Under suitable restrictions on the potential $V = V(x)$, for example, $V$ bounded below, the fundamental solution of Eq. (2.1), denoted by $h(s,x,t,y) = h(x,t-s,y)$, is strictly positive. We consider exclusively this situation afterwards. In these conditions, it is easy to see that

$$h(s,x;t,\xi;u,z) \equiv \frac{h(s,x,t,\xi)h(t,\xi,u,z)}{h(s,x,u,z)} \ , \quad -\frac{T}{2} \le s < t < u \le \frac{T}{2} \tag{2.8}$$

has the properties required to be the density of a Bernstein transition.

So, a Bernstein process $Z_t$ is specified, according to (2.7), by the Bernstein transition (2.8) and an arbitrary density $\mathscr{m}$ of joint probability. But our experimental data are the initial and final probabilities $p_{-T/2}(x)dx$ and $p_{T/2}(y)dy$, and many joint densities (therefore many Bernstein processes) are compatible with them. However, there is a distinguished choice of joint density, denoted by $M = M(x,y)$. For this choice, and only this one, the Bernstein process $Z_t$ is also Markovian. It is shown in Ref. 17 that this Markovian joint density is

$$\int_{B_S \times B_E} M(x,y)dx\,dy = \int_{B_S \times B_E} \theta^*_{-T/2}(x) h(x,T,y) \theta_{T/2}(y) dx\,dy \ , \tag{2.9}$$

where $\theta^*_{-T/2}$ and $\theta_{T/2}$ are two (yet unspecified) real functions. Clearly, in order to preserve the positivity of this joint probability, they have to be of the same signs on $\mathscr{M}$. The confusion of our notation for $\theta^*_{-T/2}$ and the initial condition of the heat equation (2.1) is not fortuitous, as we shall see. The substitution of (2.8) and (2.9) in the general finite-dimensional distribution for a Bernstein process on $I$ reduces this one to

$$p_M(x_1,t_1,x_2,t_2,\ldots,x_n,t_n)=\int_{\mathcal{M}\times\mathcal{M}} dx\, dy\, \theta^*_{-T/2}(x)h\left[x,t_1+\frac{T}{2},x_1\right]\cdots h\left[x_n,\frac{T}{2}-t_n,y\right]\theta_{T/2}(y)\ . \quad (2.10)$$

Although this is not immediately evident, (2.10) is indeed nothing but the finite-dimensional distribution of a Markovian diffusion, also denoted by $Z_t$. It is shown in Ref. 17 that its backward transition probability density is given by

$$q_*(s,x,t,y)=\frac{\theta^*(x,s)}{\theta^*(y,t)}h(x,t-s,y) \quad (2.11)$$

for $-T/2\le s<t\le T/2$ and $\theta^*_t\equiv\theta^*(x,t)$ the solution of the initial value problem (2.1). This yields for the backward drift

$$B_*(y,s)=\lim_{\Delta s\downarrow 0}\frac{1}{\Delta s}\int_{S_\epsilon(y)}(y-x)q_*(s-\Delta s,x,s,y)dx$$
$$=-\hbar\frac{\nabla\theta^*}{\theta^*}(y,s) \quad (2.11')$$

[here $S_\epsilon(y)$ denotes the sphere of center $y$ and radius $\epsilon$] and the diffusion matrix, for $\mathcal{M}=\mathbb{R}^N$, $\mathbb{1}=N\times N$ identity matrix,

$$C_*(y,s)=\lim_{\Delta s\downarrow 0}\frac{1}{\Delta s}\int_{S_\epsilon(y)}(y-x)^2q_*(s-\Delta s,x,s,y)dx$$
$$=\hbar\mathbb{1}\ . \quad (2.11'')$$

Symmetrically, the forward transition probability of $Z_t$ is

$$q(s,x,t,y)=h(x,t-s,y)\frac{\theta(y,t)}{\theta(x,s)} \quad (2.12)$$

for $-T/2\le s<t\le T/2$ and $\theta_s\equiv\theta(x,s)$ the solution of the final value problem $\hbar(\partial\theta/\partial t)=H\theta$. The associated forward drift is

$$B(x,s)=\lim_{\Delta s\downarrow 0}\frac{1}{\Delta s}\int_{S_\epsilon(x)}(y-x)q(s,x,s+\Delta s,y)dy$$
$$=\hbar\frac{\nabla\theta}{\theta}(x,s) \quad (2.12')$$

for the same diffusion matrix. After normalization, the probability density of $Z_t$ reduces to

$$p(x,t)dx=\theta^*(x,t)\theta(x,t)dx,\quad t\in I\ . \quad (2.13)$$

Finally, it is clear from (2.10) that the positive measure of this Markovian Bernstein process is not known before we specify a pair $\theta^*_{-T/2}$ and $\theta_{T/2}$. The marginals of the Markovian joint probability (2.9) give the following constraints:

$$\begin{aligned}\theta^*_{-T/2}(x)\int_{\mathcal{M}}h(x,T,y)\theta_{T/2}(y)dy&=p_{-T/2}(x)\ ,\\ \theta_{T/2}(y)\int_{\mathcal{M}}\theta^*_{-T/2}(x)h(x,T,y)dx&=p_{T/2}(y)\ .\end{aligned} \quad (2.14)$$

This system of equations for $\theta^*_{-T/2}$ and $\theta_{T/2}$, in the case $V=0$, is due to Schrödinger.[15] For $p_{-T/2}(x)$ and $p_{T/2}(y)$ without zeros, the proof of existence and uniqueness of its positive solutions is known[23–25] and therefore a unique Markovian Bernstein process is specified. This construction is completely symmetric in $\theta^*_t$ and $\theta_t$, $t\in I$. To each $\theta^*_t$ solution of the heat equation (2.1) with initial condition $\theta^*_{-T/2}$ corresponds a unique $\theta_t$ solution of $\hbar(\partial\theta/\partial t)=H\theta$ with final condition $\theta_{T/2}$. This relation for each pair $\theta^*_t$ and $\theta_t$ associated with a solution of the Schrödinger system (2.14) is the Euclidean version of complex conjugation. It plays a fundamental role afterwards. From the dynamical point of view, it is crucial that (2.10) defines a time-symmetric measure for the Bernstein diffusion. Nevertheless it also is useful, sometimes, to regard $Z_s$, $s\in I$, only as a Markovian diffusion. For example, if we need to impose a condition in the future, say $Z(t)=z$, $t\in I$ and $t>s$, the natural point of view is to consider the process as a Markovian backward diffusion, denoted by $Z^z(s)$, $-T/2\le s\le t$, and to use the backward transition (2.11) in which this future conditioning is implicit,

$$q_*(s,A,t,z)=q_*(Z_s\in A\mid Z_t=z)\ ,$$

for $A$ a Borel set.

### C. A physical Hilbert space

Let $I=[-T/2,T/2]$ be the fixed (closed but arbitrary) time interval of the Schrödinger *Gedankenexperiment.* A positive function $\theta^*(x,t)$ for $x\in\mathcal{M}$, $t\in I$, will be the analogue, in EQM, of the quantum wave function $\psi(x,t)$. According to Eq. (2.13), the density of the probability of presence for a diffusive particle is given by

$$p(x,t)dx=\theta^*(x,t)\theta(x,t)dx,\quad t\in I\ . \quad (2.15)$$

(Here, $\mathcal{M}=\mathbb{R}^3$ for example, so $dx\equiv d^3x$; afterwards, if this is not specified, $\mathcal{M}=\mathbb{R}^{3n}=\mathbb{R}^N$, $dx\equiv d^Nx$.)

The relation between $\theta^*$ and $\theta$, analogous here to the quantum complex conjugation, is called Euclidean conjugation. If $\theta^*$ solves an initial value problem

$$-\hbar\frac{\partial\theta^*}{\partial t}=H\theta^*,\quad t\in I,\ x\in\mathcal{M} \quad (2.16)$$

the relation (2.12) involves the associated solution $\theta$ of a final value problem

$$\hbar\frac{\partial\theta}{\partial t}=H\theta,\quad t\in I,\quad x\in\mathcal{M}\ . \quad (2.17)$$

We define the Euclidean conjugation $k$ by

$$k\theta^*(x,t)=\theta(x,t)$$

or by the shortest notation* such that

$$(\theta^*)^*(x,t)=\theta(x,t)\ . \quad (2.18)$$

For Eq. (2.15) to make sense we have, of course, to consider exclusively the set of $\theta^*$ such that

$$0<\int k\theta^*\theta^*dx=\int\theta\theta^*dx<\infty\ . \quad (2.19)$$

We illustrate the meaning of this constraint, for simplicity, when the potential $V$ of the Hamiltonian $H$ is such that the spectrum of $H$ is purely discrete, $\{E_n\}_{n\in\mathbb{N}}$. Let $\{\varphi_n(x)\}_{n\in\mathbb{N}}$ be the corresponding orthonormal basis of a real space $L^2(m)$ (this is not a restriction). Then, if $\langle\ ,\ \rangle$ denotes the scalar product in this function space, and if $\theta^*_{-T/2}\in L^2(m)$,

$$\theta^*(x,t)=\sum_j \alpha_j e^{-E_j(t+T/2)}\varphi_j(x)$$

with $\sum_j \alpha_j^2<\infty$ and $\alpha_j=\langle\theta^*_{-T/2},\varphi_j\rangle$.

Under Euclidean conjugation $k$, $\theta^*$ changes into

$$\theta(x,t)=\sum_i \alpha_i e^{E_i(t+T/2)}\varphi_i(x) \ .$$

So, using the orthonormality of the basis,

$$\int \theta\theta^*(x,t)dx=\sum_i \alpha_i^2<\infty \ . \qquad (2.19')$$

Since the set of $\theta^*(x,t)=(e^{-(t+T/2)H}\chi)(x)$ for arbitrary $\chi$ forms a real linear space, denoted by $v^*$, we define a scalar product in $v^*$ by

$$(\varphi^*\mid\theta^*)=\int k\varphi^*\theta^*dx$$
$$=\int \varphi\theta^*dx \ . \qquad (2.20)$$

After competition, we obtain a Hilbert space, denoted by $\bar{v}^*(m)$ or simply $\bar{v}^*$.[18]

Let us emphasize that the integrand of (2.19′) is, in general, not positive for any $t$ in $I$ and so cannot be interpreted as a probability density. We call *physical states* the collection of $\theta^*$ in $\bar{v}^*$ compatible with the probabilistic interpretation summarized in Sec. II B. They are positive functions in $\bar{v}^*$, represented by $\theta^*(x,t)=e^{-A_*(x,t)}$. Let us denote by $T$ the time-reversal operator defined by $T\theta^*(x,t)=\theta(x,-t)$. Since any function $A_*$ can be decomposed into even and odd parts under time reversal, we write $\theta^*(x,t)=e^{(\bar{R}-\bar{S})(x,t)}$ where $\bar{R}$ and $\bar{S}$ denote, respectively, the even and odd terms such that $T\bar{R}(x,t)=\bar{R}(x,-t)$ and $T\bar{S}(x,t)=-\bar{S}(x,-t)$. Then we have clearly $k\theta^*(x,t)\equiv\theta(x,t)=e^{(\bar{R}+\bar{S})(x,t)}$. The constraint (2.19) reduces, therefore, to $\int e^{2\bar{R}(x,t)}dx<\infty$. A (real) linear combination of physical states produces a physical state only if the coefficients are all positive. So the physical states form a cone in $\bar{v}^*$.

