Driven sine-Gordon breathers as kink-antikink bound states of nonrelativistic constant wave number

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We give a noncovariant Hamiltonian collective-coordinate description of the sine-Gordon breather driven by constant forces. Thresholds for breather decomposition are calculated, including their dependence on the initial phase. Quantitative agreement is obtained with the numerical results presented in a paper by Lomdahl, Olsen, and Samuelsen [Phys. Rev. A **29**, 350 (1984)].

I. INTRODUCTION

In a recent paper,[1] the problem of "breather-kink-antikink-pair conversion in the driven sine-Gordon system" was considered through an ansatz method and a potential energy argument. Lomdahl, Olsen, and Samuelson[1] checked their theoretical predictions by means of highly accurate numerical simulations, and observed rather good agreement in only half the range of the initial phase φ of the breather. The reason emphasized by the authors themselves is that none of their analyses took φ into account. Such a flaw is a serious drawback as the energy gain of the system due to the perturbation strongly depends on φ. The attempts to include the φ effect in a breather perturbation theory are not numerous in the literature. In connection with this point, we draw the attention of the reader to a recent paper[2] in which the study of a periodically driven SG breather in the nonlinear Schrödinger regime reveals an important phase effect.

One should mention the tentative approach of Ref. 3 in which the phase dependence is alluded to through a collective coordinate method leading to a Hamiltonian description. However, this description is inappropriate inasmuch as the chosen ansatz, consisting of the superimposition of one kink and one antikink, both relativistic, cannot describe a breather except in the limiting case of a zero frequency. In Ref. 4 a similar point of view is adopted, and consequently only small threshold values $\epsilon_{\rm cr}$ (corresponding to small breather frequencies) of the driving force ϵ for breather decay into the kink-antikink pair are obtained (typically, values less than 0.15, in dimensionless units). We checked that they fit the numerical simulations performed in Ref. 1 when $\varphi=0$. But no other φ dependence of the breather decay is displayed.

The situation drastically changes with Ref. 5, in which a first-order perturbation of the SG inverse scattering theory (SGIST) leads to the kinetic equations of the breather parameters including its phase. This theory is quite extensive and gives the general equation that determines the critical value of the force $\epsilon_{\rm cr}$ as a function of the initial state of the breather. We checked that the corresponding values of $\epsilon_{\rm cr}$ for $\varphi=0,\ -\pi/2,\ +\pi/2$ fit rather well the threshold values numerically determined in Ref. 1.

The aim of our work is to reconsider a collective-coordinate Hamiltonian description *a la Inoue* but this time breaking the Lorentz covariance of each component of the ansatz. In fact, it amounts to keeping the internal degree of freedom of the (anti)kink constant, thus allowing only the relative position of the components to vary. The success of this simple theory is spectacular; our predictions of breather breakup for various values of φ and ω_B, though being not far from those which come from the formulas in Ref. 4, are unquestionably better fitting those in Ref. 1. It suggests that a driven breather in its center-of-mass frame can be regarded as a kink-antikink bound state in which the (anti)kink internal degree of freedom—its wave number k (see Fig. 6)—is "frozen" and remains equal to a given value less than unity [actually, the breather parameter is $(1-\omega_B^2)^{1/2}$]. This remarkable property [remember that k for a single (anti)kink is always greater than unity] seems to proceed from the fact that in the pure SG case, a breather is *exactly* the algebraic sum of a kink and of an antikink profiles in which the Lorentz factor is replaced by a constant parameter equal to $(1-\omega_B^2)^{1/2}$. Hence our result may also appear as an extension of this property to given multisoliton systems.

This extension is quite natural. Indeed, Matsuda has shown (see Ref. 7) that the latter property of the unperturbed breather is related to a simple scheme based on a Lagrangian variational principle. Our result then means that an initial breather described in its center-of-mass frame responds to an external driving force so that Matsuda's variational principle will still be relevant. The strategy used in Ref. 7 aims to reduce the field equations to equations of motion for classical particles where the center of a solitary solution is the particle's coordinate. Therefore it actually seems quite sensible to use a simple collective-coordinate Hamiltonian method in the presence of perturbation.

