Output power in guided modes for amplified spontaneous emission in a single-pass free-electron laser

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(Received 19 November 1986)

Treating diffraction effects within the paraxial approximation, we solve the initial-value problem determining the start-up of a single-pass free-electron laser from shot noise in the electron beam. Linearized Vlasov-Maxwell equations are used to derive an equation for the three-dimensional slowly varying envelope function of the radiated electric field. In the high-gain regime before saturation, the output power is expressed in terms of Moore's exponentially growing guided modes. For a cylindrical monoenergetic electron beam with step-function profile, explicit numerical and analytical calculations have been performed, determining the power in the guided modes. The condition for the dominance of the fundamental mode is discussed. Our solution of the initial-value problem is based upon a Green's-function technique, and our results are derived despite the lack of orthogonality and completeness of the guided modes. The Green's function is expanded in terms of an orthonormal set of eigenfunctions of a two-dimensional Schrödinger equation with non-self-adjoint Hamiltonian. In the limit of a long wiggler, the asymptotic representation of the Green's function is found to be dominated by the contribution of the guided modes. The radiated electric field, and hence the output power, is determined with use of the Green's function.

I. INTRODUCTION

This paper is concerned with the theory of amplified spontaneous emission in a long wiggler magnet. When the gain is high enough, the incoherent emission from individual electrons can be amplified to saturation in a single pass. This approach to the generation of high-intensity coherent radiation at short wavelengths has the attractive feature that the use of an optical resonator is avoided.[1] In the theoretical description of the amplification of a coherent initial signal in a free-electron laser, it is reasonable to keep only a single wavelength in the analysis. On the other hand, in the case of amplified spontaneous emission the initial signal is neither coherent nor monochromatic, hence bandwidth must be taken into account and a single-wavelength analysis is not sufficient. The evolution of coherent radiation from the incoherent emission of individual electrons must be described.

This is accomplished by considering the spontaneous emission in the wiggler as resulting from the shot noise in the electron beam, and carrying out an average over the ensemble of initial conditions of the electrons. Such an analysis was originally performed within a one-dimensional model,[2,3] in which individual electrons were treated as charge sheets. Later, three-dimensional effects were included for the special case of an electron beam of infinite transverse extent.[4] Here, we treat an electron beam with finite transverse dimension, extending the earlier work of Moore[5] on mode guidance in the amplification of a coherent initial signal. We solve the initial-value problem describing the start-up of the amplified spontaneous emission process from the shot noise in the electron beam, and we express the output radiation field as a superposition of Moore's exponentially growing self-similar modes.[6] The output power is computed in the high-gain regime before saturation.

In a free-electron laser, gain plays an important role in the mode guidance. Unlike guiding in a system having purely real index of refraction, Moore's self-similar modes have power flowing radially away from the electron beam. Field energy is created inside of the electron beam at a fast enough rate to maintain the transverse mode distribution, with maximum field strength at the electron beam center and exponentially decaying field strength outside of it.

Consider a cylindrical electron beam with a step-function transverse profile, the electron density having constant value n_0 within radius $r<r_0$, and vanishing for $r>r_0$. We denote the period length of the wiggler magnet by λ_w and the radiation wavelength by λ_0. The corresponding wave numbers are $k_w=2\pi/\lambda_w$ and $k_0=2\pi/\lambda_0$. From Moore's work[5] we know that diffraction effects are important when the Rayleigh range $k_0r_0^2$ (corresponding to the electron beam radius) is small compared to the gain length $l_G(r_0)$ (wiggler length for power multiplication by e) in the free-electron-laser amplifier. On the other hand, when the Rayleigh range is long compared to the gain length, before the radiation has diffracted significantly, the central core of the radiation has increased enough in intensity due to the gain, to make the diffraction at the outskirts of the beam unimportant.

For sufficiently large electron beam radius, diffraction is negligible and the gain length is well approximated by the result of the one-dimensional theory, $l_G(r_0) \simeq l_G^{1D} \propto (2\rho k_w)^{-1}$, where ρ is the Pierce[7] parameter depending on the electron beam density n_0. It is useful to introduce a dimensionless scaled electron beam radius $\tilde{a}$ defined up to a multiplicative constant by $\tilde{a}^2 \propto k_0 r_0^2/l_G^{1D}$, the ratio of the Rayleigh range to the one-dimensional gain length. To be specific, we define

$$\tilde{a}^2 = 2\rho(2k_0 k_w) r_0^2 \ . \tag{1.1}$$

For $\tilde{a} \gg 1$, the gain is accurately given by the result of one-dimensional theory. However, as noted by Moore,[5] there is a large degeneracy of the growth rates of the self-similar modes, so a single mode does not dominate, and full transverse coherence is not achieved before saturation. The total power S per unit cross-sectional area of the electron beam should be close to the result[4] recently obtained in the limit $\tilde{a} = \infty$,

$$S = \frac{\rho S_e}{9 n_0 V_c} \exp(\sqrt{3} 4\pi N_w \rho) \ , \tag{1.2}$$

where V_c is the coherence volume, $S_e = (\gamma_0 mc^2) n_0 c$ is the power per unit area in the electron beam, and N_w is the number of wiggler periods.

Now suppose we hold the electron density fixed and reduce $\tilde{a}$. The degeneracy in the growth rates of the modes is broken, because diffraction decreases the growth rates of the higher-order modes more than that of the fundamental mode. As the electron beam radius becomes smaller, fewer modes are needed to describe the output radiation field, and the transverse coherence improves. At about $\tilde{a} \approx 6$, the fundamental mode is dominant and its gain has been reduced only slightly from the one-dimensional value. In this case, the total power P is close to the one-dimensional result,[2,3]

$$P \approx P^{(1\mathrm{D})} = \frac{\rho P_e}{9 N_c} \exp(\sqrt{3} 4\pi N_w \rho) \ , \tag{1.3}$$

where N_c is the number of electrons in a coherence length and $P_e = S_e \pi r_0^2$ is the electron beam power.

When the electron beam radius is reduced below $\tilde{a} \approx 6$, density still held fixed, the domination of the fundamental mode becomes more complete, and full transverse coherence is achieved. However, the growth rate of the fundamental mode is now significantly reduced due to diffraction, and the total radiated power becomes less than predicted by the one-dimensional theory. For $\tilde{a} \lesssim 1$, the radius r_{em} of the fundamental mode of the electromagnetic field is large compared to the radius r_0 of the electron beam. Moore's[5] result in the small electron beam size limit can be re-expressed as (see Appendix A)

$$k_0 r_{\mathrm{em}}^2 \simeq l_G(r_0) \ . \tag{1.4}$$

In summary, for $\tilde{a}$ small the fundamental mode dominates, but the output power is less than the prediction of one-dimensional theory, because diffraction reduces the growth rate. For $\tilde{a}$ slightly larger than unity, a single mode still dominates and its growth rate is only slightly reduced from the one-dimensional value, so one-dimensional theory gives a good approximation to the output power. For $\tilde{a}$ large, many modes are important, and the output power is larger than predicted by one-dimensional theory, but full transverse coherence is not achieved before saturation.

Our paper is organized as follows: In Sec. II, we discuss the coupled Vlasov-Maxwell equations, and for the case of an initially monoenergetic electron beam, we derive a partial differential equation describing the evolution of the radiation field [Eq. (2.39)]. Next, in Sec. III, we briefly review the exponentially growing self-similar modes introduced by Moore. In Appendix A, we provide a correspondence between our notation and that of Moore.[5] The guided modes are solutions of a two-dimensional Schrödinger equation with non-self-adjoint Hamiltonian [Eqs. (3.4)–(3.6)].