Of course, as far as physics is concerned, we only use some subspaces of $\bar{v}^*$ with additional regularity conditions (cf. Ref. 18, also Secs. II E and IV). We shall also use occasionally the following Euclidean version of Dirac notation. Since (by Riesz theorem) any linear functional on $\bar{v}^*$ is a scalar product, a linear functional $(\varphi^*\mid$ (a bra) is defined by its image (2.20) on the ket $\mid\theta^*)$. In the EQM, the correspondence between ket and bra is linear since the space is real.

If $A$ denotes a linear operator on $\bar{v}^*$, its adjoint $A^\dagger$ is such that $(A^\dagger\varphi^*\mid\theta^*)=(\varphi^*\mid A\theta^*)$. [This is actually sufficient only for $A$ bounded, and in general for $\theta^*$ in the domain $\mathscr{D}_A$ and $\varphi^*$ in $\mathscr{D}_{A^\dagger}$. This will not be specified afterwards (cf. Ref. 18).] The Dirac notation is defined by

$$(A\varphi^*\mid=(\varphi^*\mid A^\dagger \ . \qquad (2.21)$$

Let us consider, for example, the time-reversal operator $T$. It is easy to check that

$$(\varphi^*\mid\theta^*)=(T\varphi^*\mid T\theta^*)=(T\theta^*\mid T\varphi^*) \qquad (2.22)$$

so $T$ is unitary in $\bar{v}^*$. Now let $\mid\gamma^*)$ be $A^\dagger\mid\theta^*)$ and consider the expression $(T\varphi^*\mid TA^\dagger T^{-1}T\theta^*)=(T\varphi^*\mid T\gamma^*)$. By (2.22) this is $(\gamma^*\mid\varphi^*)=(\theta^*\mid A\varphi^*)$. An operator $A$ in $\bar{v}^*$ is called even (respectively, odd) under time reversal according to the following:

$$TA^\dagger T^{-1}=\begin{cases}+A, & \text{even}\\ -A, & \text{odd}\ .\end{cases}$$

If $\mid\varphi^*)$ and $\mid\theta^*)$ are identical, that is, for an expectation value, we therefore have, according to the parity of $A$,

$$(\theta^*\mid A\theta^*)=\pm(T\theta^*\mid AT\theta^*) \ . \qquad (2.23)$$

### D. Path-integral formulation of Euclidean dynamics

The starting object in Feynman's investigation of quantum dynamics is the classical action functional

$$J[Z(\ )]=\int_{-T/2}^{T/2}L(Z(s),\dot{Z}(s))ds \qquad (2.24)$$

for the Lagrangian

$$L:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}\ ,L(q,p)=\tfrac{1}{2}p^2-V(q)$$
$$=\textstyle\sum_{i=1}^N p_i^2/2-V(q)$$

of the considered classical system. In the Feynman-Kac approach, one has to do an analytical continuation in time and therefore the relevant Lagrangian becomes proportional to $\bar{L}(q,p)=\frac{1}{2}p^2+V(q)$. A common funny feature of the two usual path-integral representations (sum over all the paths) of $U_\tau=e^{-i\tau H}$ and $T_t=e^{-tH}$ is that they both involve a kinetic energy term which is formally infinite on the trajectories which contribute essentially to the path integrals. For example, it is well known that the Wiener measure involved in the Feynman-Kac formula is concentrated on continuous but nowhere differentiable trajectories. Even though this is without consequence on the validity of the path-integral formula itself, it strongly suggests that the really natural starting action has to be some regularized version of (2.24), or of its Euclidean version.

The right kind of regularization is suggested by the kinematical hypothesis associated naturally with our physical situation. Since we start from the diffusion Eq. (2.1) (with $\lambda=\hbar$) we may try to characterize dynamically a process in the class of the diffusion processes $\hat{Z}^z$ such that (here $m=\mathbb{R}^N$)

$$d_*\hat{Z}^z(s)=\hat{B}_*(\hat{Z}^z(s),s)ds+\hbar^{1/2}\mathbb{1}d_*W_*(s)\ ,$$
$$-\frac{T}{2}\le s<t<\frac{T}{2} \qquad (2.25)$$

where $d_*$ denotes the backward differential defined by $d_*f(s)=f(s)-f(s-ds)$ and $W_*(s)$ is a backward Brownian motion. When $\hat{B}_*$, the drift, is known explicitly (this is not the case here) Eq. (2.25) is an Itô stochastic differential equation with respect to the future sigma algebra $\mathscr{F}_s=\sigma\{\hat{Z}_\xi, s\leq \xi \leq t < T/2\}$.[26] We impose upon all these processes the condition that $\hat{Z}^z(t)=z$, a given future position. Since the diffusion matrix of these processes is $\hbar \mathbb{1}$ ($\mathbb{1}$ stands for unit matrix) they all have the irregularities of the quantum "trajectories" described by Feynman[7] (but without the imaginary factor $i$ responsible for the mathematical trouble). So Eq. (2.25) is a natural kinematical hypothesis in our context. In order to be more specific, in Eq. (2.25), we consider the following class of diffusion processes:

$$d_* Z^\epsilon(s)=B_*^\epsilon[Z^\epsilon(s),s]ds+h^{1/2}\mathbb{1} d_* W_*(s)\ ,\quad -T/2\leq s\leq t<T/2 \qquad (2.25')$$

with $Z^\epsilon(t)=z$ and the drift depending on the parameter $\epsilon$,

$$B_*^\epsilon(x,t)=-\left[\frac{\nabla\theta}{\theta^*}+\epsilon\nabla g^*(x,t)\right] .$$

According to Eq. (2.11′), for $\epsilon=0$, this is the backward Markovian Bernstein process of Sec. II B. The drift has clearly the units of a velocity. Actually, it also may be defined as the mean backward velocity $D_* Z^\epsilon(s)=B_*^\epsilon[Z^\epsilon(s),s]$, a particular case of the mean backward derivative

$$D_* g[Z^\epsilon(s),s] \equiv \lim_{\Delta s\downarrow 0} E_s\left[\frac{g[Z^\epsilon(s),s]-g[Z^\epsilon(s-\Delta s),s-\Delta s]}{\Delta s}\right] = \left[\frac{\partial}{\partial s}+B_*^\epsilon\nabla-\frac{h}{2}\Delta g\right][Z^\epsilon(s),s]\ , \qquad (2.26)$$

where $E_s$ denotes the conditional expectation knowing the future position $Z^\epsilon(s)$. Let us define the backward variation $\delta_* Z(s)$ by the derivative of $Z^\epsilon(s)$ at $\epsilon=0$, $\delta_* Z(s)=\partial Z^\epsilon/\partial\epsilon(s)\,|_{\epsilon=0}$. From Eq. (2.25′) we get

$$\delta_* Z(s)=-\int_{-T/2}^{s}\nabla g^*[Z(u),u]du+\int_{-T/2}^{s}\nabla B_*[Z(u),u]\delta_* Z(u)du\ .$$

In particular, this process is differentiable and

$$D_*\delta_* Z(s)=\frac{d}{ds}\delta_* Z(s) = -\nabla g^*[Z(s),s]+\nabla B_*[Z(s),s]\delta_* Z(s)\ . \qquad (2.27)$$

Since $\delta_* Z(s)$ solves an ordinary differential equation of the first order, it is uniquely determined by the final boundary condition $\delta_* Z(t)=0$.

In term of this variation, any process in the class defined by Eq. (2.25′) is of the form

$$Z^\epsilon(s)=Z(s)+\epsilon\delta_* Z(s)+O(\epsilon),\quad -T/2\leq s\leq t \qquad (2.28)$$

where $O(\epsilon)$ is infinitesimal compared to $\epsilon$. The action function is defined, for any such $Z^\epsilon$, by

$$A_*[Z^\epsilon(s),s]=-\log\theta^*[Z^\epsilon(s),s]\ . \qquad (2.29)$$

It solves the stochastic differential equation

$$d_* A_*[Z^\epsilon(s),s] = D_* A_*[Z^\epsilon(s),s]ds+\nabla A_*[Z^\epsilon(s),s]d_* W_*(s)\ .$$

Now it has been shown in Ref. 17 that, for any $\epsilon\geq 0$,

$$D_* A_*[Z^\epsilon(s),s]\leq \tfrac{1}{2}\,|B_*^\epsilon|^2[Z^\epsilon(s),s]+V[Z^\epsilon(s)] \qquad (2.30)$$

and that the equality holds only for $\epsilon=0$ and $B_*(x,s)=\nabla A_*(x,s)$. Applying $E_t\int_{-T/2}^{t}(\ )ds$ to the inequality (2.30), we obtain

$$A_*[Z^\epsilon(t),t]-E_t A_*[Z^\epsilon(-T/2)] \leq E_t\int_{-T/2}^{t}\{\tfrac{1}{2}\,|B_*^\epsilon|^2[Z^\epsilon(s),s]+V[Z^\epsilon(s)]\}ds\ . \qquad (2.30')$$

On the other hand, the variation of $A_*[Z^\epsilon(t),t]$ solves

$$\delta_* A_*[Z(t),t]-\delta_* A_*[Z(-T/2),-T/2] = -\int_{-T/2}^{t}B_*\nabla g^*[Z(s),s]ds +\int_{-T/2}^{t}(B_*\nabla B_*+V)[Z(s),s]\delta_* Z(s)ds +\int_{-T/2}^{t}\nabla B_*[Z(s),s]\delta_* Z(s)d_* W_*(s)\ .$$

Therefore, after integration, and using Eq. (2.27),

$$\delta_* A_*[Z(t),t]-E_t\delta_* A_*[Z(-T/2),-T/2] = E_t\int_{-T/2}^{t}\{B_* D_*\delta_* Z+\nabla V\delta_* Z\}ds\ . \qquad (2.31)$$

Let us define the (Euclidean and finite) action functional with initial condition by

$$J[Z(\ )]=E_t\int_{-T/2}^{t}\bar{L}[Z(s),D_* Z(s)]ds+E_t A_*(Z_{-T/2}) \qquad (2.32)$$

for $\bar{L}$ the classical (Euclidean) Lagrangian.

Now the inequality (2.30′) means that

$$E_t\int_{-T/2}^{t}[\tfrac{1}{2}\,|B_*^\epsilon|^2+V(Z^\epsilon)]ds \geq E_t\int_{-T/2}^{t}[\tfrac{1}{2}\,|B_*|^2+V(Z)]ds +\epsilon E_t\int_{-T/2}^{t}(B_* D_*\delta_* Z+\nabla V\delta_* Z)ds+O(\epsilon)\ .$$

After an integration by parts in the second term of the rhs and the use of $\delta_* Z(t)=0$, this reduces to

$$J(Z+\epsilon\delta_* Z)-J(Z) \geq \epsilon\left[E_t\int_{-T/2}^{t}(-D_* B_*+\nabla V)\delta_* Z(s)ds +E_t[(-B_*+\nabla A_*)\delta_* Z(-T/2)]\right]+O(\epsilon) \qquad (2.30'')$$

with equality only for $\epsilon=0$. In other words, since $B_*[Z(s),s]=D_* Z(s)$, each solution of

$$D_* D_* Z(s)=\nabla V[Z(s)],\quad -T/2\leq s\leq t \qquad (2.33)$$

minimizes the action functional $J$ on the class of varied processes (2.25′), with

$$D_*Z(-T/2)=\nabla A_*(Z_{-T/2}),\quad Z(t)=z\ . \tag{2.33′}$$

This idea is basically due to Yasue.[28] The term of Eq. (2.30″) in the large parentheses is the Gâteau variation of $J$ in direction $\delta_*Z$, and can be denoted by $\delta J[Z(\ )](\delta_*Z)$. Notice that the boundary conditions (2.33′) are sufficient for the minimization. Reciprocally, we define a (local) minimum for $J$ as a diffusion $Z$ such that $J(Z^\epsilon)\geq J(Z)$ for any neighboring diffusion $Z^\epsilon$ such that

$$\|Z^\epsilon-Z\|\equiv\sup_{-T/2\leq s\leq t}E[\,|Z^\epsilon(s)-Z(s)|^2] \tag{2.34}$$

is smaller than a positive number $r$.