The very fact that this Hamiltonian method yields results at least as good as an explicit first-order SGIST perturbation scheme does, is of a true theoretical interest and emphasizes the highly specific wave-particle duality of the

SG kink. Indeed, there are various cases in which these kinks may be considered as particles—in fact, for time values much greater than a phonon-wave period (see Ref. 8). When interacting with a driving force alone, they behave like purely classical relativistic particles (see Refs. 9 and 10) whereas in the presence of a confining potential well, their dynamical behavior is still somewhat relativistic but in a rather involved manner (see Ref. 6). Finally, when interacting in bound-state systems with external forces, the covariance of the corresponding fields is broken.

II. THE HAMILTONIAN COLLECTIVE-COORDINATE TRANSFORMATION AT CONSTANT WAVE NUMBER

Consider the following algebraic equality:

$$4\tan^{-1}\exp[k(x+z)]+4\tan^{-1}\exp[-k(x-z)]-2\pi = 4\tan^{-1}\left[\frac{\sinh(kz)}{\cosh(kx)}\right], \quad (1)$$

where $z=z(t)$, which expresses the sum of a kink and of an antikink profile with wave number k, as a breatherlike envelope profile. Note that for the (anti)kink in the pure SG case, one would have

$$k=(1-\dot{z}^2)^{-1/2}\geq 1 , \quad (2a)$$

$$\dot{z}=\frac{dz}{dt}=\text{const}<1 , \quad (2b)$$

while the breather would be determined by

$$\sinh[kz(t)]=\frac{k}{\omega_B}\sin[\omega_B t+\varphi],\quad k\leq 1 \quad (3a)$$

where

$$\omega_B=(1-k^2)^{1/2} \quad (3b)$$

is the breather frequency and φ is the breather initial phase which plays a crucial role in the present work. Insert the right-hand side of (1) as an *ansatz* function ϕ in the Hamiltonian

$$H=\int_{-\infty}^{+\infty}dx\,(\tfrac{1}{2}\phi_t^2+\tfrac{1}{2}\phi_x^2+1-\cos\phi-\epsilon\phi) \quad (4)$$

of the field obeying the driven SG equation,

$$\phi_{tt}-\phi_{xx}+\sin\phi=\epsilon\quad(\epsilon>0) . \quad (5)$$

Now assume

$$\dot{k}=0 . \quad (6)$$

Then, defining

$$X=2kz , \quad (7)$$

one obtains,

$$H=2k^{-1}\left[1+\frac{X}{\sinh X}\right][\dot{X}^2+4\tanh^2(X/2)] +8k\left[1-\frac{X}{\sinh X}\right]-2\pi k^{-1}\epsilon X . \quad (8)$$

Since the Hamiltonian (4) of the system does not explicitly depend on time, the total energy is conserved and if one takes $X(0)$ and $\dot{X}(0)$ such that ϕ $(t=0,x)$ and ϕ_t $(t=0,x)$ are those of an unperturbed breather at $t=0$, one has

$$H=E=16k-2\pi k^{-1}\epsilon X(0) , \quad (9)$$

which reads

$$\dot{X}^2+4\tanh^2(X/2)+\left[1+\frac{X}{\sinh X}\right]^{-1} \times\left[4k^2\left[1-\frac{X}{\sinh X}\right]-\pi\epsilon X-\tfrac{1}{2}Ek\right]=0 . \quad (10a)$$

Equation (10a) is the equation of motion for the unique degree of freedom X of the system (1) and (5) [cf. (6) and (7)]. Using (9), (10a) becomes

$$\dot{X}^2+4\tanh^2(X/2)-\pi\epsilon\frac{X-X(0)}{1+X/\sinh X}-4k^2=0 , \quad (10b)$$

where $X(0)$ is given by (3a) and (7),

$$X(0)=2\sinh^{-1}\left[\frac{k}{\omega_B}\sin\varphi\right] . \quad (11)$$

For a given value of the wave number k (i.e., of the breather frequency ω_B), the potential

$$W_k(X,\varphi,\epsilon)=4\tanh^2(X/2)-\pi\epsilon\frac{X-X(0)}{1+X/\sinh X}-4k^2 \quad (12)$$