We base the solution of the initial-value problem describing the start-up of the amplified spontaneous emission process upon Green's theorem (Appendix B). In Sec. IV, we show that even though the effective Hamiltonian operator is not self-adjoint, the Green's function of the two-dimensional Schrödinger equation can still be expanded in terms of an orthonormal set of eigenfunctions [Eqs. (4.4)–(4.6)]. In the high-gain regime before saturation, the Green's function can be represented as a superposition of Moore's self-similar modes [Eq. (4.13)]. This result for the Green's function is used, in Sec. V, to solve the start-up problem, and a general expression [Eq. (5.15)] for the output power expressed in terms of the self-similar modes is derived. This general result is applied to the special case of an electron beam having step function profile, in Sec. VI. We present both a numerical calculation (Fig. 5) and an analytical approximation [Eq. (6.50), Fig. 5] for the output power. Certain technical details of the analysis presented in Sec. VI can be found in Appendixes C and D.

In Sec. VII, it is shown that the formalism developed for a monoenergetic electron beam, in Sec. IV and V, is easily generalized to allow the inclusion of initial energy spread. The results obtained are in agreement with the recent work of Kim,[6] who has applied a different method of solution, originally introduced by van Kampen.[8] Finally, in Sec. VIII, we make some concluding remarks.

II. ENVELOPE EQUATION

Using linearized Vlasov-Maxwell equations, we derive the partial differential equation determining the three-dimensional slowly varying envelope function of the emitted radiation. We suppose the electron beam to be highly relativistic and moving in the z direction through a periodic left-hand circularly polarized helical wiggler, whose vector potential is given by

$$\mathbf{A}_w = A_w(\hat{\mathbf{e}}_- e^{ik_w z} + \mathrm{c.c.})/\sqrt{2} \ , \tag{2.1}$$

where $\hat{\mathbf{e}}_\pm = (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)/\sqrt{2}$ and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are orthogonal unit vectors transverse to $\hat{\mathbf{z}}$. The transverse electron velocity is approximated by

$$\mathbf{v}_\perp \simeq = -e\mathbf{A}_w / m\gamma \ , \tag{2.2}$$

and the longitudinal velocity by

$$v_\parallel \simeq c \left[1 - \frac{1+K^2}{2\gamma^2} \right] , \tag{2.3}$$

where $K = eA_w/mc$ is the wiggler magnetic strength parameter.

The radiation electric field $\boldsymbol{\varepsilon}$ satisfies the wave equation, in mks units,

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \boldsymbol{\varepsilon} = \mu_0 \frac{\partial \mathbf{j}_\perp}{\partial t} \ . \tag{2.4}$$

The transverse current density $\mathbf{j}_\perp$ is expressed as

$$\mathbf{j}_\perp = e n_0 \int \mathbf{v}_\perp f \, d\gamma \ , \tag{2.5}$$

with n_0 being the peak density of the electron beam and $n_0 f(z,\mathbf{r},\gamma,t) dz\, d^2r\, d\gamma$ being the number of electrons in element $dz\, d^2r\, d\gamma$. (Transverse coordinates denoted by $\mathbf{r}$.)

The electron beam is assumed to be initially monoenergetic with all electrons having energy γ_0 and longitudinal velocity $v_\parallel(\gamma_0) = v_0$. The spontaneous radiation emitted by the electrons in the forward direction is left circularly polarized with wave number k_0 and frequency $\omega_0 = k_0 c$. The combined action of the static wiggler field and the radiation field produces a ponderomotive potential, which has the dependence $e^{ik_0 z - i\omega_0 t} e^{ik_w z}$. Because the electron beam moves with velocity v_0 the modulation of the distribution function should have the form $e^{ik_r(z - v_0 t)}$. To be in resonance, these two exponential expressions must be the same, hence

$$k_r = k_0 + k_w \tag{2.6}$$

and

$$k_r v_0 = k_0 c = \omega_0 \ . \tag{2.7}$$

It follows that

$$\frac{k_0}{k_w} = \frac{v_0}{c - v_0} \simeq \frac{2\gamma_0^2}{1 + K^2} \tag{2.8}$$

and

$$k_r = \frac{\omega_0}{v_0} = \frac{\omega_w}{c - v_0} \ , \tag{2.9}$$

where $\omega_w = k_w c$.

Let us now return to the wave equation (2.4) and introduce the slowly varying envelope function $E(\mathbf{r},z,t)$ by

$$\boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}} E e^{ik_0 z - i\omega_0 t} \hat{\mathbf{e}}_+ + \text{c.c.} \tag{2.10}$$

The wave equation is simplified by using the paraxial approximation,

$$\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \simeq 2ik_0 \left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} \right] . \tag{2.11}$$

Since the rapid variation of the distribution f is described by the exponential factor $e^{ik_r z - i\omega_0 t}$, the time derivative of f is well approximated by $\partial f/\partial t \simeq -i\omega_0 f$. Therefore, using Eqs. (2.4) and (2.5), together with (2.10) and (2.11), we derive

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{1}{2ik_0}\nabla_T^2 \right] E = \frac{n_0 \mu_0 e^2 c A_w}{2m} e^{-ik_r(z - v_0 t)} \int \frac{d\gamma}{\gamma} f \ , \tag{2.12}$$

where ∇_T^2 is the transverse Laplacian.

Equation (2.12) determines the slowly varying envelope function E in terms of the electron distribution f. The Vlasov equation will provide a second relation between E and f, completing the description of the free-electron laser. The Vlasov equation can be written as

$$\frac{\partial f}{\partial t} + v_\parallel(\gamma)\frac{\partial f}{\partial z} + (\mathbf{v}_\perp \cdot \nabla_T) f + \dot{p}_z \frac{\partial f}{\partial p_z} = 0 \ . \tag{2.13}$$

In Eq. (2.13) we have used $\dot{p}_\perp = 0$, and in the discussion which follows we shall neglect the rapidly oscillating term $(\mathbf{v}_\perp \cdot \nabla_T) f$ (an explanation of this follows). Taking $p_z \simeq mc\gamma$, and introducing

$$\xi = z - v_0 t \ , \tag{2.14}$$

Eq. (2.13) becomes

$$\frac{\partial f}{\partial t} + [v_\parallel(\gamma) - v_0]\frac{\partial f}{\partial \xi} + \dot{\gamma}\frac{\partial f}{\partial \gamma} = 0 \ . \tag{2.15}$$

Eq. (2.15) is nonlinear, because $\dot{\gamma}$ depends on the electric field, and hence on f, via

$$\dot{\gamma} = \frac{e}{mc^2}\mathbf{v}\cdot\boldsymbol{\varepsilon} = -\frac{e^2 A_w}{2m^2 c^2 \gamma}(e^{ik_r \xi} E + \text{c.c.}) \ . \tag{2.16}$$

In order to linearize Eq. (2.15), we write

$$f = f_0 + f_1 \ , \tag{2.17}$$

where f_0 is a solution of

$$\frac{\partial f_0}{\partial t} + [v_\parallel(\gamma) - v_0]\frac{\partial f_0}{\partial \xi} = 0 \ , \tag{2.18}$$

whose initial value is the ensemble average of the initial distribution $\langle f(t=0) \rangle$. Equation (2.18) describes the time evolution of the slowly varying component of the electron distribution in the absence of the radiation field.

The linearized Vlasov equation takes the form

$$\frac{\partial f}{\partial t} + [v_\parallel(\gamma) - v_0]\frac{\partial f}{\partial \xi} + \dot{\gamma}\frac{\partial f_0}{\partial \gamma} = 0 \ , \tag{2.19}$$

where $\dot{\gamma}$ is given by Eq. (2.16).