It is easy to show that the condition $\delta J[Z(\ )](\delta_*Z)=0$ for any admissible $\delta_*Z(s)$, $-T/2\leq s\leq t$, with $\delta_*Z(t)=0$, is necessary for the minimality of $J$. This means that

$$E_t\int_{-T/2}^{t}(-D_*B_*+\nabla V)\delta_*Z(s)ds+E_t[(-B_*+\nabla A_*)\delta_*Z(-T/2)]=0 \tag{2.35}$$

for any such $\delta_*Z$, and then that Eq. (2.33) and (2.33′) hold. In particular, the boundary conditions (2.33′) are therefore also necessary for the minimization of the action functional (2.32). Let us observe that this is not the most general variational derivation of Eq. (2.33). In Ref. 17 it has been shown that the conclusion remains unchanged even if we do not restrict, as here, the class of variations to a Markovian one.

However, Eq. (2.33) cannot be a complete specification of the dynamics of the Markovian Bernstein process $Z(s)$ since it is not time symmetric. Indeed, it can immediately be checked that under time reversal, a backward derivative such as (2.26) changes into

$$D_*g(Z(s),s)\to -Dg(Z(s),s)\ ,$$

where the forward derivative $D$ is defined by

$$Dg(Z(s),s)=\lim_{\Delta s\downarrow 0}E_s\left[\frac{g(Z(s+\Delta s),s+\Delta s)-g(Z(s),s)}{\Delta s}\right]=\left[\frac{\partial}{\partial s}+B\cdot\nabla+\frac{\hbar}{2}\Delta\right]g(Z(s),s)\ . \tag{2.36}$$

The time a symmetry of Eq. (2.33) is not surprising; our variational principle uses only the Markovian (backward) property of $Z(s)$ and its (backward) transition probability $q_*$. And such a description is essentially asymmetric. It is possible to restore the natural symmetry of the (unconditioned) Bernstein process $Z(s)$, $s\in I$, in observing that a completely analogous variational characterization is available with respect to the past sigma algebra $\mathscr{P}_s$, the forward transition probability $q$, and a class of varied processes with a common given position in the past ("forward" variation $\delta$). The shortest way is the following. According to (2.12′) the forward drift of $Z(s)$ reduces to

$$B(x,s)=\hbar\frac{\nabla\theta}{\theta}(x,s)\ . \tag{2.37}$$

Now let us define

$$\overline{V}(x,s)=\tfrac{1}{2}[B(x,s)+B_*(x,s)] \tag{2.38}$$

and

$$\overline{U}(x,s)=\tfrac{1}{2}[B(x,s)-B_*(x,s)]\ . \tag{2.39}$$

In contrast to $B$ and $B_*$, $\overline{V}$ and $\overline{U}$ have well-defined symmetry under time reversal: $\overline{V}$ is odd and $\overline{U}$ is even. They are therefore more natural in our framework. By Eqs. (2.13), (2.11′), and (2.12′), $\overline{V}$ and $\overline{U}$ may also be expressed as the gradients

$$\overline{V}(x,s)=\hbar\nabla\ln\left[\frac{\theta}{\theta^*}\right]^{1/2}(x,s)=\nabla\overline{S}(x,s) \tag{2.38′}$$

and

$$\overline{U}(x,s)=\hbar\nabla\ln p^{1/2}(x,s)=\nabla\overline{R}(x,s)\ . \tag{2.39′}$$

On the other hand, knowing the form of the probability density (2.13) and the equations of motion of $\theta$ and $\theta^*$ [Eqs. (2.16) and (2.17)] it is easy to find the equation expressing the local conservation of probability,

$$\frac{\partial p}{\partial t}+\mathrm{div}(p\overline{V})=0 \tag{2.40}$$

or, equivalently,

$$\frac{\partial\overline{U}}{\partial t}=-\frac{\hbar}{2}\mathrm{grad}\,\mathrm{div}\overline{V}-\mathrm{grad}\overline{V}\cdot\overline{U}\ . \tag{2.41}$$

According to the definition (2.40) and (2.41),

$$B_*(x,s)=(\overline{V}-\overline{U})(x,s)=\nabla(\overline{S}-\overline{R})(x,s)\ . \tag{2.42}$$

Therefore, on the (unconditioned) Bernstein process $Z_s$, Eq. (2.33) modifies to

$$D_*\overline{V}-D_*\overline{U}=\nabla V \tag{2.43}$$

or, using the definition (2.26) and Eq. (2.41),

$$\frac{\partial\overline{V}}{\partial t}=-\frac{\hbar}{2}\Delta\overline{U}-\overline{U}\nabla\overline{U}-\overline{V}\nabla\overline{V}+\nabla V\ . \tag{2.44}$$

Now a straightforward computation shows that Eq. (2.44) reduces to

$$\tfrac{1}{2}(DDZ+D_*D_*Z)(t)=\nabla V(Z(t)),\quad t\in I\ . \tag{2.45}$$

This is the time-symmetric (probabilistic) law of motion we are looking for. We call (2.45) the Euclidean Newton equation (notice the positive sign of the right-hand side). Mathematically, it can be interpreted as a "differential" version of the Bernstein property (2.3). Equation (2.45) is not new (cf. Ref. 29). However, it has never been associated with a probabilistic interpretation of the heat equation (2.16). The initial gradient condition associated with (2.33) is preserved during the evolution. So, the comparison with Eq. (2.42) shows that we can identify (up to an irrelevant additive time dependence) $\overline{S}-\overline{R}$ and the action function (2.29)

$$(\overline{S}-\overline{R})(z,t)=A_*(z,t),\quad t\in I\ . \tag{2.46}$$

On the other hand, Eqs. (2.41) together with (2.44) constitute a coupled nonlinear partial differential system. This one is linearized by the change of variables

$$\theta^*(z,t)=e^{(\bar{R}-\bar{S})(z,t)/\hbar}\equiv p^{1/2}e^{-\bar{S}(z,t)/\hbar}\ , \tag{2.47}$$

where $\theta^*$ is the solution of the heat equation (2.16) involved in this construction. So, Eqs. (2.32), (2.46), and (2.47) give us the following path-integral representation for $\theta^*$:

$$\begin{aligned}\theta^*(z,t)&=\exp-\frac{1}{\hbar}E_t\int_{-T/2}^{t}\bar{L}(Z^z(s),D_*Z^z(s))ds-\frac{1}{\hbar}E_tA_*(Z^z_{-T/2})\\&=\exp-\frac{1}{\hbar}E_t\int_{-T/2}^{t}\{\tfrac{1}{2}D_*Z^z(s)\,|^2+V(Z^z(s))\}ds-\frac{1}{\hbar}E_tA_*(Z^z_{-T/2})\ .\end{aligned}\tag{2.48}$$

This is the Euclidean version of Feynman's path-integral representation for the solution of the Schrödinger equation (1.2). The basic difference is, of course, that the measure used in (2.48) is the well-defined probability measure of the unique Markovian Bernstein process $Z(s)$ described in Sec. II B. Also notice that we need a sum over history (from time $-T/2$ to time $t$) to characterize $\theta^*(z,t)$.

Before elaborating somewhat the claimed analogy with Feynman's results, let us observe that a direct derivation of the Euclidean-Newton equation (2.45) is also possible, using a method discovered by Yasue.[28,30] Starting from the reasonable time-symmetric Euclidean concept of action

$$J[Z(\ )]=-E\left[\int_{-T/2}^{T/2}[\tfrac{1}{2}DZD_*Z+V(Z)]dt\right]\ , \tag{2.49}$$

where $E$ is the absolute (that is unconditioned) expectation, Yasue defines a notion of criticality for the process $Z_t$ which turn out to be equivalent to the validity of Eq. (2.45) on $I$. Here, thanks to the use of the absolute expectation, a time-symmetric concept for any Bernstein process, there is no need for an *a posteriori* symmetrization. Notice, however, that the Lagrangian of Eq. (2.49) is not the one used before. It shall be easier to understand its meaning after Secs. II C and III. Yasue's idea is at the origin of the stochastic variational approaches of quantum dynamics. In particular, he suggested that in this context it is natural to construct a diffusion process from the data of two boundary probabilities.

One of the key concepts of Feynman's path-integral method in quantum mechanics is the concept of transition elements. According to the (still partially unjustified) interpretation of $\theta^*(\theta)$ as the Euclidean analogue of the quantum wave function $\psi(\psi^*)$, a *transition amplitude* in EQM becomes (for convenience we adopt here Feynman's backward-in-time convention)

$$\begin{aligned}\int\int_{B_S\times B_E}\theta\left[y,\frac{T}{2}\right]h(x,T,y)\theta^*\left[x,-\frac{T}{2}\right]dx\,dy\\=E[\chi_{B_S}(Z_{-T/2})\chi_{B_E}(Z_{T/2})]\ ,\end{aligned}\tag{2.50}$$

where $\chi_A(x)$ is the characteristic function of the set $A$, which is 1 if $x\in A$ and 0 otherwise. Equations (2.5) and (2.9) show that this is nothing else than the joint probability $P_M(Z_{-T/2}\in B_S,\ Z_{T/2}\in B_E)$. Similarly, an Euclidean *transition element* involving a function of one time, $F(Z(t_1))$, for $-T/2<t_1<T/2$, reduces to

$$\begin{aligned}\int\int\int\theta\left[y,\frac{T}{2}\right]h\left[y,\frac{T}{2}-t_1,x_1\right]F(x_1,t)h\left[x,t_1+\frac{T}{2},x_1\right]\theta^*\left[x,-\frac{T}{2}\right]dx\,dx_1dy=\int\theta(x_1,t_1)F(x_1,t_1)\theta^*(x_1,t_1)dx_1\\=E[F(Z(t_1))]\ ,\end{aligned}\tag{2.51}$$

where Eq. (2.13), $\theta^*_{t_1}$ a solution of $-\partial\varphi^*/\partial t_1=H\varphi^*$ and $\theta_{t_1}$ the associated time-reversed solution have been used. The Euclidean version of transition elements involving functions of the process at several separate times is found similarly.

As observed by Feynman (Sec. 7 in Ref. 7) it is difficult to understand intuitively quantum transition elements. He proposes the help of a classical analogy using a small particle in Brownian motion. It is comforting that EQM specifies the nature of this analogy in a frame where all the "averages" are well-defined real expectations.

In Feynman's theory, there also is a concept of functional derivative convenient for the analysis of transition elements. One shows easily, along the line used before, that our variational approach is compatible with Feynman's one. In particular, it follows rather directly that

$$E_t\{[Z(t)-Z(t-\Delta t)]^2\}=\hbar\mathbb{1}\Delta t\ . \tag{2.52}$$

This is the Euclidean version of the well-known Feynman's characterization [cf. (7.50) in Ref. 7] of the "paths" for a quantum-mechanical particle (but without the famous, and meaningless, imaginary factor $i$). So, we do not have to postulate this kinematical characterization as in Eq. (2.25); it is a consequence of the variational approach.