has an absolute maximum on the positive X axis (the only physically relevant part of the X axis since $\epsilon\geq 0$) at $X=X_M(\varphi,\epsilon)>0$. The condition [deduced from (10b)]

$$W_{\max}=W_k(X_M,\varphi,\epsilon)>0 \quad (13)$$

allows oscillations of the degree of freedom X in the range $X\leq X_0<X_M$ [X_0 being the smallest positive zero of $W_k(X,\varphi,\epsilon)$], which correspond to the existence of a driven oscillating bound state of constant wave number k. Clearly the condition

$$W_{\max}=W_k(X_M,\varphi,\epsilon)=0 \quad (14)$$

defines a critical value

$$\epsilon=\epsilon_{\text{cr}}(\omega_B,\varphi) , \quad (15)$$

which is an upper bound for the values of ϵ allowing the existence of such driven *quasi*breathers. When $\epsilon>\epsilon_{\text{cr}}$, the breather breaks up into a kink-antikink pair.

Choosing the representation $(-\pi/2,+\pi/2)$ for $\sin^{-1}$, it may be seen that, given a fixed ω_B, $\epsilon_{\text{cr}}(\varphi)$ monotonically increases with φ while $X_M-X(0)$ decreases and even vanishes at $\varphi=\pi/2$. The latter situation is exceptional since the "particle" whose dynamics is described by (10) and (11) stands on the top of the potential (12) and is unstable. The corresponding wave (neglecting the $\sin^{-1}\epsilon$ vacuum state) is the (unstable) stationary solution of the driven SG equation (5), called a "boxon" (see Ref. 11) because of its rectangular boxlike profile. This boxon dynamical state is defined by

$$X(t)=X(0)=X_M \ . \tag{16}$$

Actually, in this case, (14) is identically verified for $\varphi=\pm\pi/2$ since one has

$$\tanh^2\sinh^{-1}(k/\omega_B)=k^2 \ . \tag{17}$$

Since $X_M>0$, only the value $\varphi=+\pi/2$ is relevant. Equating the X derivative of $W_k(X,\varphi,\epsilon)$ evaluated at $X=X(0)$ to zero and replacing $X(0)$ by its value (11) at $\varphi=\pi/2$, yields

$$\epsilon_{cr}\left[\omega_B,\frac{\pi}{2}\right]=\frac{4}{\pi}\omega_B^2\left[(1-\omega_B^2)^{1/2}+\omega_B^2\sinh^{-1}\left[\frac{(1-\omega_B^2)^{1/2}}{\omega_B}\right]\right] . \tag{18}$$

The critical value ϵ_{cr} of the force is plotted in Fig. 1 as a function of ω_B for $\varphi=-\pi/2,0,+\pi/2$ and compared to the numerical values obtained in Ref. 1. Table I gives a comparison between the values from Ref. 1 and the theoretical ones given in Ref. 5 as well as those predicted in the present paper. Concerning the comparison with the numerically determined threshold forces of Ref. 1 at $\varphi=\pi/2$, the agreement is acceptable. Note that the function defined by (14) and (15) is flat in ϵ at $\epsilon\simeq\epsilon_{cr}(\omega_B,\pi/2)$. Indeed (12), (14), and (16) imply

$$\left.\frac{dW_{max}}{d\epsilon}\right|_{X=X_M}=\left.\frac{\partial W_{max}}{\partial\epsilon}\right|_{X=X_M}+\left.\frac{\partial W_k}{\partial X}\right|_{X=X_M}\frac{\partial X_M}{\partial\epsilon}=\left.\frac{\partial W_{max}}{\partial\epsilon}\right|_{X=X_M} \tag{19}$$

and

$$\left.\frac{\partial W_{max}}{\partial\epsilon}\right|_{X=X_M=X(0)}=0 \ .$$

Therefore the function $W_k(X_M,\pi/2,\epsilon)$ has a minimum value equal to zero for $\epsilon=\epsilon_{cr}(\omega_B,\pi/2)$ [see (15)]. Hence (19) leaves a rather important uncertainty in the physical determination of $\epsilon_{cr}(\omega_B,\pi/2)$, although the mathematical one is unambiguously obtained from (14) and (15). Direct numerical simulations of the driven SG equation (5) show that this range of uncertainty is indeed comparable to the interval in ϵ values separating our theoretical values (18) from the numerical ones obtained in Ref. 1. Qualitatively speaking, what happens when $\epsilon\lesssim\epsilon_{cr}(\omega_B,\pi/2)$ is a rather fast damping of the breather oscillations (over a few periods) and the simultaneous appearance of large-amplitude nonlinear periodic waves which rapidly become as important as the original breather (see Fig. 2).