The rapid variation of the perturbed distribution $f_1 \sim e^{ik_r \xi}$. Due to Eq. (2.18),

$$\frac{\partial f}{\partial t} + [v_\parallel(\gamma) - v_0]\frac{\partial f}{\partial \xi} = \frac{\partial f_1}{\partial t} + [v_\parallel(\gamma) - v_0]\frac{\partial f_1}{\partial \xi} \ ,$$

whose rapid variation is also $e^{ik_r \xi}$. If f_0 has no rapid density modulation, then $\dot{\gamma}(\partial f_0/\partial \gamma)$ again has the dependence $e^{ik_r \xi}$, as seen from Eq. (2.16). The neglected term $(\mathbf{v}_\perp \cdot \nabla_T) f$ has the dependence $e^{ik_w z} e^{ik_r \xi}$, because of Eq. (2.2), hence it is rapidly oscillating for fixed ξ, and its contribution is expected to be small.

It is now convenient to introduce dimensionless variables measuring spatial and temporal variations:

$$\tau = \omega_w t, \quad \zeta = k_r(z - v_0 t) \ ,$$
$$\mathbf{x} = \sqrt{2k_0 k_w}\,\mathbf{r}, \quad \nabla_\perp^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \ . \tag{2.20}$$

For an initially monoenergetic electron beam, the unperturbed distribution is

$$f_0 = u(\zeta,\mathbf{x})\delta(\gamma - \gamma_0) \ , \tag{2.21}$$

where the smooth function $u(\zeta,\mathbf{x})$ describes the average properties of the initial electron distribution, neglecting the high-frequency shot noise due to the discrete nature of the individual electrons comprising the beam. The distribution f is determined from Eqs. (2.12) and (2.19), subject to the initial condition at $t=0$,

$$f(t=0)=\frac{1}{n_0}\sum_i \delta(z-z_i)\delta(\mathbf{r}-\mathbf{r}_i)\delta(\gamma-\gamma_0) . \tag{2.22}$$

The shot noise is taken into account by treating the initial coordinates $z_i,\mathbf{r}_i$ of the ith electron as stochastic variables and determining physical quantities as averages over the ensemble of possible $z_i,\mathbf{r}_i$.

Let us now turn to the derivation of the partial differential equation determining the slowly varying envelope function E. To proceed, we first rewrite the wave equation (2.12) and the linearized Vlasov equation (2.19) in terms of the dimensionless variables introduced in Eq. (2.20):

$$\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}-i\nabla_\perp^2\right]E=D_1e^{-i\zeta}\int\frac{d\gamma}{\gamma}f , \tag{2.23}$$

$$\frac{\partial f}{\partial\tau}+\eta\frac{\partial f}{\partial\zeta}-D_2(Ee^{i\zeta}+E^*e^{-i\zeta})\frac{1}{\gamma}\frac{\partial f_0}{\partial\gamma}=0 , \tag{2.24}$$

where

$$D_1=\frac{n_0\mu_0e^2c^2A_w}{2m\omega_w} , \tag{2.25}$$

$$D_2=\frac{e^2A_w}{2m^2c^2\omega_w} , \tag{2.26}$$

$$\eta=\frac{v_\parallel(\gamma)-v_0}{c-v_0}=1-\frac{\gamma_0^2}{\gamma^2} . \tag{2.27}$$

In what follows, we shall drop the nonresonant term $E^*e^{-i\zeta}$ in Eq. (2.24).

We define

$$I=\int\frac{d\gamma}{\gamma}f , \tag{2.28}$$

and use the Vlasov equation (2.24) to compute the partial derivatives of I with respect to τ. In the special case of an initially monoenergetic electron beam described by the distribution f_0 specified in Eq. (2.21), we obtain

$$\frac{\partial I}{\partial\tau}=\int\frac{d\gamma}{\gamma}\left[-\eta\frac{\partial f}{\partial\zeta}+\frac{D_2}{\gamma}e^{i\zeta}Eu\,\delta'(\gamma-\gamma_0)\right] , \tag{2.29}$$

$$\frac{\partial^2 I}{\partial\tau^2}=\frac{\partial^2}{\partial\zeta^2}\int\frac{d\gamma}{\gamma}\eta^2 f+\frac{2D_2}{\gamma_0^3}e^{i\zeta}\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}+i\right](uE) . \tag{2.30}$$

We shall now show that

$$\int\frac{d\gamma}{\gamma}\eta^2 f=0 . \tag{2.31}$$

To see this, note that the linearized Vlasov equation (2.24) has the form

$$\left[\frac{\partial}{\partial\tau}+\eta\frac{\partial}{\partial\zeta}\right]f=\phi(\tau,\zeta,\gamma) . \tag{2.32}$$

The solution of Eq. (2.32) is

$$f(\tau,\zeta,\gamma)=h(\zeta-\eta\tau,\gamma)+\int_0^\tau d\tau'\,\phi(\tau',\zeta-\eta(\tau-\tau'),\gamma) . \tag{2.33}$$

For an initially monoenergetic electron beam, h and ϕ have the forms $h=\tilde{h}\delta(\gamma-\gamma_0)$ and $\phi=\tilde{\phi}\delta'(\gamma-\gamma_0)$, so

$$f(\tau,\zeta,\gamma)=\delta(\gamma-\gamma_0)\tilde{h}(\zeta-\eta\tau)+\delta'(\gamma-\gamma_0)\int_0^\tau d\tau'\,\tilde{\phi}(\tau',\zeta-\eta(\tau-\tau'),\gamma) . \tag{2.34}$$

Equation (2.34) clearly demonstrates the vanishing of the integral of Eq. (2.31), and this discussion reminds us that the linearized Vlasov equation does not provide a description of the development of energy spread.

Use of Eqs. (2.23) and (2.24) together with Eqs. (2.30) and (2.31) shows that for an initially monoenergetic electron beam the coupled Vlasov-Maxwell equations can be written in the form

$$\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}-i\nabla_\perp^2\right]E=J , \tag{2.35}$$

$$\frac{\partial^2 J}{\partial\tau^2}=\alpha\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}+i\right](uE) , \tag{2.36}$$

with

$$J\equiv\frac{n_0\mu_0e^2c^2A_w}{2m\omega_w}e^{-i\zeta}\int\frac{d\gamma}{\gamma}f . \tag{2.37}$$

The constant $\alpha=2D_1D_2/\gamma_0^3$ in Eq. (2.36) is related to the Pierce parameter ρ of Bonifacio *et al.*[7] by

$$\alpha=(2\rho)^3=\frac{n_0\mu_0e^4A_w^2}{2m^3\gamma_0^3\omega_w^2} . \tag{2.38}$$

Now Eqs. (2.35) and (2.36) lead immediately to the envelope equation

$$\frac{\partial^2}{\partial\tau^2}\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}-i\nabla_\perp^2\right]E=\alpha\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}+i\right](uE) , \tag{2.39}$$

which provides the basis for the discussion in the following sections of this paper.

Because E is a slowly varying function, the terms $\alpha(\partial E/\partial\tau)$ and $\alpha(\partial E/\partial\zeta)$, in Eq. (2.39), are small compared with iE. If we ignore them and consider only single frequency Ω (replace $\partial/\partial\tau$ by $-i\Omega$), we obtain an equation equivalent to Moore's (see Appendix A). However, for very large electron beam size, $\tilde{a}\gg 1$, we found the term $\alpha(\partial E/\partial\zeta)$ is important, determining the divergence angle of the radiation field.[4] In the present paper, we are mostly concerned with only a few lowest-order guiding modes, and the contribution of the $\alpha(\partial E/\partial\zeta)$ term can be neglected, it becomes important only for modes of higher order.