Before concluding this section on the path-integral formulation of EQM, let us summarize our results. Given the (Euclidean) Lagrangian of a mechanical system, and nothing else, we define in (2.32) the regularized version of action, natural for the class of diffusion processes we consider here. A variational principle gives us the law of motion of the Markovian process, the Euclidean Newton equation (2.45). The notion of classical limit is therefore rather different here from Feynman's path-integral method. At this limit $\hbar=0$ (i.e., $\lambda=0$) there is no longer physical cause for the diffusion of the classical system. So the (mean) forward and backward velocities coincide with the classical velocity and Eq. (2.45) reduces to the classical (Euclidean) Newton equation. And it is indeed true that the state of such a classical mechanical system can be determined by this Newton equation and the data of two boundary points on $I$ (a pair of $3n=N$ position coordinates) which constitute the classical limit of our data for the Schrödinger system. On the other hand, all the Feynman's results specific of his approach have here their Euclidean, and well-defined, analogues. The path-integral representation of the probability amplitude [Eq. (2.48)], the transition amplitude [Eq. (2.50)], the transition element [Eq. (2.51)] are valid. The intuitive appeal of Feynman's formulation of quantum mechanics is therefore completely preserved in this purely classical physical context. Nevertheless, in order to justify the claim that the resulting theory is the Euclidean version of quantum mechanics, we have to describe now its Hilbert-space formulation.

### E. Hilbert-space formulation of EQM

We have to find the operators on the Hilbert space $\overline{\upsilon}^*$ (cf. Sec. II C) which are the Euclidean versions of quantum-mechanical observables. The most natural way to do it is to come back to the classical limit of EQM, namely, according to Eq. (2.45) when $\hbar=0$,

$$\ddot{z}(t)=\nabla V(z(t)),\quad t\in I\ . \tag{2.53}$$

Since the Euclidean version of the Hamiltonian has to be

$$H(p,z)=-\tfrac{1}{2}p^2+V(z)\ . \tag{2.54}$$

(From now on, we do not distinguish anymore in the notations between the real-time and imaginary-time functions or operators. It will be clear from the context.) The relevant Hamilton's equations take the form

$$\begin{aligned}\dot{z}&=-\frac{\partial H}{\partial p}\ ,\\ \dot{p}&=\frac{\partial H}{\partial z}\ ,\end{aligned} \tag{2.55}$$

so the symplectic structure of classical mechanics is preserved. In particular, the concept of the classical Poisson bracket is unaltered, and the equation of motion of a classical observable $U=U(z,p,t)$ on the phase space $\mathbb{R}^{6n}=\mathbb{R}^{3n}\times\mathbb{R}^{3n}$, with $z_i,p_i,i=1,\ldots,3n$ is

$$\frac{dU}{dt}=-\{U,H\}+\frac{\partial U}{\partial t}\ , \tag{2.56}$$

where

$$\{U,H\}=\sum_{i=1}^{3n}\frac{\partial U}{\partial z_i}\frac{\partial H}{\partial p_i}-\frac{\partial U}{\partial p_i}\frac{\partial H}{\partial z_i}\ . \tag{2.57}$$

The observables form an algebra of functions on phase space. The Poisson bracket (2.57) gives to this algebra a structure of Lie algebra.

Since EQM has to contain this theory as a particular limit, the symmetry under time reversal of the three operators associated with the basic classical dynamical variables $z$, $p$, and $H$ will be preserved. So, if we denote by capitals these operators, $Z$ and $H$ will be even and $P$ will be odd under time reversal. Now, by analogy with quantum mechanics, it is easy to find these differential operators on $\overline{\upsilon}_*$. Equations (2.16) and (2.17) show that the Hamiltonian $H$ is necessarily identical to the quantum one,

$$H=-\frac{\hbar^2}{2}\Delta+V(z)\ . \tag{2.58}$$

It follows from the comparison of Eqs. (2.54) and (2.58) that

$$P=-\hbar\nabla \tag{2.59}$$

and the position operator is, as in quantum mechanics, the multiplication operator

$$Z=z\ . \tag{2.60}$$

The presence of partial differential operators in (2.58) and (2.59) shows that $H\theta^*$ and $P\theta^*$ cannot be defined for all vectors $\theta^*$ in $\overline{\upsilon}^*$. So, we have to be prepared to deal only with densely defined operators, i.e., operators with domains dense in $\overline{\upsilon}^*$. Any other quantum-mechanical operator is transferred in EQM along the same line.

In the (Euclidean) Schrödinger picture of the dynamics, the (real) probability amplitude $\theta^*=\theta_t^*(x)$, solution of an initial value problem on $\mathcal{M}\times I$,

$$-\hbar\frac{\partial\theta^*}{\partial t}=H\theta^*\ , \tag{2.61}$$

is considered as a vector in the time-dependent Hilbert space $\overline{\upsilon}_t^{*s}$ of the $\theta^*$ such that (2.19) is satisfied. The scalar product of $\overline{\upsilon}_t^{*s}$ is defined by Eq. (2.20), and $H$, $P$, and $Z$ are three fixed (unbounded) operators on this space.

$H$ and $Z$ are symmetric operators, but $P$ is skew symmetric. Indeed,

$$\begin{aligned}(\theta^*\mid P\varphi^*)&=-\hbar\int\theta\nabla\varphi^*dx\\ &=+\hbar\int\nabla\theta\varphi^*dx\\ &=-(P\theta^*\mid\varphi^*)\ .\end{aligned}$$

So

$$P^+=-P\ . \tag{2.62}$$

Since $v_t^{*s}$ is a real Hilbert space, a theorem of von Neumann tells us that $H$ and $Z$ have self-adjoint extensions. $P$ is not symmetric, but it is normal and closable.[18]

Let us examine the relation between the expectation of the dynamical characteristics of the Bernstein process introduced in Sec. II C and the operators $H$, $P$, and $Z$. For the Hamiltonian, integrations by parts give

$$(\theta^* \mid H\theta^*)=\int \theta H\theta^* dx$$
$$=\int \left[\frac{\hbar^2}{2}\nabla\theta\nabla\theta^* + V\theta\theta^*\right]dx$$
$$=E[-\tfrac{1}{2}BB_* + V] , \tag{2.63}$$

where relations (2.13), (2.11′) and (2.12′) have been used. Let us assume that this energy is positive, namely, for all $\theta_*$ such that the expectation is meaningful,

$$(\theta^* \mid H\theta^*) \geq 0 . \tag{2.63′}$$

In other words, $H$ is a positive operator on $\overline{v}^*$. Notice that Eq. (2.63) is consistent with the form of Yasue's Lagrangian in Eq. (2.46), namely, $-\frac{1}{2}BB_* - V$. On the other hand,

$$(\theta^* \mid P\theta^*)=-\hbar\int \theta\nabla\theta^* dx$$
$$=E[B_*] , \tag{2.64}$$

using the definition (2.11′) and (2.13). In the same way, we obtain

$$(\theta^* \mid Z\theta^*)=E[Z] . \tag{2.65}$$

The symmetry of the operators $H$, $P$, and $Z$, as defined in Sec. II B using the time-reversal operator $T$, is built into these relations. For example, $P$ is odd because $(\theta^*(x,-t) \mid P\theta^*(x,-t))=-(T\theta^*(x,t) \mid PT\theta^*(x,t))$ but this is obvious since $E[B_*]=E[\overline{V}]$ and the velocity of (2.38) is odd under time reversal.

In case we have to consider the product of two operators, for example $ZP$, we notice that this product has no symmetry under time reversal. So we introduce the following Euclidean analogue of symmetrization procedure:

$$ZP=\tfrac{1}{2}(ZP-PZ)+\tfrac{1}{2}(ZP+PZ)$$

and we verify, using Eqs. (2.38)–(2.39), that

$$(\theta^* \mid ZP\theta^*)=-E[Z\overline{U}]+E[Z\overline{V}]$$
$$=E[ZB_*] .$$

In the Euclidean Heisenberg picture of the dynamics, the probability amplitude $\theta^*$ is regarded as constant in the space $\overline{v}^{*H}$ of the function $\theta^*=\theta^*(x,t)$ solution of Eq. (2.61) and such that (2.19) is satisfied. The scalar product of the two vector of $\overline{v}^{*H}$ is defined by

$$(\varphi^*(x,t) \mid \theta^*(x,t))=\int \varphi(x,t)\theta^*(x,t)dx ,$$

which is indeed time independent, by Eqs. (2.16) and (2.17). Our three basic operators, but now functions of the initial time $-T/2$, are defined as before.

In order to describe the relation between these two pictures, we introduce the dynamical semigroup $T_{t+T/2}$ by

$$|\,\theta_t^*)=e^{-(t+T/2)H/\hbar}\,|\,\theta_{-T/2}^*)$$
$$\equiv T_{t+T/2}\,|\,\theta_{-T/2}^*) .$$

Since $H$ is self-adjoint and positive, a theorem of functional analysis attests to the fact that this is well defined. Notice that we have conservation of probability: $\|T_{t+T/2}\theta_{-T/2}^*\|=\|\theta_{-T/2}^*\|$, that is, $T_{t+T/2}$ is norm preserving but not one to one (therefore not unitary) since the heat equation is not time symmetric. If the index $H$ refers to the Heisenberg picture and $S$ to Schrödinger picture, we set

$$|\,\theta^*(t))_H=|\,\theta^*(-T/2))_S, \quad t\in I . \tag{2.66}$$

This is a constant vector. For an operator, formally,

$$A_H(t)=e^{(t+T/2)H/\hbar}A_S e^{-(t+T/2)H/\hbar}, \quad t\in I . \tag{2.67}$$

The Euclidean version of the Heisenberg equation of motion for operators follows formally from Eq. (2.67),

$$-\hbar\frac{d}{dt}A_H=[A_H,H]-\hbar\frac{\partial A_H}{\partial t}, \quad t\in I . \tag{2.68}$$

The comparison with the classical equation (2.56) yields, therefore, that, as shown by Dirac for quantum mechanics, EQM is founded on a very simple relation between the concept of Poisson bracket and commutator,

$$[A,B]=\hbar\{A,B\}_{\rm op} , \tag{2.69}$$

where $\{A,B\}_{\rm op}$ is the operator corresponding to the classical Poisson bracket defined in (2.57).