When $\varphi\neq\pi/2$, such a continuous decay of the original breather into a train of large-amplitude perturbed cnoidal waves does not occur because $(dW_{max}/d\epsilon)_{X=X_M,\epsilon=\epsilon_{cr}}\neq 0$ (actually it is negative). So at $\epsilon=\epsilon_{cr}(\omega_B,\varphi)$ for $\varphi<\pi/2$, the driven breather clearly separates into an asymptotically free kink-antikink pair. In the general case, ϵ_{cr} is implicitly given by the formulas (12), (14), and (15) of the theory. Our results are obtained through a trivial numerical solution of the system (14) and (15), and we note (as shown in Fig. 1) a good agreement between our ϵ_{cr} values and the experimental values given in Ref. 1.

We recover all these results in Fig. 3 which shows a comparison of the numerically obtained critical values ϵ_{cr} of Ref. 1, with those of the present theory for a rather large value of ω_B ($\omega_B=0.6$). The agreement is quite concluding.

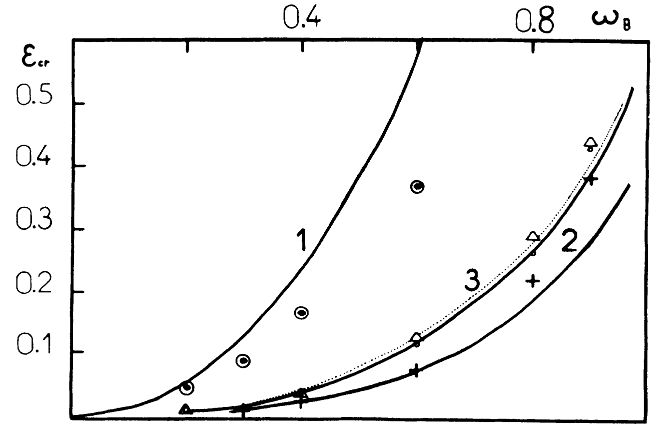

FIG. 1. Comparisons between our ϵ_{cr} curves (solid curves) at $\varphi=+\pi/2(1),-\pi/2(2),0(3)$, and the numerical threshold obtained in Ref. 1 as well as the theoretical curve (dotted curve) obtained through a potential energy argument in the same paper. The numerically determined points of Ref. 1 are for $\varphi=-\pi/2$ (+), 0 (○), $+\pi/2$ (⊙), π(△).

TABLE I. Comparison of threshold values ϵ_{cr} for $\omega_B=0.2$, 0.3, 0.4, 0.6, 0.8 and $\varphi=-\pi/2$, 0, $+\pi/2$. Our theoretical predictions are labeled (1). The numerical values of Ref. 1 are labeled (2) and are given within the precision we could obtain by a direct reading of Fig. 5 presented in Ref. 1. We calculated the threshold values [labeled (3)] of the theory presented in Ref. 5, through the formulas given in the same paper. Our results and those of Ref. 5 are akin for $\omega_B<0.5$. The agreement is not as good at $\varphi=+\pi/2$.

φ	ω_B	0.2	0.3	0.4	0.6	0.8
$-\frac{\pi}{2}$	(1)	0.004	0.011	0.022	0.068	0.173
	(2)	0.005	0.010	0.020	0.075	0.215
	(3)	0.005	0.010	0.022	0.063	0.145
0	(1)	0.007	0.018	0.036	0.108	0.260
	(2)	0.010	0.015	0.035	0.115	0.265
	(3)	0.008	0.018	0.038	0.102	0.235
$+\frac{\pi}{2}$	(1)	0.055	0.129	0.238	0.548	0.850
	(2)	0.045	0.080	0.160	0.360	
	(3)	0.055	0.133	0.257	0.725	1.876