III. SELF-SIMILAR MODES

We suppose the electron beam to be initially uniform in the z direction, and rotationally symmetric about the z axis. The transverse profile $u=u(x)$ is a function only of the modulus $x=|\mathbf{x}|$. Following Moore,[5] we seek solutions of the envelope equation (2.39) having the form

$$E=e^{-i\Omega\tau}e^{iq_{\parallel}\zeta}\psi(\mathbf{x}) . \tag{3.1}$$

Substituting this expression into Eq. (2.39), one derives

$$\left[\Omega-q_{\parallel}+\nabla_{\perp}^2+\frac{\alpha}{\Omega^2}(\Omega-q_{\parallel}-1)u(x)\right]\psi(\mathbf{x})=0 . \tag{3.2}$$

This is a generalized eigenvalue problem determining the eigenfrequencies $\Omega=\Omega_n(q_{\parallel})$ and the eigenfunctions $\psi=\psi_n(q_{\parallel},\mathbf{x})$. The self-similar modes are given by

$$E=e^{-i\Omega_n(q_{\parallel})\tau}e^{iq_{\parallel}\zeta}\psi_n(q_{\parallel},\mathbf{x}) . \tag{3.3}$$

The term "self-similar" refers to the fact that the transverse dependence of the mode is independent of the axial coordinate z. Of greatest interest are those modes with complex eigenfrequencies having positive imaginary part, $\mathrm{Im}\Omega_n>0$, since these correspond to exponential growth in $\tau=\omega_w t=2\pi N_w$, where N_w is the number of wiggler periods.

Equation (3.2), which determines the transverse profile of the guiding modes, has the form of a two-dimensional Schrödinger equation with a complex potential, since Ω is complex. Let us rewrite Eq. (3.2) in the form

$$[\Lambda+\nabla_{\perp}^2+Vu(x)]\psi(\mathbf{x})=0 \tag{3.4}$$

with

$$\Lambda=\Omega-q_{\parallel} \tag{3.5}$$

and

$$V=\frac{\alpha}{\Omega^2}(\Omega-q_{\parallel}-1) . \tag{3.6}$$

There is, of course, the additional complication that the potential depends on the eigenvalue.

At the beginning of a long wiggler magnet, the radiation field is certainly not described by a single guided mode. If the growth rate of one mode is sufficiently greater than the others, however, that mode can dominate at the end of the magnet. In the following sections of this paper, we shall determine the output power in the guided modes.

IV. GREEN'S FUNCTION

The initial value problem, describing the start-up of the amplified spontaneous emission process can be solved by utilizing a Green's-function technique (see Appendix B). We introduce the Green's function $g(\tau,\zeta,\mathbf{x},\mathbf{x}')$ via

$$\left[\frac{\partial^2}{\partial\tau^2}\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}-i\nabla_{\perp}^2\right]-\alpha u(x)\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}+i\right]\right]g(\tau,\zeta,\mathbf{x},\mathbf{x}')=\delta(\tau)\delta(\zeta)\delta(\mathbf{x}-\mathbf{x}') . \tag{4.1}$$

In order to determine the Green's function, we make use of the following Fourier-Laplace expansion:

$$g(\tau,\zeta,\mathbf{x},\mathbf{x}')=\int_{-\infty+is}^{\infty+is}\frac{d\Omega}{2\pi i\Omega^2}\times\int\frac{dq_{\parallel}}{2\pi}e^{-i\Omega\tau}e^{iq_{\parallel}\zeta}G(\Omega,q_{\parallel},\mathbf{x},\mathbf{x}') , \tag{4.2}$$

where $s>0$ is chosen so that the integration path is above all singularities of the integrand. Inserting (4.2) into (4.1) leads to

$$\left[\Omega-q_{\parallel}+\nabla_{\perp}^2+\frac{\alpha}{\Omega^2}(\Omega-q_{\parallel}-1)u(x)\right]G(\Omega,q_{\parallel},\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}') . \tag{4.3}$$

The solution of Eq. (4.3) can be expressed in terms of the eigenfunctions $\Phi_n(\mathbf{x})$ and eigenvalues Λ_n of the associated homogeneous eigenvalue problem:

$$\left[\Lambda_n+\nabla_{\perp}^2+\frac{\alpha}{\Omega^2}(\Omega-q_{\parallel}-1)u(x)\right]\Phi_n(\mathbf{x})=0 . \tag{4.4}$$

If we were to impose the restriction $\Lambda_n=\Omega_n-q_{\parallel}$, then Eq. (4.4) would reduce to Moore's eigenvalue problem [Eq. (3.2)] for the guiding modes, having nonorthogonal eigenfunctions. However, in Eq. (4.4) we have *not* restricted $\Lambda_n=\Omega_n-q_{\parallel}$, therefore Eq. (4.4) has the form of a Schrödinger equation with Hamiltonian $H=\nabla_{\perp}^2+(\alpha/\Omega^2)(\Omega-q_{\parallel}-1)u(x)$. We consider Ω to be complex, so the differential operator H is not self-adjoint, and consequently, the eigenvalues Λ_n are complex. Because H is not self-adjoint, the solutions of Eq. (4.4) are not orthogonal relative to the scalar product $\int\phi_n(x)\phi_m^*(x)d^2x$. However, according to the bi-orthogonality theorem,[9] for fixed Ω and $q_{\parallel}$ the eigenfunctions of H are orthogonal to those of the adjoint operator H^+, which clearly are simply $\Phi_n^*(x)$. Therefore, we can normalize the eigenfunctions according to

$$\int d^2x\,\Phi_n(\Omega,q_{\parallel},\mathbf{x})[\Phi_m^*(\Omega,q_{\parallel},\mathbf{x})]^*=\delta_{mn}$$

or

$$\int d^2x\,\Phi_n(\Omega,q_{\parallel},\mathbf{x})\Phi_m(\Omega,q_{\parallel},\mathbf{x})=\delta_{mn} , \tag{4.5}$$

and assume the validity of the completeness relation

$$\sum_n\Phi_n(\Omega,q_{\parallel},\mathbf{x})\Phi_n(\Omega,q_{\parallel},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}') . \tag{4.6}$$

The sum over n may include an integral over the continuous portion of the spectrum, however, we are only interested in a few modes which have the fastest growth rates. These will dominate at the end of a long wiggler and are elements of the discrete spectrum. We do not prove completeness, and subtleties of the spectrum which are important only near the beginning of the wiggler are not addressed.

In this spirit, we expand the Green's function in terms of the eigenfunctions Φ_n as

$$G(\Omega,q_{\parallel},\mathbf{x},\mathbf{x}')=\sum_n G_n(\Omega,q_{\parallel},\mathbf{x}')\Phi_n(\Omega,q_{\parallel},\mathbf{x}) , \tag{4.7}$$

and employ this expansion in Eq. (4.3) to obtain

$$\left[\Omega-q_{\parallel}+\nabla_{\perp}^2+\frac{\alpha}{\Omega^2}(\Omega-q_{\parallel}-1)u(x)\right]G(\Omega,q_{\parallel},\mathbf{x},\mathbf{x}')$$
$$=\sum_n G_n(\Omega,q_{\parallel},\mathbf{x}')[\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})]\Phi_n(\Omega,q_{\parallel},\mathbf{x})$$
$$=\sum_n \Phi_n(\Omega,q_{\parallel},\mathbf{x}')\Phi_n(\Omega,q_{\parallel},\mathbf{x}) . \tag{4.8}$$

The last equality results from the completeness relation given in Eq. (4.6). Using the orthonormality condition of Eq. (4.5), we find

$$G_n(\Omega,q_{\parallel},\mathbf{x}')=\frac{\Phi_n(\Omega,q_{\parallel},\mathbf{x}')}{\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})} . \tag{4.9}$$