## III. PHYSICAL INTERPRETATION

### A. Euclidean quantum mechanics and probability

In EQM, the knowledge of a single positive function $\theta^*=\theta_t^*(x)$ in $\overline{v}^{*s}$ enables us to solve completely the theoretical problem associated to the classical Schrödinger's *Gedankenexperiment.* In particular, the dynamics of these diffusive particles, for $t\in I$, follows from $\theta_t^*$ and can be described in a probabilistic way [Eq. (2.45)], using the properties of the Bernstein processes, or in an analytical way [Eq. (2.61)], by Hilbert-space methods. Since $\theta^*$ has all the qualitative properties of the quantum wave function $\psi$, we call it the (pure) physical state of Schrödinger diffusive particles. This is an operational definition; the knowledge of $\theta_t^*$ is necessary and sufficient to solve our problem. This does not mean that we get in this way a complete description of the elementary processes involved in Schrödinger's experiment. By nature, a statistical description like ours is not relevant for this purpose. In spite of this physical "incompleteness," it is highly unlikely that there is some hidden variable theory able to complete EQM. Indeed, the data of EQM are inherently probabilistic and, given these data and the forces, the physical state $\theta_t^*$, $t\in I$, is all the information we need. Notice, however, that the present concept of state is not at all the one used in conventional statistical mechanics. But, according to our comment on the classical limit (end of Sec. II D), EQM concept of state can indeed be understood as a probabilistic generalization of a classical (pure) state regarded from the variational point of view. $\theta_t^*$ is also the exponential of a path integral, according to Eq. (2.48), that is, the exponential of a sum over history. But the distinctive feature of EQM is that $\theta_t^*$, $t \in I$, can be known, in principle, only from the data of a pair of boundary probabilities and then after the solution of Schrödinger's system (2.14). This aspect has important consequences for the physical interpretation of the theory (cf. Sec. III B).

Also notice that one of the lessons of EQM is that the unitarity of the evolution is not a necessary requirement for the conservation of probability [cf. Eq. (2.40)]. To paraphrase Feynman,[7] the concept of probability is not modified in EQM. What is changed, and changed radically, is the method of calculating probabilities. As in quantum mechanics, the specificity of the theory is connected to the distinction between probability amplitude $\theta_t^*$, solution of the deterministic initial-value problem (2.16), and probability $p_t = \theta_t \theta_t^*$, $t \in I$. It is worth observing that the emergence of this probability, in EQM, does not correspond to a more or less mysterious interpretation but to a theoretical fact, due to the physical nature of Schrödinger's *Gedankenexperiment.*

The key particularity of quantum probabilities is the existence of interference. We consider now the Euclidean version of this effect. Let $\theta^*(x,t)$, with $t \in I$, the (normalized) state of a free Schrödinger particle, for example,

$$\theta^*(x,t) = \left[\pi\left[\frac{b^2-t^2}{b}\right]\right]^{-1/4} \times e^{-bx^2/2(b^2-t^2)+x^2t/2(b^2-t^2)} = p^{1/2}(x,t)e^{-\bar{S}(x,t)} \qquad (3.1)$$

for $I \subset ]-b,b[$. From Eq. (2.47), the probability of the associated Bernstein process $Z_t$ is a Gaussian whose width at $t=0$ is determined by $b$ (Gaussian slit of width $b$). Using Eqs. (2.28), (2.39), and (2.39′), one checks that $Z_t$ indeed solves the Newton equation (2.45) for $V=0$.

In Sec. III B it shall be shown that the expectation of the position for a general free Schrödinger particle moves with a constant velocity. This constant is zero here, so we shall interpret (3.1) from a reference frame moving with the particles.

Now we consider the two-slits state

$$\theta_{\mathrm{TS}}^*(x,t) = \mathcal{N}[\theta^*(x-L,t)+\theta^*(x+L,t)] , \qquad (3.2)$$

where $\mathcal{N} = \frac{1}{2}(1+e^{-L^2/b})$ and $L$ is a constant. By linearity of the heat equation (2.16), $\theta_{\mathrm{TS}}^*$ is also the physical state of a free particle (superposition principle in the cone of the physical states) through two Gaussian slits at $\pm L$. With our choice of $\mathcal{N}$, $\theta_{\mathrm{TS}}^*$ is normalized. Using the notation of (2.47) and Eq. (2.13) one gets

$$p_{\mathrm{TS}}(x,t) = \mathcal{N}^2[p(x-L,t)+p(x+L,t) + 2p^{1/2}(x-L,t)p^{1/2}(x+L,t) \times \cosh[\bar{S}(x-L,t)-\bar{S}(x+L,t)] . \qquad (3.3)$$

This is the Euclidean version of the quantum interference of probabilities. Although the effect of the last term of Eq. (3.3) is less spectacular than in quantum theory (there is no constructive and destructive pattern because the quantum cosine is replaced by a hyperbolic cosine) it is without analogue in conventional probabilistic models. What is important is not the form of this interference term but the fact that

$$p_{\mathrm{TS}}(x,t) \neq p(x-L,t)+p(x+L,t) .$$

The real-time version of this is generally presented as not understandable at all from a classical point of view. It has been shown in Ref. 16 that to Eq. (3.3) corresponds a simple relation between the transition probabilities of the two Bernstein processes associated with the right-hand side of Eq. (3.1) and the transition probability of the superposed process associated with $\theta_{\mathrm{TS}}^*$. As in quantum theory, the superposition $\theta_{\mathrm{TS}}^*$ contains information not present in $\theta^*(x-L,t)$ and $\theta^*(x+L,t)$ taken separately, namely, the relative phase of these two vectors [cf. Eq. (2.47)], $\bar{S}(x-L,t)-\bar{S}(x+L,t)$. It is indeed one of the surprising lessons of EQM that some classical particles (Schrödinger particles) may have a phase. Also notice that according to the definition of the state for a Schrödinger particle, this state is a ray, i.e., it is actually defined up to an arbitrary phase. Another specific aspect of quantum probability is generally used to deny the existence of particle trajectories in this theory. In classical theories of Markov processes, if $P(b \mid a)$ denotes the probability of an event $b$, given that the event $a$ occurred, then we have the Chapman-Kolmogorov relation

$$\int_{m} P(Z_u \in B \mid Z_t = y)P(Z_t \in dy \mid Z_s = x) = P(Z_u \in B \mid Z_s = x) \qquad (3.4)$$

for any $t$ in between $s$ and $u$ (cf. Fig. 2). Now, in conventional quantum mechanics, a formula like (3.4) only exists for the probability amplitude denoted by $K$, i.e., the integral kernel of the Schrödinger equation (1.2),

$$\int_{m} K(u,z;t,y)K(t,y;s,x)dy = K(u,z;s,x) . \qquad (3.5)$$

The point is, of course, that $K$ is not a probability; only the square of its absolute value can be interpreted as a probability. So the classical intuition behind the very similar looking formula (3.4) cannot be transposed to Eq. (3.5). In Euclidean quantum mechanics, both aspects depicted by Eqs. (3.4) and (3.5) are simultaneously valid, without contradiction or difficulties of interpretation. The analogue of Eq. (3.5) is the semigroup property for $h = h(s,x,t,y)$, the integral kernel of $T_{t-s} = e^{-(t-s)H/\hbar}$,

$$\int_{m} h(u,z;t,y)h(t,y;s,x)dy = h(u,z;s,x) . \qquad (3.5')$$

As soon as the potential $V$ is not zero, $h$ is not a probability because, for any $x$ in $m$, $t$ in $I$,

$$h(s,x,t,m) = \int_{m} h(s,x,t,dy) < 1 . \qquad (3.6)$$

The official interpretation of this is that $1-h(s,x,t,m)$ represents the chance that the particle, starting from $x$, has vanished from the state space $m$ before the time $t$. Actually, there is no physics behind this interpretation; it is just a verbalization that the dynamics of the particle is poorly understood.

But, in EQM, the semi-group property for $h$ implies the Chapman-Kolmogorov equation for the (forward, for example) transition probability of our process since, by Eq. (2.12),

$$q(s,x,t,y)=h(s,x,t,y)\frac{\theta(y,t)}{\theta(x,s)} . \tag{3.7}$$

So, we have indeed the right here to use the intuition suggested by Eq. (3.4). In particular, the distinction between $h$ and $q$ cannot be used to deny the existence of trajectories (but cf. Sec. III B).

### B. Physical content and questions of interpretation

Let $A$ be a linear operator in $\overline{v}^*$. According to Sec. II E the expectation value of $A$ in the physical state $\theta_t^*$ is given by

$$(\theta_t^* \mid A\theta_t^*)\equiv\langle A\rangle(t) . \tag{3.8}$$

The evolution of $\langle A\rangle$ is described by the expectation of the Heisenberg equation of motion (2.68),

$$-\hbar\frac{d}{dt}\langle A\rangle=\langle [A,H]\rangle-\hbar\left\langle\frac{\partial A}{\partial t}\right\rangle . \tag{3.9}$$

This gives the connection between EQM and (Euclidean) classical mechanics. Let us apply (3.9) to the interesting case where [cf. Eq. (2.54)]

$$H=-\tfrac{1}{2}P^2+V(Z) . \tag{3.10}$$

Then, by Eq. (3.9),

$$\frac{d}{dt}\langle Z\rangle=-\frac{1}{\hbar}\langle [Z,H]\rangle , \tag{3.11}$$

$$\frac{d}{dt}\langle P\rangle=-\frac{1}{\hbar}\langle [P,H]\rangle .$$

To compute these commutators, it is sufficient to know their classical form and to use the relation (2.69). Since

$$\{z,H\}=-p \text{ and } \{p,H\}=-\nabla V$$

we get

$$\frac{d}{dt}\langle Z\rangle=\langle P\rangle ,$$
$$\frac{d}{dt}\langle P\rangle=\langle\nabla V(Z)\rangle . \tag{3.12}$$

This is the Euclidean version of the Ehrenfest theorem, to compare with the classical Hamilton equation (2.55). Its physical content is analogous to that of its quantum version. It is interesting to compare also Eq. (3.12) with the Newton law of motion (2.45) derived from the variational principle. Taking the expectation of this Newton law and permuting a time derivative with the expectation we find

$$\frac{d}{dt}E[D_*Z]=E[\tfrac{1}{2}(DDZ+D_*D_*Z)]=E[\nabla V] .$$

Since, on the other hand,

$$\frac{d}{dt}E[Z]=E[D_*Z]$$

the expectation of the Newton law is reduced to Eq. (3.12) if we associate the momentum $P$ with $D_*Z$. By the relation (2.64) we already know that this can be done. In other words, the Newton law of motion (2.45) can be interpreted as a stronger (Euclidean) version of the Ehrenfest theorem.

By definition, a (Euclidean) constant of motion $A$ is such that

$$[A,H]=0 \text{ and } \frac{\partial A}{\partial t}=0 . \tag{3.13}$$

It follows from (3.9) that for any physical state $\theta_t^*$,

$$\frac{d}{dt}\langle A\rangle=0 . \tag{3.14}$$

For example, the Hamiltonian $H$ of a conservative system is a constant of motion. Of course, it is also possible to directly verify this via the relation (2.63) and the equations of motion (2.41) and (2.44). As another example, for a free evolution ($V=0$), we find $\langle P\rangle=$const, as in quantum mechanics, so the expectation of the position, $\langle Z\rangle$, moves linearly in time. In this sense, the dynamical structure of EQM is really an analogue of the dynamical structure of quantum mechanics.

Now we come to some points of physical interpretation in EQM. From the probabilistic point of view, in EQM, we are dealing with a new class of diffusions, the Bernstein processes. Since, mathematically, any Markov process is Bernstein[23] and since the starting distribution (2.10) of our Bernstein process $Z_t$ can be understood in Markovian terms (forward *and* backward) one could interpret the Bernstein property as a minor aspect of the construction. It would result in a physical mistake. Bernstein processes are intrinsically time symmetric, in contrast with the description of a Markov process in terms of the transition function, which is essentially asymmetric. It may be technically useful to take into consideration this partial information associated with the Markovian point of view, and this is what we did in Sec. II D in order to define the most natural concept of action. But afterwards, we have to restore the time symmetry lost in using this partial information. Let us emphasize that the equation of motion of EQM is the Newton law (2.45) and not the heat equation. Of course, it is also possible to symmetrize *a priori* the diffusion processes in insisting on the simultaneous validity of the forward and backward Markov descriptions. But no physical insight is gained by this *a priori* requirement, of purely mathematical nature. If we know (how?) before beginning the construction that the resulting theory has to be time symmetric, this is sufficient but, from the operational point of view, the really natural question is, from what kind of experimental data can we get time-symmetric diffusion processes?