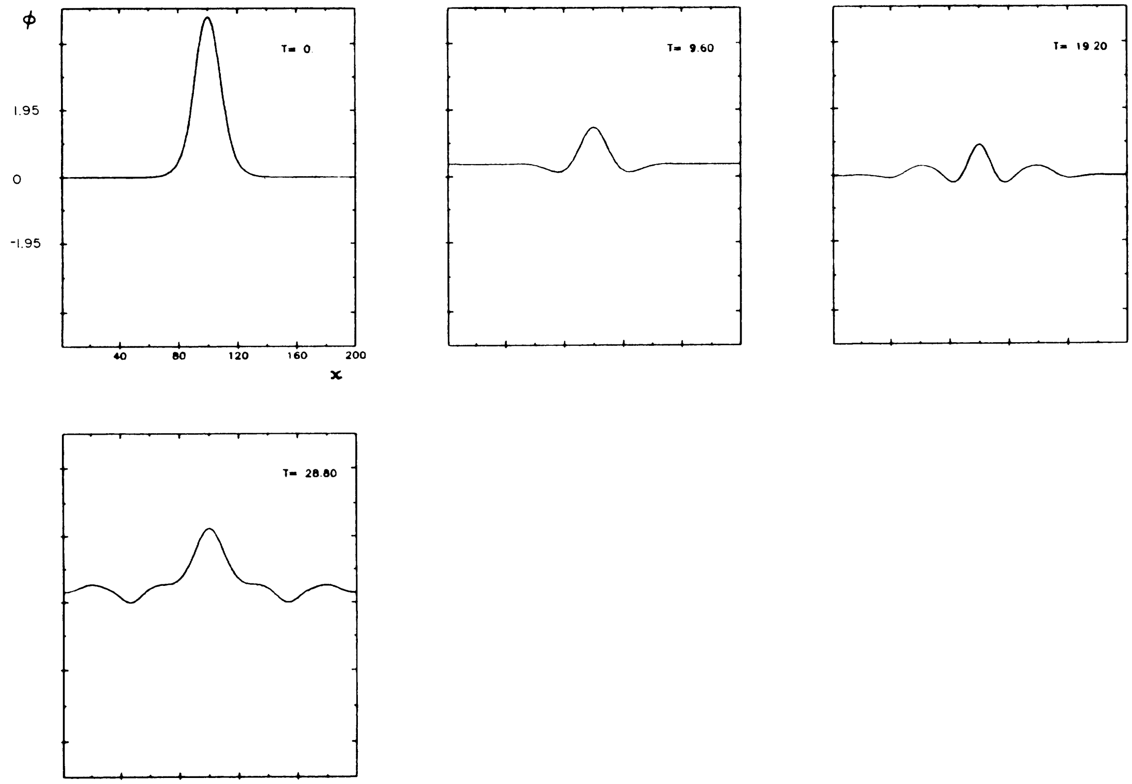

FIG. 2. A time sequence shows the field ϕ. The parameters are $\varphi=+\pi/2$, $\omega_B=0.4$ and $\epsilon=0.18$ (i.e., just at half distance between our predicted threshold value and the one numerically obtained in Ref. 1). This sequence was obtained through a direct numerical solution of Eq. (5). The initial breather does not break up but decays into large-amplitude nonlinear waves.

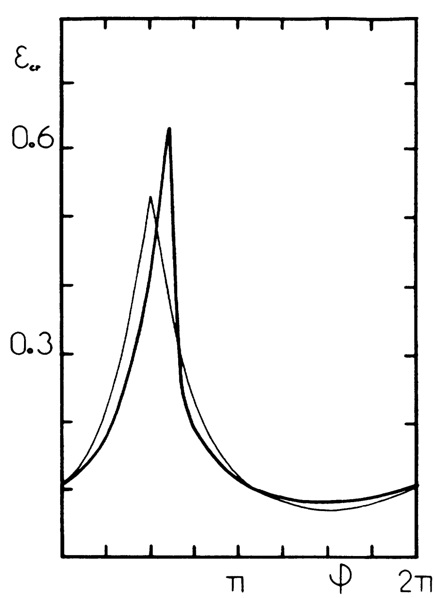

FIG. 3. Influence of the initial phase of the breather. Comparison between our theoretical curve (thin solid curve), and the one numerically obtained in Ref. 1 (thick solid curve). Our curve reaches its maximum at $\varphi=\pi/2$. Here $\omega_B=0.6$.