Finally, Eqs. (4.2), (4.7), and (4.9) establish the following representation for the Green's function:

$$g(\tau,\zeta,\mathbf{x},\mathbf{x}')$$
$$=\int\frac{dq_{\parallel}}{2\pi}e^{iq_{\parallel}\zeta}\int_{-\infty+is}^{\infty+is}\frac{d\Omega}{2\pi i\Omega^2}e^{-i\Omega\tau}\times\sum_n\frac{\Phi_n(\Omega,q_{\parallel},\mathbf{x})\Phi_n(\Omega,q_{\parallel},\mathbf{x}')}{\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})} . \tag{4.10}$$

The asymptotic behavior for large τ of the Green's function is determined by the singularities in the complex Ω plane of the integrand appearing in Eq. (4.10). It is reasonable to assume that the dominant behavior is described by poles corresponding to the solutions of

$$\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})=0 . \tag{4.11}$$

We keep only those solutions with $\mathrm{Im}\Omega_n>0$, which will dominate for large τ, and we denote these solutions by

$$\Omega=\Omega_n(q_{\parallel}) . \tag{4.12}$$

At the end of a long wiggler, the following asymptotic representation for the Green's function is appropriate:

$$\ddot{g}(\tau,\zeta,\mathbf{x},\mathbf{x}')$$
$$=\int\frac{dq_{\parallel}}{2\pi}e^{iq_{\parallel}\zeta}\sum_n e^{-i\Omega_n(q_{\parallel})\tau}\frac{\psi_n(q_{\parallel},\mathbf{x})\psi_n(q_{\parallel},\mathbf{x}')}{1-F_n(q_{\parallel})} , \tag{4.13}$$

where $\ddot{g}=\partial^2 g/\partial\tau^2$,

$$\psi_n(q_{\parallel},\mathbf{x})=\Phi_n(\Omega_n(q_{\parallel}),q_{\parallel},\mathbf{x}) , \tag{4.14}$$

$$F_n(q_{\parallel})=[\partial\Lambda_n(\Omega,q_{\parallel})/\partial\Omega]_{\Omega=\Omega_n(q_{\parallel})} . \tag{4.15}$$

The functions $\psi_n(q_{\parallel},\mathbf{x})$ are seen to be the self-similar modes discussed in Sec. III. The eigenfunctions $\Phi_n(\Omega,q_{\parallel},\mathbf{x})$ introduced in Eq. (4.4) have the orthogonality and completeness properties specified in Eqs. (4.5) and (4.6). These properties allowed us to derive the representation of the Green's function given in Eq. (4.10). The self-similar modes $\psi_n(q_{\parallel},\mathbf{x})=\Phi_n(\Omega_n(q_{\parallel}),q_{\parallel},\mathbf{x})$ are not orthogonal, because for each n, the function Φ_n is evaluated at a different frequency $\Omega_n(q_{\parallel})$, whereas orthogonality and completeness of the Φ_n holds when all Φ_n are evaluated at the same frequency Ω.

V. START-UP FROM SHOT NOISE

We wish to solve the envelope equation (2.39) subject to initial conditions specified at $t=0$. (See Appendix B.) In particular, we specify $E(0,\zeta,\mathbf{x})=E_0(\zeta,\mathbf{x})$, $J(0,\zeta,\mathbf{x})=J_0(\zeta,\mathbf{x})$, and $\dot{J}(0,\zeta,\mathbf{x})=\dot{J}_0(\zeta,\mathbf{x})$, where the dot denotes $\partial/\partial\tau$ and the current J was introduced in Eqs. (2.35)–(2.37). The envelope function is then determined by

$$E(\tau,\zeta,\mathbf{x})$$
$$=\int d\zeta'\,d^2x'[E_0(\zeta',\mathbf{x}')\ddot{g}(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}')+J_0(\zeta',\mathbf{x}')\dot{g}(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}')+\dot{J}_0(\zeta',\mathbf{x}')g(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}')] , \tag{5.1}$$

where $g(\tau,\zeta,\mathbf{x})$ is the Green's function defined in Eq. (4.1). Here, E_0 represents an initial electric field possibly due to an external laser; J_0 describes the initial spatial bunching of the electron beam and $\dot{J}_0$ corresponds to an initial energy modulation of the electron beam.

We assume the absence of an external radiation field, $E_0=0$, and describe the shot noise by

$$J_0=\frac{n_0\mu_0 e^2c^2A_w}{2m\omega_w}e^{-i\zeta}\int\frac{d\gamma}{\gamma}f(\tau=0) , \tag{5.2}$$

$$f(\tau=0)=\frac{1}{n_0}\sum_i\delta(z-z_i)\delta(\mathbf{r}-\mathbf{r}_i)\delta(\gamma-\gamma_0) , \tag{5.3}$$

where the coordinates $z_i,\mathbf{r}_i$ of the ith electron are treated as independent random variables. For the purposes of the present discussion we ignore the spread in energies of the electrons, hence $\dot{J}_0=0$. Although $\langle E\rangle=0$, averages of quantities quadratic in E do not vanish.

For Eqs. (5.2) and (5.3) we see that

$$J_0(\zeta,\mathbf{x})=\frac{b\kappa}{n_0}\sum_i e^{-i\zeta_i}\delta(\zeta-\zeta_i)\delta(\mathbf{x}-\mathbf{x}_i) , \tag{5.4}$$

where we have defined

$$b=2k_wk_0k_r \tag{5.5}$$

and

$$\kappa=\frac{n_0\mu_0e^2c^2A_w}{2m\omega_w\gamma_0} . \tag{5.6}$$

Using Eq. (5.1), the correlation function of the electric field at two different spatial points can be expressed as

$$\langle E(\tau,\zeta,\mathbf{x})E^*(\tau,\zeta',\mathbf{x}')\rangle = \int d\zeta_1 d^2x_1 d\zeta_2 d^2x_2 \dot{g}(\tau,\zeta-\zeta_1,\mathbf{x},\mathbf{x}_1) \times \dot{g}^*(\tau,\zeta'-\zeta_2,\mathbf{x}',\mathbf{x}_2) \times \langle J_0(\zeta_1,\mathbf{x}_1)J_0^*(\zeta_2,\mathbf{x}_2)\rangle \ . \tag{5.7}$$

Applying Eq. (5.4) to (5.7) results in

$$\begin{aligned}\langle E(\tau,\zeta,\mathbf{x})E^*(\tau,\zeta',\mathbf{x}')\rangle &= b^2\frac{\kappa^2}{n_0^2}\sum_{i,j}\langle e^{-i(\zeta_i-\zeta_j)}\dot{g}(\tau,\zeta-\zeta_i,\mathbf{x},\mathbf{x}_i) \times \dot{g}^*(\tau,\zeta'-\zeta_j,\mathbf{x}',\mathbf{x}_j)\rangle \\ &\cong b^2\frac{\kappa^2}{n_0^2}\sum_i \dot{g}(\tau,\zeta-\zeta_i,\mathbf{x},\mathbf{x}_i)\dot{g}^*(\tau,\zeta'-\zeta_i,\mathbf{x}',\mathbf{x}_i) \ .\end{aligned} \tag{5.8}$$

Now replacing the sum over i by an integral according to

$$\sum_i \rightarrow \int dz\, d^2r\, n_0 u(r) = \frac{1}{b}\int d\zeta\, d^2x\, n_0 u(x) \ , \tag{5.9}$$

the correlation function is finally expressed in terms of the Green's function by

$$\langle E(\tau,\zeta,\mathbf{x})E^*(\tau,\zeta',\mathbf{x}')\rangle = \frac{b}{n_0}\kappa^2\int d\zeta_1 d^2x_1 u(x_1)\dot{g}(\tau,\zeta-\zeta_1,\mathbf{x},\mathbf{x}_1) \times \dot{g}^*(\tau,\zeta'-\zeta_1,\mathbf{x}',\mathbf{x}_1) \ . \tag{5.10}$$