The concept of Bernstein processes answers clearly this question: these processes can be constructed only from the data of two boundary probabilities $p_{-T/2}(x)dx$ and $p_{T/2}(y)dy$. So we have to draw interpretative consequences from this unusual requirement.

The first one is that the definition of the Bernstein process $Z_t$, for any $-T/2\leq t<T/2$ requires knowing information only available after the completion of the experi-

ment. Indeed, since we use in a crucial way the final probability density $p_{T/2}(y)$ to solve the Schrödinger system (2.14), already at $t=-T/2+\epsilon$ (for $\epsilon$ arbitrarily small) the probability (2.13) of the process and its backward drift involve this future information. For example, in the two slits experiment, $p_{T/2}(y)$ certainly depends on the distance $2L$ between the slits so, at $t=-T/2+\epsilon$, the process $Z_t$ shall depend on this parameter, even if $T$ is large and the screen with the two slits is set up far from the source of diffusive particles. Therefore, the construction of the process $Z_t$ requires us to take into consideration the whole experimental setup (the black box of Fig. 1).

A consequence of this first observation is that the concept of trajectories naturally associated with the diffusion process $Z_t$, $t\in I$, that is a continuous function $t\to Z_t(\omega)$ for a fixed $\omega$ in the path space $\Omega$, cannot be used in this theory without any warning against a naively realistic interpretation. Most of the questions about the localization of the particle in the setup are indeed relevant only after the experiment is concluded. And if the trajectories are regarded as physically real, they are determined by some future information regarding the setup. This strongly suggests that in the physical context of Schrödinger experiment, the random trajectories of the classical particles are just *a posteriori* theoretical construction associated with the solution of our theoretical problem.

The direct physical meaning of the superposition principle for physical states (cf. two-slit experiment, Sec. III A) may also be questioned. EQM is the statistical theory of some relatively rare classical probabilistic events. It is hardly credible that there is some kind of physical "interaction" between them responsible for the superposition effect [Eq. (3.3)]. This effect certainly appears at the statistical level as a consequence of the linearity of the heat equation (2.16) but not as the expression of any kind of physical interference.

Also notice that the same kind of argument leads to a strong limitation of any naive conception of measurement. For example, given a couple of boundary probabilities it is clearly out of question to do an extra measurement somewhere in between the source and the screen without disturbing the final probability. This extra measurement would correspond to a change in the content of the black box (as the introduction of a new slit) and this was excluded by hypothesis.

Actually, a nontrivial question has been avoided until now: What is really a measurement in EQM? In the frame of conventional quantum mechanics, the problem appears at the very beginning of the theory, namely, in the interpretation of the uncertainty relations. So, before describing the Euclidean version of the problem, we first show the analogue of the famous Heisenberg result ($m=\mathbb{R}$, for simplicity). As suggested by Eq. (2.63), we define

$$\Delta z^2=\int z^2\theta\theta^* dz \tag{3.15}$$

and

$$\Delta p^2=\hbar^2\int \nabla\theta\nabla\theta^* dz \ . \tag{3.16}$$

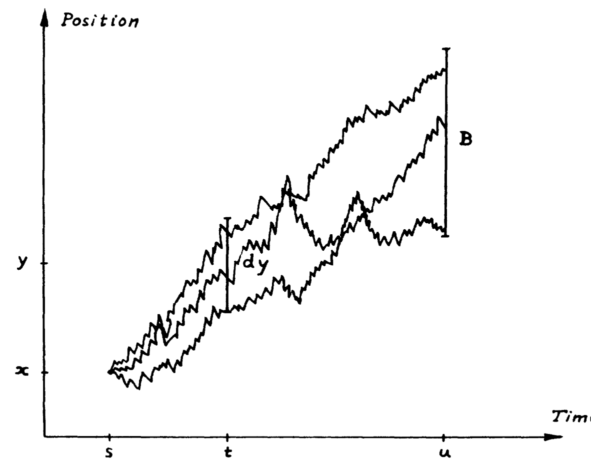


FIG. 2. Chapman-Kolmogorou equation.

Whereas (3.15) is positive by construction, this is *a priori* not the case for (3.16). It is therefore natural to emphasize the hypothesis of positive and finite kinetic energy,

$$0<\int \nabla\theta\nabla\theta^* dz<\infty \ . \tag{3.17}$$

In the definitions (3.15) and (3.16) it has been implicitly assumed that

$$\bar{z}=\int z\theta\theta^* dz \tag{3.18}$$

and [cf. Eq. (2.64)]

$$\bar{p}=-\int \theta\nabla\theta^* dz \tag{3.19}$$

are both zero. If this is not the case, but (3.18) and (3.19) are, respectively, $\bar{z}_0$ and $\bar{p}_0$, replace $z$ by $z-\bar{z}_0$ in (3.18) and change the phase of $\theta^*$ by $\theta^* e^{\bar{p}_0\cdot z}$ in (3.19). Then, the new means will vanish. We also can assume that $\Delta z=\Delta p$. Indeed, if $\gamma$ is a positive constant, the substitution $\theta^*_{\text{new}}(z)=\sqrt{\gamma}\theta^*(\gamma z)$ (which preserves the normalization of the original state) produces $\Delta z_{\text{new}}=\gamma\Delta z$ and $\Delta p_{\text{new}}=(1/\gamma)\Delta p$. So, $\Delta z_{\text{new}}\Delta p_{\text{new}}=\Delta z\cdot\Delta p$ and we can choose $\gamma$ such that $\Delta z=\Delta p$. When this is the case, $[\Delta z-\Delta p]^2=0$ so

$$\begin{aligned}\Delta z\cdot\Delta p&=\tfrac{1}{2}(\Delta p^2+\Delta z^2)\\ &=\int\left[\frac{\hbar^2}{2}\nabla\theta\nabla\theta^*+\tfrac{1}{2}z^2\theta\theta^*\right]dx\\ &=(\theta^*\mid H_{\text{os}}\theta^*)\end{aligned} \tag{3.20}$$

where Eq. (2.63) has been used, for $H_{\text{os}}$ the Hamiltonian of the harmonic oscillator,

$$H_{\text{os}}=\frac{\hbar^2}{2}\Delta+\tfrac{1}{2}Z^2 \ .$$

The positive [by (3.17)] functional (3.20) is minimized by the analytically continued fundamental state of $H_{\text{os}}$,

$$\theta^*_{\min}=\varphi_0(x)e^{-E_0t/\hbar}$$
$$=(\pi\hbar)^{-1/4}e^{-x^2/2\hbar}\cdot e^{-(1/2)t} . \quad (3.21)$$

The reason for this will be clear in Sec. IV. The corresponding minimal value of the functional (3.20) is $\hbar/2$. So, we obtain in the general case

$$\Delta z\cdot\Delta p\geq\frac{\hbar}{2} . \quad (3.22)$$

In the quantum-mechanical context, Heisenberg interpreted this relation as referring to mutually exclusive experimental arrangements, in which position and momentum cannot be simultaneously measured. This is not the only possible interpretation. The relation (3.22) can also be regarded as a statistical result without reference to the measurement of $z$ and $p$. In the classical context of EQM, this seems to be the only natural point of view since the lower bound $\hbar/2$ (actually the classical diffusion constant $\lambda/2$) can be made, in principle, arbitrarily small.

Nevertheless, let us come back to the case of a Gaussian slit (Sec. III A) from a different point of view. Suppose that free diffusion particles start at time $-T/2$ from the position 0 and that it is known that, at time 0 these particle are within distance $\pm\sqrt{b/2}$ from the origin. How can we be *a priori* sure of this localization property at time 0? In knowing the existence in the setup of a slit of width related to the parameter $b$, and able to constrain the particles. So, after the slit (at positive time $t$) it is reasonable to expect, from (3.5′), that the state of the free particles is given (up to a normalization) by

$$\theta^*(x,t)=\int_{-\xi}^{\xi}h_0\left[-\frac{T}{2},0,0,y\right]h_0(0,y,t,x)dy , \quad (3.23)$$

where $h_0$ is the kernel of the heat equation for $V=0$ (the free kernel) and $\xi=\xi(b)$ is a constant ensuring the localization of the particles at time 0. The introduction of the characteristic function $\chi_{[-\xi,\xi]}(y)$ in between the two free kernels enables us to replace the limits of integration on $y$ by $-\infty$ and $+\infty$. One checks easily that, replacing $\chi_{[-\xi,\xi]}(y)$ by the more manageable localization function $e^{-y^2/2\xi^2}$ for

$$\xi=\left[\frac{b}{1-\frac{b}{T/2}}\right]^{1/2}$$

with $0\leq b<T/2$, $\theta^*(x,t)$ is the one given in Eq. (3.1). Since this $\theta^*(x,t)$ is already decomposed in an even and an odd term under time reversal, according to (2.44), then we also know explicitly the associated $\theta(x,t)$.

The drifts of the corresponding Markovian Bernstein process are given by Eqs. (2.11′) and (2.12′)

$$B_*(x,t)=\frac{x}{b+t},\quad B(x,t)=-\frac{x}{b-t} . \quad (3.24)$$

They solve the free Newton equation (2.45) for $|t|<b$. It follows from Eqs. (3.15) and (3.16) that

$$\Delta p^2=\frac{1}{2b},\quad \Delta z^2=\frac{b}{2}-\frac{t^2}{2b} \quad (3.25)$$

(since $|t|<b$, $\Delta z^2>0$). The point is that, according to this $\Delta p^2$, if the slit is very narrow, $\Delta p$ is indeed very large. This could be interpreted as the effect of an "observation" of the particle through the slit at time 0. Notice in particular that there is no contradiction between the validity of the Newton law of motion and this *a priori* localization of the position. So, one could argue that the equation of motion of EQM can describe some "measurement" if we content ourselves with this very indirect notion of measurement.

It is also worth observing that, in the condition of validity of Eq. (3.3) for the two-slits case, we have no indication which slit the free particles go through, although these classical particles certainly go through either the first or the second slit. To have such an indication is equivalent to modifying the content of the setup, introducing an extra physical device whose effect will be to modify (at least) the final probability. A measurement corresponds to an interaction between our system and the setup in the black box. A free evolution with one slit between $S$ and 0 (Fig. 1) is different from a free evolution with two or three slits. In EQM, it is clear from the beginning that we are describing the whole setup. Is it necessary to develop a measurement theory for EQM as counterintuitive as the quantum one (assuming that this one deserves the appellation of "theory")? According to von Neumann, only the observation process is at the origin of the probabilistic nature of quantum mechanics; when the system is isolated, its behavior is deterministic. But nothing is deterministic in EQM (for nonzero diffusion constant) and the system is never isolated; it is in interaction with a medium responsible for its diffusion. So, whereas von Neumann concludes that the quantum-mechanical equations of motion do not describe the measurement process, it is not evident that the analogue is true in EQM.

Is it necessary to postulate, in EQM, that only operators defined according to the rules of Sec. II E can be measured? But, if so, is it possible really to describe these measurements? On the other hand, such a postulate would strongly reduce the number of observable random variables in EQM. What about the physical meaning of random variables not associated with operators?