III. COMPARING WITH THE COLLECTIVE-COORDINATE TRANSFORMATION AT RELATIVISTIC WAVE NUMBER

At this stage of the present study, it seems necessary to emphasize the differences between our theory and the relativistic collective-coordinate Hamiltonian treatment given in Refs. 3 or 4. Both methods lead, of course, to Eq. (8) since they both assume (6) when starting from an original ansatz of type (1). Then in Refs. 3 and 4, (2a) is assumed but not (2b). As a consequence (8) becomes (see Ref. 3)

$$H=16k(t)-8k(t)^{-1}\left[\left[1+\frac{X}{\sinh X}\right]\Big/[\cosh^2(X/2)]\right] -2k(t)^{-1}\pi\epsilon X\ . \qquad (20)$$

The further treatment of this dynamical problem is not obvious since the corresponding equation of motion (9) reads

$$16k(t)^2-8\left[\left[1+\frac{X}{\sinh X}\right]\Big/[\cosh^2(X/2)]\right] -2\pi\epsilon X-Ek(t)=0\ , \qquad (21a)$$

where

$$\dot{X} \simeq 2(k^2-1)^{1/2} \quad (21b)$$

(see Ref. 7). In order to reduce the strong contradiction consisting in assuming (6) and rejecting (2b), one has to further assume

$$|\dot{z}| \ll 1 . \quad (22)$$

Strictly speaking, the only possible *real* value for ω_B is then zero, i.e., for $\dot{z}=0$. Therefore such a "hybrid" theory as (20) and (21) can account for the existence of only small threshold values. Indeed, Ref. 3 considers values of ϵ_{cr} lower than 0.15, which fit the values obtained by the present theory since approximation (22) then basically reduces the dynamical system (21a) to (10a).

The greatest flaw of this adiabatic relativistic kink-antikink ansatz method lies in its basic impossibility to describe the phase effects involved in the mechanism of a breather breakup. The point is to match, at $t=0$, the ansatz (1)

$$\phi\,(t=0,x)=4\tan^{-1}\{\sinh[kz(0)]/\cosh(kx)\} \quad (23)$$

(where k is the Lorentz factor and therefore is greater than or equal to 1), to the initial breather profile (see Ref. 4)

$$\phi_{Br}\,(t=0,x)=4\tan^{-1}\left[\frac{\tilde{k}}{\omega_B}\sin(\varphi)/\cosh(\tilde{k}x)\right] \quad (24)$$

[where $\tilde{k}=(1-\omega_B)^{1/2}\le 1$]. The only possibility is obviously $\varphi=0$ and $z(0)=0$, and indeed, Ref. 4 displays threshold values ϵ_{cr} versus $2\omega_B^2$ which fit our values for $\varphi=0$.

IV. SOME REMARKS ON THE FURTHER STEP TOWARD A MORE DETAILED THEORY

Concluding this paper, one should notice that we did not take the vacuum into account [in fact, we fixed ϕ $(t,x=\pm\infty)$ to zero]. As stressed in Ref. 1, none of the numerical results presented there were sensitive to the choice of the vacuum. Nevertheless, if one separates the ansatz function into a soliton part ϕ_0 [the right-hand side of (1)] plus an x-independent background u_0, one obtains a system of coupled equations of motion for X and u_0. At the moment, this system is being investigated and there is some evidence that it leads to threshold values still closer to the experimental ones obtained in Ref. 1. We also have some hint that introducing the vacuum should give results that would be slightly shifted (in comparison with the results of the theory presented in this paper), but that would not depend on the initial value of the vacuum for sufficiently small values of ω_B.

In the present paper we showed how one can describe the breatherlike part of the solution of Eq. (5) when starting with an initial breather, in terms of a simple (only one degree of freedom) collective-coordinate Hamiltonian method. That this description works even without including the vacuum is already very encouraging. We feel that this is backing up the notion that SG solitons tend to behave like particles in many perturbation cases.

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