To proceed, we shall express $\dot{g}$ in terms of the eigenmodes, as discussed in Sec. IV. From Eq. (4.13), it follows that

$$\dot{g}(\tau,\zeta,\mathbf{x},\mathbf{x}') = \int \frac{dq_{\|}}{2\pi}e^{iq_{\|}\zeta}\sum_n \dot{G}_n(q_{\|},\tau)\psi_n(q_{\|},\mathbf{x})\psi_n(q_{\|},\mathbf{x}') \ , \tag{5.11}$$

where

$$\dot{G}_n(q_{\|},\tau) = \frac{e^{-i\Omega_n(q_{\|})\tau}}{-i\Omega_n(q_{\|})[1-F_n(q_{\|})]} \ . \tag{5.12}$$

Inserting the expansion for $\dot{g}$ given in Eq. (5.11) into (5.10), we derive

$$\langle E(\tau,\zeta,\mathbf{x})E^*(\tau,\zeta',\mathbf{x}')\rangle = \frac{b}{n_0}\kappa^2\int\frac{dq_{\|}}{2\pi}e^{iq_{\|}(\zeta-\zeta')}\sum_{n,l}\int d^2x_1 u(x_1)\psi_n(q_{\|},\mathbf{x}_1)\psi_l^*(q_{\|},\mathbf{x}_1)\dot{G}_n(q_{\|},\tau)\dot{G}_l^*(q_{\|},\tau)\psi_n(q_{\|},\mathbf{x})\psi_l^*(q_{\|},\mathbf{x}') \ . \tag{5.13}$$

The total radiated power P is given by

$$P = \frac{1}{Z_0}\int d^2r\langle E(\tau,\zeta,\mathbf{x})E^*(\tau,\zeta,\mathbf{x})\rangle \ , \tag{5.14}$$

hence

$$P = \sum_{n,l} P_{nl} \tag{5.15a}$$

with

$$P_{nl} = \frac{\kappa^2 k_r}{n_0 Z_0}\int\frac{dq_{\|}}{2\pi}\sum_{n,l}\dot{G}_n(q_{\|},\tau)\dot{G}_l^*(q_{\|},\tau)\int d^2x\,\psi_n(q_{\|},\mathbf{x})\psi_l^*(q_{\|},\mathbf{x})\int d^2x_1 u(x_1)\psi_n(q_{\|},\mathbf{x}_1)\psi_l^*(q_{\|},\mathbf{x}_1) \ , \tag{5.15b}$$

where $Z_0=\sqrt{\mu_0/\epsilon_0}=377\Omega$ is the impedance of free space, and, as is easy to verify,

$$\frac{\kappa^2}{n_0 Z_0} = \tfrac{1}{2}mc^2\gamma_0(2\rho)^3 c \ . \tag{5.16}$$

VI. OUTPUT POWER FOR ELECTRON BEAM WITH STEP-FUNCTION PROFILE

A. Self-similar modes

In Sec. III, we briefly discussed the guided modes, noting in Eq. (3.4) that they are determined by the eigenvalue problem ($x=|\mathbf{x}|$):

$$[\Lambda+\nabla_\perp^2+Vu(x)]\psi(\mathbf{x})=0 \tag{6.1}$$

with

$$\Lambda=\Omega-q_{\|} \tag{6.2}$$

and

$$V=\frac{\alpha}{\Omega^2}(\Omega-q_{\|}-1) \ . \tag{6.3}$$

Here, we shall consider the special case of a step-function electron beam profile,

$$u(x)=\begin{cases}1, & x<a\\ 0, & x>a\end{cases} \tag{6.4}$$

which was originally treated by Moore.[5] Our goal is the determination of the output power in the guided modes, using the formula of Eq. (5.15).

With the ansatz,

$$\psi(\mathbf{x})=e^{im\theta}R(x) \ , \tag{6.5}$$

one easily derives from Eq (6.1) the radial equation

$$R''+\frac{1}{x}R'+\left[\Lambda+Vu(x)-\frac{m^2}{x^2}\right]R=0 \; . \tag{6.6}$$

For the step function profile of Eq. (6.4), the radial function is expressed in terms of Bessel and Hankel functions,

$$R(x)=\begin{cases} CJ_m\left[\mathcal{X}\dfrac{x}{a}\right], & x\le a \\ DH_m^{(1)}\left[\phi\dfrac{x}{a}\right], & x\ge a \end{cases} \tag{6.7}$$

where

$$\mathcal{X}=a\sqrt{\Lambda+V},\quad \mathrm{Re}\mathcal{X}>0 \; , \tag{6.8}$$

$$\phi=a\sqrt{\Lambda},\quad \mathrm{Im}\phi>0 \; . \tag{6.9}$$

Matching the radial function and its derivative at $x=a$, we derive

$$CJ_m(\mathcal{X})=DH_m^{(1)}(\phi) \; , \tag{6.10}$$

$$C\mathcal{X}J_m'(\mathcal{X})=D\phi H^{(1)\prime}(\phi) \; . \tag{6.11}$$

It is convenient to define new scaled variables:

$$\lambda=\Omega/2\rho,\quad \Delta=q_{\parallel}/2\rho,\quad \tilde{a}=\sqrt{2\rho}a \; . \tag{6.12}$$

Note that since $x=\sqrt{2k_0k_w}r$ [Eq. (2.20)], it follows that $a=\sqrt{2k_0k_w}r_0$, where r_0 is the radius of the electron beam, and

$$\tilde{a}^2=2\rho(2k_0k_w)r_0^2 \; , \tag{6.13}$$

in agreement with Eq. (1.1) of the Introduction. The equations determining $\mathcal{X}$ and ϕ can now be written in the form

$$\frac{\phi^2}{\tilde{a}^2}-\frac{\mathcal{X}^2}{\tilde{a}^2}=\frac{1-2\rho\dfrac{\phi^2}{\tilde{a}^2}}{\left[\dfrac{\phi^2}{\tilde{a}^2}+\Delta\right]^2} \tag{6.14}$$

and

$$\mathcal{X}\frac{J_m'(\mathcal{X})}{J_m(\mathcal{X})}=\phi\frac{H_m^{(1)\prime}(\phi)}{H_m^{(1)}(\phi)} \; . \tag{6.15}$$

Given a solution of the coupled Eqs. (6.14) and (6.15), one determines λ via

$$\lambda=\frac{\phi^2}{\tilde{a}^2}+\Delta \; . \tag{6.16}$$

As mentioned in the Introduction, in the large beam size limit, $\tilde{a}\to\infty$, the growth rates of the guided modes are expected to approach the result of the one-dimensional theory. Let us now see how this follows from Eqs. (6.14)–(6.16). We combine Eqs. (6.14) and (6.16) to obtain

$$\lambda^3-\left[\Delta+\frac{\mathcal{X}^2}{\tilde{a}^2}\right]\lambda^2+2\rho\lambda-(1+2\rho\Delta)=0 \; . \tag{6.17}$$

If $\mathcal{X}$ remains finite in the limit $\tilde{a}\to\infty$, then Eq. (6.17) will reduce to the well-known cubic dispersion relation of the one-dimensional theory, and λ will remain finite and approach the one-dimensional value. When λ remains finite, Eq. (6.16) implies that $\phi\to\infty$ in the limit $\tilde{a}\to\infty$.