These are some of the interesting open problems in the interpretation of EQM. Not accidentally, they are very reminiscent of the difficulties we meet in quantum mechanics.[31]

In summary, starting from the theoretical problem associated with the Schrödinger *Gedankenexperiment,* it has been shown that this problem is well posed and that its solution is Euclidean quantum mechanics, a statistical theory of some diffusive particles. Although this theory is perfectly classical, it reveals particularities without equivalent in classical probabilistic models, suggesting limitations in radically realistic interpretations of its results. In particular, the state $\theta^*$ of Schrödinger's particle appears to express more a kind of statistical relation between the system (the particles) and the black box (the experimental setup) than the exclusive properties of the system. Such a limitation of the realistic interpretation of this particular theory does not mean that we have to renounce realism (even in quantum mechanics this is not necessary[32]) but that, at the level of description associated with our starting hypotheses, some elementary phenomena are not accessible.

## IV. RELATIONS WITH QUANTUM MECHANICS

### A. From quantum mechanics to EQM

The main technical difficulty of EQM is that we cannot determine the Bernstein diffusion process $Z_t$, $t \in I$, before solving Schrödinger's system (2.14). Except for few simple cases (mainly when the two given boundary probabilities are Gaussian) this is very hard. Of course, the theory is consistent anyway since we have existence and uniqueness of the solution for (2.14). But it would be interesting to know how to produce systematically some dynamical Bernstein processes without solving this complicated nonlinear integral system. We are going to describe such a method; the result also will justify our appellation of "Euclidean quantum mechanics" for the theory presented before. Nevertheless, let us emphasize that this is far from being the only way to produce these processes.[17]

Suppose that we are given a (strict) *solution* $\psi_\tau$ of the Schrödinger equation (1.2) on the same time interval $I$,

$$i\hbar\frac{\partial\psi}{\partial\tau}=H\psi \tag{4.1}$$

for $H$ the Hamiltonian of Eq. (2.16). We also assume the (weak) regularity condition that the expectation of its energy is positive and finite, i.e.,

$$0<\int\left[\frac{\hbar^2}{2}\,|\nabla\psi_\tau|^2+V\,|\psi_\tau|^2\right]dx\equiv\langle\psi_\tau\,|\,H\psi_\tau\rangle<\infty\ , \tag{4.2}$$

where $\langle\ |\ \rangle$ is the inner product in the quantum-mechanical space $L^2(\mathscr{M})$. In analogy with Eq. (2.47) we represent this solution $\psi_\tau$ by

$$\psi(x,\tau)=e^{(R+iS)(x,\tau)/\hbar}\ . \tag{4.3}$$

Going further in the analogy, we define, as in (2.38′) and (2.39′) two vector fields $u$ and $v$ on $\mathscr{M}\times I$ such that

$$\begin{aligned}u(x,\tau)&=\nabla R(x,\tau)\ ,\\ v(x,\tau)&=\nabla S(x,\tau)\ .\end{aligned} \tag{4.4}$$

Now, the gradient of the quantum phase, $\nabla S$, is the analogue of the classical velocity. By assumption, we know in particular the initial velocity,

$$v_0=\nabla S(x,0)\ . \tag{4.5}$$

So we regard the representation (4.3) of $\psi_\tau$ as a function of $v_0$, denoted by $\psi_{v_0}(x,\tau)$ and define its analytical continuation in time, $t=i\tau$, ($\tau$ real) by

$$\begin{aligned}\theta^*(x,t)&=\psi_{-iv_0}(x,-it)\\&=e^{[R_{-iv_0}(x,-it)+iS_{-iv_0}(x,-it)]/\hbar}\\&\equiv e^{[\overline{R}(x,t)-\overline{S}(x,t)]/\hbar}\ .\end{aligned} \tag{4.6}$$

Knowing the dynamics of $\psi_\tau$ for $\tau\in I$ [i.e., the Schrödinger equation (4.1)] one obtains the following (real) equations of motion for $\overline{R}$ and $\overline{S}$,

$$\frac{\partial\overline{R}}{\partial t}=-\frac{\hbar}{2}\Delta\overline{S}-\nabla\overline{R}\cdot\nabla\overline{S}\ , \tag{4.7}$$

$$\frac{\partial\overline{S}}{\partial t}=-\frac{\hbar}{2}\Delta\overline{R}-\tfrac{1}{2}(\nabla\overline{R})^2-\tfrac{1}{2}(\nabla\overline{S})^2+V\ . \tag{4.7′}$$

Taking gradients and using the notations introduced in (2.38′) and (2.39′), Eq. (4.7) reduces to Eq. (2.41) and Eq. (4.7′) to Eq. (2.44). This means that Eq. (4.6) is nothing but indeed that a solution $\theta_t^*$ of the heat equation (2.16) in terms of which a Bernstein process $Z(t)$, $t\in I$, is defined. Since, under time reversal, $\overline{R}\to\overline{R}$ and $\overline{S}\to-\overline{S}$, the relation (2.18) shows that we also get in this way the associated solution of the backward heat equation (2.17), namely,

$$\theta(x,t)=e^{[\overline{R}(x,t)+\overline{S}(x,t)]/\hbar},\quad t\in I\ . \tag{4.6′}$$

But, by construction of the Bernstein processes (cf. Sec. II B), $\theta_t^*$ and $\theta_t$, $t\in I$, also are, respectively,

$$\theta_t^*=e^{-(t+T/2)H/\hbar}\theta_{-T/2}^*\quad\text{and}\quad\theta_t=k\theta_t^*$$

for $\theta_{T/2}$ and $\theta_{-T/2}^*$ the solutions of the Schrödinger system (2.14) with our imposed (by hypothesis) boundary densities of probability,

$$\begin{aligned}&p_{-T/2}(x)=e^{2\overline{R}(x,-T/2)}\\&\text{and}\\&p_{T/2}(y)=e^{2\overline{R}(y,T/2)}\ ,\end{aligned} \tag{4.8}$$

as far as they fulfill the conditions of existence and uniqueness for the solution of this system.

In summary, to each (regular enough) solution of the Schrödinger equation (4.1) is associated, after analytical continuation in time, a unique Markovian Bernstein process $Z_t$, $t\in I$. The point is that this process is explicitly determined in this way without solving the associated Schrödinger system. This gives us an easy and systematical way to produce Bernstein processes needed for EQM, and then the solution of the Newton equation (2.45), whose existence is already ensured by the results of Sec. II B. The energy condition (4.2) for any square integrable $\psi$ [with the self-adjointness of $H$ in $L^2(\mathscr{M})$] is the condition under which the analytical continuation in time of the quantum unitary group of evolution is indeed mathematically licit. It is simple to verify that its Euclidean version corresponds to the positive energy condition of Sec. II E,

$$0<\int\left[\frac{\hbar^2}{2}\nabla\theta_t\nabla\theta_t^*+V\theta_t\theta_t^*\right]dx<\infty \tag{4.9}$$

or, according to Eq. (2.63),

$$0<(\theta_t^*\,|\,H\theta_t^*)<\infty\ . \tag{4.9′}$$

Notice that, when $V=0$, this constraint is fulfilled, according to our positive and finite kinetic energy condition (3.17). So (4.9) is a condition on the potential $V$, satisfied, for example, if $V$ is bounded below.

A crucial qualitative feature of the Bernstein process follows from this property (it has already been shown in Sec. III B that $(\theta_t^* \mid H\theta_t^*)$, $t\in I$, is a constant of motion). By its very physical nature, EQM is a local theory in time. According to the hypothesis of Schrödinger's *Gedankenexperiment* Sec. II A, the experiment takes place in an arbitrary, but finite, time interval $I$ and our aim is to construct a diffusion process $Z_t$ on $I$ compatible with given boundary probabilities. We do not care about what happens outside $I$, because our experimental data are irrelevant there. Nevertheless, suppose that we insist on extending the Bernstein process $Z_t$, solution of our problem on $I$, outside this time interval. It follows from Eq. (2.63) and the definitions (2.38) and (2.39) that the conserved energy (4.9) may also be written as

$$e=\int (\tfrac{1}{2}\overline{U}^2-\tfrac{1}{2}\overline{V}^2+V)(x,t)p(x,t)dx \; . \tag{4.10}$$

Let us pick a value for $e$, necessarily positive and finite by (4.9). As time develops outside $I$, $\overline{U}$ and $\overline{V}$ can get large in such a way that their contribution to $e$ cancel. Therefore, the definitions (2.38) and (2.39) show that nothing prevents the drifts of $Z_t$, $B$, and $B_*$, from getting arbitrarily large in a finite time. In this case, the solution of the underlying stochastic differential equation can reach infinity in finite time with positive probability. This violated the hypothesis under which the existence and uniqueness of this solution is guaranteed. This phenomenon of "explosion" shows that $Z_t$ is only a local solution of the Euclidean Newton equation (2.45) in general.

Now we consider independently the question of "backward" analytical continuation in time, from EQM to quantum mechanics.

### B. From EQM to quantum mechanics

Suppose that we know a Bernstein process $Z_t$, $t\in I$, constructed from a solution of the Schrödinger equation (4.1) according to the method of Sec. IV A. In other words, we have existence and uniqueness of its measure $P_M$, consistent with the finite-dimensional distribution (2.10).

Taking the analytical continuation backward in (4.6), $\tau=-it$, we get the following equations of motion for $R$ and $S$ associated to $\psi$ by (4.3):

$$\frac{\partial R}{\partial \tau}=-\frac{\hbar}{2}\Delta S-\nabla R\cdot\nabla S \; , \tag{4.11}$$

$$\frac{\partial S}{\partial \tau}=\frac{\hbar}{2}\Delta R+\tfrac{1}{2}(\nabla R)^2-\tfrac{1}{2}(\nabla S)^2-V \; . \tag{4.11'}$$

Notice the changes of sign in (4.11′) with respect to (4.7′). Taking gradients as before and using the notations introduced in (4.4), one gets the real-time analogue of Eqs. (2.41) and (2.41′), namely,

$$\frac{\partial u}{\partial \tau}=-\frac{\hbar}{2}\,\mathrm{grad}\,\mathrm{div}\, v-\mathrm{grad}\, v\cdot u \; , \tag{4.12}$$

$$\frac{\partial v}{\partial \tau}=\frac{\hbar}{2}\Delta u+u\nabla u-v\nabla v-\nabla V \; . \tag{4.12'}$$

As Eq. (2.41′) is shown to be equivalent to the Euclidean Newton equation (2.45), Eq. (4.12′) modifies to

$$\tfrac{1}{2}(DD_*X+D_*DX)(\tau)=-\nabla V(X(\tau)) \; , \tag{4.13}$$

where $X_\tau$ is a Markovian diffusion process whose forward and backward drifts, denoted by $b$ and $b_*$ are related to $u$ and $v$ by the same formula as in EQM [Eqs. (2.38) and (2.39)],

$$v(x,\tau)=\tfrac{1}{2}[b(x,\tau)+b_*(x,\tau)] \; , \tag{4.14}$$

$$u(x,\tau)=\tfrac{1}{2}[b(x,\tau)-b_*(x,\tau)] \; . \tag{4.15}$$

The diffusion coefficient of $X_\tau$ is the same as the one of $Z_t$. Notice that the acceleration of the Newton equation (4.13) is different from the one involved in EQM [Eq. (2.45)] and that the classical force appears with the correct sign for a real-time theory.