These considerations instruct us to search for a solution of Eqs. (6.14) and (6.15) with the property that as $\tilde{a}\to\infty$, one has $\mathcal{X}$ finite and ϕ divergent. We define

$$\xi=\phi\frac{H_m^{(1)\prime}(\phi)}{H_m^{(1)}(\phi)} \; , \tag{6.18}$$

and note that if $\phi\to\infty$ as $\tilde{a}\to\infty$, then

$$\xi\simeq i\phi \quad (\tilde{a}\to\infty) \; . \tag{6.19}$$

This shows that the right-hand side of Eq. (6.15) is divergent as $\tilde{a}\to\infty$, hence it follows that $J_m(\mathcal{X})$ must vanish in this limit. Therefore,

$$\mathcal{X}\to\mu_{mn},\quad (\tilde{a}\to\infty) \; , \tag{6.20}$$

where μ_{mn} is a zero of the Bessel function, $J_m(\mu_{mn})=0$. In Eq. (6.15), we may use $J_m(\mathcal{X})\simeq J_m'(\mu_{mn})(\mathcal{X}-\mu_{mn})$, together with (6.19) to derive

$$\frac{\mathcal{X}}{\mathcal{X}-\mu_{mn}}\simeq i\phi \quad (\tilde{a}\to\infty) \tag{6.21}$$

or

$$\mathcal{X}\simeq\mu_{mn}\frac{\phi}{\phi+i} \quad (\tilde{a}\to\infty) \; . \tag{6.22}$$

The solution valid for large $\tilde{a}\gg 1$ can be obtained by iterating. Begin with $\mathcal{X}\simeq\mu_{mn}$. Then Eq. (6.17) provides a finite result for $\lambda=\lambda_0(\Delta)$. Employing this value in Eq. (6.16) leads to the approximation $\phi=\phi_0(\Delta)=\tilde{a}\sqrt{\lambda_0(\Delta)-\Delta}$. Now inserting this approximate value of ϕ into Eq. (6.22) yields an improved value of $\mathcal{X}=\mathcal{X}_0(\Delta)=\mu_{mn}\phi_0(\Delta)/[\phi_0(\Delta)+i]$. These approximations, accurate for large $\tilde{a}$, have been used as starting values for a numerical calculation of $\mathcal{X}$, ϕ, and λ, for given Δ, ρ, and $\tilde{a}$. Results of these numerical calculations are shown in Figs. 1–4 and are in agreement with the

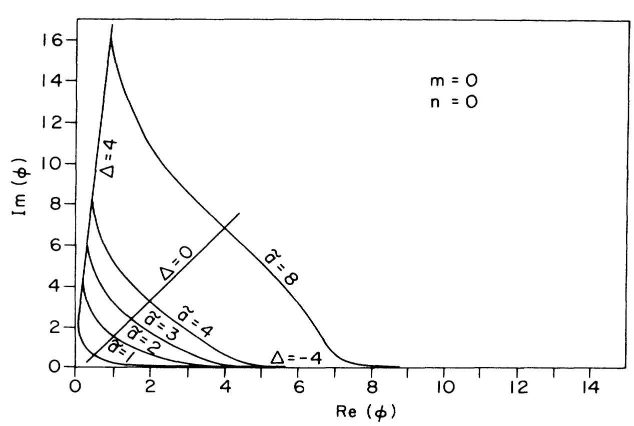

FIG. 1. Real and imaginary parts of ϕ as functions of $\tilde{a}$ and Δ, determined by solving Eqs. (6.14) and (6.15), for fundamental mode $m=0$, $n=0$. In this plot we have taken $\rho=0.7\times10^{-3}$; however, because the term containing ρ in Eq. (6.14) is negligible except for very large Δ, the results are valid for all $\rho\ll 1$. One can see that $\phi\simeq\tilde{a}e^{\pi i/3}$ for $\Delta=0$ and large $\tilde{a}$.

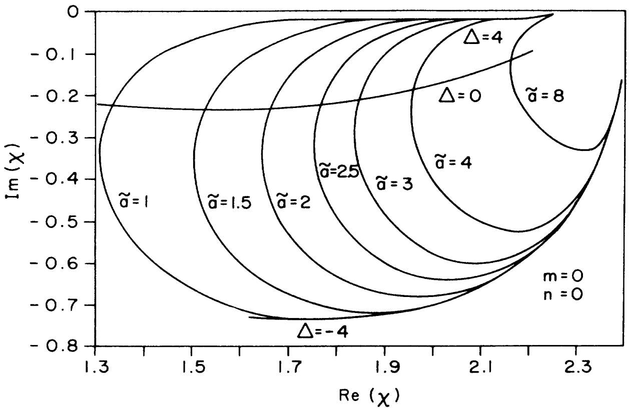

FIG. 2. Real and Imaginary parts of χ as functions of $\tilde{a}$ and Δ. Same conditions as Fig. 1.

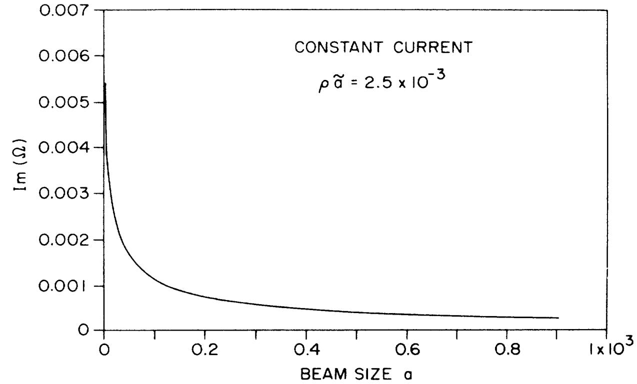

FIG. 4. ImΩ as function of a $=\sqrt{2k_0k_w}r_0$ for constant current: $\rho\tilde{a}=2.5\times10^{-3}$.

work of Moore.[5]

Figures 1–3 are calculated for constant ρ, which is convenient from a theoretical point of view. In practical cases, when we reduce beam size $\tilde{a}$, electron density will increase, and it is reasonable to assume constant current I_0. It is easy to show that

$$\rho\tilde{a}=\left[\frac{eZ_0}{2\pi mc^2}\frac{I_0}{\gamma_0}\frac{K^2}{1+K^2}\right]^{1/2} .$$

Hence for constant current, $\rho\tilde{a}$ is a constant too. In Fig. 4 we plot Im(Ω) versus a for constant current ($\rho\tilde{a}=2.5\times10^{-3}$) to show how the gain increases as we reduce the beam size.

B. Calculation of output power

Let us now turn to the calculation of the output power for an electron beam with step function profile. We rewrite Eq. (5.15) in the form

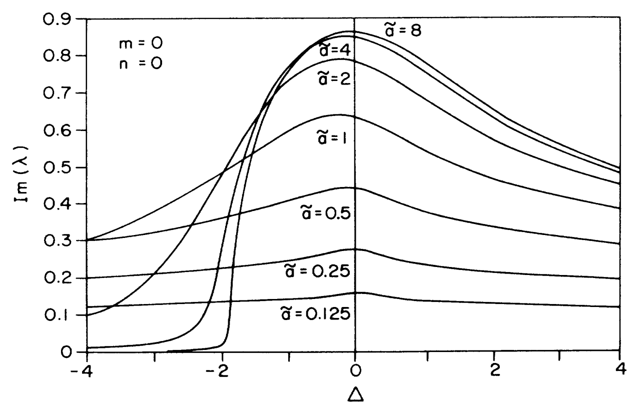

FIG. 3. Imλ as function of $\tilde{a}$ and Δ. Same conditions as Fig. 1.