In terms of $u$ and $v$, the energy condition (4.2) modifies to

$$0<\int (\tfrac{1}{2}u^2+\tfrac{1}{2}v^2+V)(x,\tau)\rho(x,\tau)dx<\infty \tag{4.2'}$$

for the quantum probability density $\rho(x,\tau)=|\psi(x,\tau)|^2$. This way to associate a Markovian diffusion process $X_\tau$ with a solution of the Schrödinger equation (4.1) has its origin in the works of Fényes and Nelson.[11-13] A functional analytical proof of existence and uniqueness of this process under the finite kinetic energy condition (its positivity is trivial) and for a rather large class of potential $V$ has been given recently by Carlen.[33] From the probabilistic point of view, the main qualitative difference between the processes $Z_t$ and $X_\tau$ is the following. In the real-time theory, the energy condition (4.2′) involves only positive signs under the integral. So, $u$ and $v$ are prevented from getting arbitrarily large since neither can get large without the (conserved) value of energy getting large. So the drifts $b$ and $b_*$ remain always bounded and the process $X_\tau$ can be arbitrarily extended outside the starting time interval $I$. This is, of course, what is expected in the quantum-mechanical context.

Both mathematical construction and physical interpretation of Nelson's theory ("stochastic mechanics") and EQM are very different. From the mathematical point of view, Nelson's theory produces global (in time) solutions of the Newton equation (4.13), itself derived from several versions of variational principles involving stochastic generalizations of the given classical Lagrangian [in the real-time analogues of the action (2.32), the Lagrangian is not the classical one; cf. (14.5) and (14.25) in Ref. 13 and (4.3) in Ref. 16]. From the physical point of view, the diffusion process $X_\tau$ is interpreted as the result of an interaction (presumably electromagnetic) of the starting classical mechanical system with a hypothetical background field. In other words, this is a remarkable attempt to describe quantum effects in terms of a realistic classical field theory, without any reference to the role of the observation.

The physical observables of the theory are random variables and the conventional quantum identification of observables with self-adjoint operators is regarded as epistemologically unsatisfactory.[13] In particular, any intrusion of noncommutativity in this probabilistic frame would be unnatural.

Euclidean quantum mechanics is the statistical theory of some (relatively improbable) classical diffusion processes. Its dynamics is contained in the local (in time) solutions of the Euclidean Newton equation (2.45), derived from a variational principle involving exclusively the classical (Euclidean) Lagrangian.

The existence of a Hilbert-space formulation of the theory is very useful. In particular, the relation (2.69) between classical Poisson bracket and commutator, used in conjunction with the Heisenberg equation (2.68), strongly limits the freedom of choice for the operators corresponding, in EQM, to classical dynamical variables and then for the definition of the associated random variables (Sec. II E). Moreover, this relation (2.69) is the key to the compatibility between the Newtonian aspect of EQM and the noncommutative aspect of its Hilbert-space formulation.

Experience has taught us that it is hard to understand the probabilistic meaning of the real-time acceleration of (4.13). In Ref. 17 the reason for this has been analyzed; the result is that, for a fixed potential $V$, the real-time analogue of the transition probability involves a kernel which depends nontrivially (but without any physical contradiction) on the starting solution of the Schrödinger equation. This means that the class of real-time processes associated to a given potential $V$ contains diffusions very different from each other. In EQM, this is not the case since, according to Eq. (2.12), all the corresponding transition probabilities are perturbations of the same kernel $h(x,t-s,y)$.

From the physical point of view, EQM is a classical statistical theory. The source of the fluctuations is the same as the one in any classical diffusion experiment [although, for convenience, the classical diffusion constant $\lambda$ of Eq. (2.1) has been denoted here by $\hbar$]. The relatively exceptional nature of these diffusions appears in the fact that they are time symmetric. Moreover, their statistical theory shows qualitative properties without analogue in regular statistical mechanics. Actually, it has been shown that the construction of this theory requires the introduction of concepts generally considered as the exclusive privilege of the quantum domain. In particular, this suggests some limitations in a truly realistic interpretation of the resulting diffusions, and also that a formulation of the observation process in this frame may be indirect and tricky.

In spite of the omnipresence of the mechanical references in EQM (trajectories, velocity, acceleration, Lagrangian, action) there is no assertion that this theory is of mechanical nature. For example, one shows in Ref. 16 that, due to the definition of the Bernstein process of EQM in the configuration space $\mathcal{M}$, EQM is a nonlocal theory, in complete analogy with quantum mechanics. [In Ref. 16, rhs of Eq. (4.26), the gradients $\nabla_1$ and $\nabla_2$ are exterior to the summation symbols $\sum$.] It is worth noticing, however, that this nonlocality is not surprising in our context since the Schrödinger *Gedankenexperiment* takes place in a medium responsible for the diffusion of the classical system. Indeed this medium is in contact with all the details of the experimental setup. From this point of view, the existence of EQM, where the reality of this underlying medium is obvious, lends weight to Nelson's background field hypothesis in quantum mechanics. The only mechanical aspect of EQM lies in the fact that nothing but the classical information is used to develop it. Nevertheless, it is more consistent, physically, to interpret EQM as an extension not of classical mechanics but of statistical mechanics.

In one class of physical situations, the results of Nelson's theory and EQM almost coincide; this is in the description of stationary states. Suppose that the potential $V$ of our Hamiltonian [Eq. (2.16)] is such that Eq. (2.16) admits a stationary solution

$$\theta_j^*(x,t)=\varphi_j(x)e^{-E_jt/\hbar} \tag{4.16}$$

for a real $\varphi_j$ (this is not a restriction) and in the one-dimensional case $\mathcal{M}=\mathbb{R}$. Equation (4.16) is clearly an analytically continued quantum stationary state. According to the method of Sec. IV A the two associated boundary densities of probability for the Schrödinger system are

$$p_j\left(x,-\frac{T}{2}\right)=p_j\left(x,\frac{T}{2}\right)=\varphi_j^2(x)\ . \tag{4.17}$$

The problem is that, except for the ground state, this invariant probability has zeros (nodes), in contradiction to the hypothesis under which the existence and uniqueness of the solution of Schrödinger's system (2.14) is assured. In Ref. 17 we show how to decompose the state space $\mathcal{M}=\mathbb{R}$ into connected domains $\Lambda$ between the nodes, in order to construct a Bernstein process on each domain $\Lambda$. It appears that the kernel $h=h_{\Lambda j}$ is different in each domain, due to the effect of the nodes at the boundaries of the domain. Let us also underline that the specific contribution of EQM, in all these stationary situations, is to describe in a completely explicit way the resulting diffusion processes. In any case, this partial coincidence of some results of stochastic mechanics and EQM shows that several outcomes of Nelson theory involving exclusively stationary processes can be reinterpreted in the context of Euclidean quantum mechanics.

Before concluding, it is interesting to compare the Feynman path-integral method, Nelson's theory, and EQM apropos of the question of time discretization of physical quantities. It is notorious, at least since Feynman's historical paper[5] that we are often faced with an excessive freedom when we are looking for the expectation of the time-discretized version of functions of operators. For example, let us consider the expectation of the kinetic energy. According to Eq. (4.2′), in Nelson's theory this energy is given by

$$E(\tfrac{1}{2}v^2+\tfrac{1}{2}u^2)=E(\tfrac{1}{4}b^2+\tfrac{1}{4}b_*^2)\ , \tag{4.18}$$

where Eqs. (4.14) and (4.15) have been used. This corresponds to a specific time discretization if we introduce the definitions (2.26) and (2.38) for $D_*X(\tau)=b_*(X(\tau),\tau)$ and $DX(\tau)=b(X(\tau),\tau)$. The point is that (4.18) is far from being the only time-symmetric candidate for a kinetic energy. Consider

$$E(\tfrac{1}{2}v^2-\tfrac{1}{2}u^2)=E(\tfrac{1}{2}bb_*)\ . \tag{4.19}$$

This is also a reasonable (time-symmetric) candidate for this energy in Nelson's theory [if we adhere to its point of view that the operators formulation of such an expectation does not necessarily give the right choice; if we do not, Yasue's kinetic energy (4.18) is the correct expression]. The form (4.19) is strongly reminiscent of the choice advocated by Feynman.[7] As a matter of fact, a kinetic term like (4.19) is only natural in EQM, according to Yasue's action (2.49) [and the energy (2.63)], namely, in the theory founded on the heat equation and not on the Schrödinger equation. This means that at least part of the freedom in these time discretizations is due to the coexistence of two (well-defined) probabilistic theories rather close in some aspects of their kinematical structure, namely, Nelson's theory and EQM.

## V. CONCLUSIONS

Fifty years ago Schrödinger concluded the paper which is at the origin of the present one in pondering over the possibility of describing quantum phenomena from the data of two boundary probability densities.[15] EQM answers positively to this question. Euclidean quantum mechanics is designed to be a tool, in the same sense as the Feynman path-integral method. The principal aim of the theory lies in its extension to Euclidean quantum fields. In this context, what is the analogue of the dynamical equation (2.45)? What is the analogue of the locality in time of the processes solving this equation, and its physical meaning? Since the action and the path-integral representation for the solutions of the heat equation are quite different here from the Feynman-Kac formula, is it possible to have a better control of the divergences? These are some of the first problems to investigate.

In any case, the existence of EQM shows that there is indeed a physical point in analytical continuation in time, but that the resulting Euclidean dynamical theory is essentially without connection to the Feynman-Kac formula.

Let us observe that the construction given here is extended without difficulty to the case where the starting classical system is subject to an external electromagnetic field. It is also virtually unchanged if the configuration space $\mathcal{M}$ is a Riemannian manifold with invariant volume $d_{\mathcal{M}}x=\sqrt{c}\,dx$, for $c=(c_{ij})$ the Riemannian metric tensor and for the starting heat equation

$$-\hbar\frac{\partial\theta^*}{\partial t}=-\frac{\hbar^2}{2}\nabla^i\nabla_i\theta^*+V\theta^*$$

considered in $\overline{v}^*(\mathcal{M},d_m x)$.

On the other hand, EQM is the closest classical analogy of quantum mechanics and may also be very useful at this level. Most of the counterintuitive particularities of quantum mechanics are already present in this classical statistical theory. The point is that the physical context of Schrödinger's *Gedankenexperiment* is far from being as trivial as the one of regular diffusion experiments. Nevertheless, it would be surprising not to be able to use EQM for purifying the foundations of quantum mechanics of some pseudoparadoxes already present in its classical analogue.

To what extent can the interpretations of Sec. III B be directly transposed in conventional quantum mechanics? Is it, or is it not, possible to avoid in EQM the elaboration of a measurement-theory analogue to the quantum one? In any case, EQM seems to be compatible with several interpretative approaches.

The most intriguing point of EQM may be that it constitutes a call for (subtle) experiments. In spite of the message of the Copenhagen school, is it not possible to set up some of the classical diffusion experiments suggested here and whose qualitative results are as puzzling as the ones of quantum theory? In a recent paper concerned, in parts, with the real meaning of quantum mechanics, Feynman considered the possibility "of things being affected not just by the past, but also by the future, and therefore that our probabilities are in some sense 'illusory.' We only have the information from the past, and we try to predict the next step, but in reality it depends upon the near future which we can't get at, or something like that. A very interesting question is the origin of the probabilities in quantum mechanics."[34]

Maybe EQM may help to reconsider this kind of question and, at the very least, the relations between quantum mechanics and classical probabilities.

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