$$P=\sum_{n,l}P_{nl} , \tag{6.23a}$$

$$P_{nl}=\tfrac{1}{2}mc^2\gamma_0(2\rho)^3ck_r \times\int\frac{dq_{\|}}{2\pi}\dot{G}_n(q_{\|},\tau)\dot{G}_l^*(q_{\|},\tau)N_{nl}(q_{\|}) , \tag{6.23b}$$

where

$$\dot{G}_n(q_{\|},\tau)=\frac{e^{-i\Omega_n(q_{\|})\tau}}{-i\Omega_n(q_{\|})[1-F_n(q_{\|})]} , \tag{6.24}$$

$$F_n(q_{\|})=[\partial\Lambda_n(\Omega,q_{\|})/\partial\Omega]_{\Omega=\Omega_n(q_{\|})} , \tag{6.25}$$

$$N_{nl}(q_{\|})=\int d^2x\,\psi_n(q_{\|},\mathbf{x})\psi_l^*(q_{\|},\mathbf{x}) \times\int d^2x_1u(x_1)\psi_n(q_{\|},\mathbf{x}_1)\psi_l^*(q_{\|},\mathbf{x}_1) . \tag{6.26}$$

For simplicity, we only disucss axial symmetric modes [$m=0$ in Eq. (6.5)]. From Eq. (6.7), we can express ψ_n as

$$\psi_n=\begin{cases}C_nJ_0\left[\chi_n\dfrac{x}{a}\right], & x\le a\\[2ex] D_nH_0^{(1)}\left[\phi_n\dfrac{x}{a}\right], & x\ge a .\end{cases} \tag{6.27}$$

The parameters χ_n and ϕ_n are determined as functions of $q_{\|}=2\rho\Delta$ from Eqs. (6.14) and (6.15), and once ϕ_n is known, $\Omega_n=2\rho\lambda_n$ is determined from Eq. (6.16). The constants C_n and D_n are specified by the conditions

$$D_n/C_n=J_0(\chi_n)/H_0^{(1)}(\phi_n) \tag{6.28}$$

and [see Eq. (4.5) and Appendix C]

$$\int_0^\infty 2\pi x\,dx\,\psi_n^2=C_n^2\pi a^2J_0^2(\chi_n)\xi_n^2\left[\frac{1}{\chi_n^2}-\frac{1}{\phi_n^2}\right]=1 . \tag{6.29}$$

When $\rho|\Delta|\ll1$ and $\rho|\lambda_n|\ll1$, an explicit expression for F_n is (see Appendix D):

$$F_n=-\frac{\phi_n^2}{\phi_n^2-\chi_n^2}\left[1+\frac{\chi_n^2}{\xi_n^2}\right]\frac{2}{\lambda_n^3} . \tag{6.30}$$

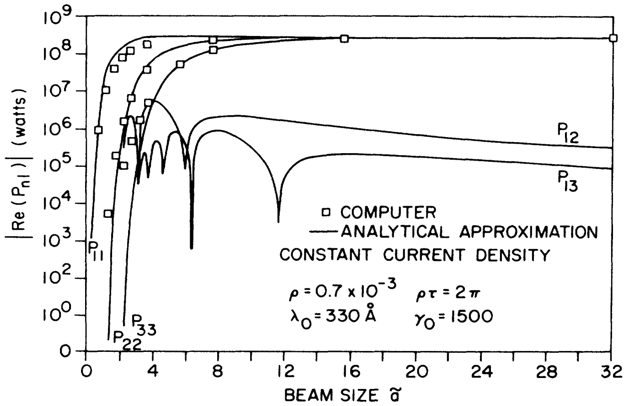

FIG. 5. Power terms P_{nl} [Eq. (6.23)] calculated by direct computer calculation and with the approximate analytic expression of Eq. (6.50).

The integrals in Eq. (6.26) for N_{nl} can be evaluated (see Appendix C) yielding

$$N_{nl} = \frac{4(\xi_l - \xi_n^*)^2}{\xi_n^2 \xi_l^{*2}} \frac{(\chi_n \chi_l^*)^2}{[\tilde{a}(\chi_n^2 - \chi_l^{*2})]^2} \frac{(\phi_n \phi_l^*)^2}{\phi_n^2 - \phi_l^{*2}} (\lambda_l^{2*} - \lambda_n^2) \; . \tag{6.31}$$

Equation (6.23b) for P_{nl} can now be written

$$P_{nl} = \frac{(2\rho)^2}{4\pi} mc^2 \gamma_0 k_r c \int d\Delta \frac{N_{nl} e^{i2\rho\tau(\lambda_l^* - \lambda_n)}}{\lambda_n \lambda_l^* (1 - F_n)(1 - F_l^*)} \; . \tag{6.32}$$

Solving Eqs. (6.14) and (6.15) yields χ_n, ϕ_n, ξ_n, and λ_n as functions of Δ, and these results can be used in Eq. (6.32) to calculate the output power numerically, with F_n and N_{nl} expressed by Eqs. (6.30) and (6.31), respectively. Results are given in Fig. 5. The parameters in our example are $\rho = 0.7 \times 10^{-3}$, $2\rho\tau = 4\pi$, $\lambda_0 = 330$ Å, $\gamma_0 = 1500$ (750 MeV). This corresponds to $\rho N_w = 1$, i.e., near saturation. If we have an electron beam with 100 A, beam diameter $2r_0 = 0.68$ mm, wigger period 3 cm, wiggler length 43 m, we would have these parameters, with $\tilde{a} = 3.6$ and wiggler parameter $K = 2$.

C. Approximation to output power

We shall now derive an approximate expression for P_{nl}, which is asymptomatically correct for large $\tilde{a}$. We consider $|2\rho\lambda_n| \ll 1$ and $|2\rho\Delta| \ll 1$, so from Eq. (6.17) it is seen that $\lambda = \lambda_n$ is a solution of the cubic equation,

$$\lambda^3 - \left[\Delta + \frac{\chi_n^2}{\tilde{a}^2}\right] \lambda^2 - 1 = 0 \; , \tag{6.33}$$

where χ_n is given by the approximate result of Eq. (6.22),

$$\frac{\chi_n}{\tilde{a}} \simeq \frac{\mu_{0n}}{\tilde{a}} \left[1 - \frac{i}{\phi_n + i}\right] , \tag{6.34}$$

and

$$\phi_n = \tilde{a}\sqrt{\lambda_n - \Delta} \; . \tag{6.35}$$

In Eq. (6.34), μ_{0n} is a zero of the Bessel function, $J_0(\mu_{0n}) = 0$.

In Eq. (6.32) for P_{nl}, the dominant contribution to the integral comes from the neighborhood of the maximum of $\text{Im}(\lambda_n - \lambda_l^*)$, see Fig. 3. From Eq. (6.33) we see that in this region $|\Delta + \chi_n^2/\tilde{a}^2|$ is small. For $\tilde{a}$ large, this means Δ is small. We shall move all factors except the exponential $e^{i2\rho\tau(\lambda_l^* - \lambda_n)}$ outside of the integration in Eq. (6.32). The factors moved out of the integral are slowly varying and we evaluate them for

$$\Delta = 0 \; , \tag{6.36a}$$

$$\lambda_n \simeq e^{2\pi i/3} \; , \tag{6.36b}$$

$$\phi_n \simeq \tilde{a} e^{i\pi/3}, \quad \xi_n \simeq i\phi_n \; , \tag{6.36c}$$

$$\chi_n \simeq \frac{\mu_{0n}}{1 + \dfrac{i}{\tilde{a}} e^{-i\pi/3}} \; . \tag{6.36d}$$

In this way, we obtain

$$P_{nl} \simeq \frac{(2\rho)^2}{4\pi} mc^2 \gamma_0 k_r c \frac{N_{nl}^{(0)}}{(1 - F_n^{(0)})(1 - F_l^{(0)*})} \times \int d\Delta \, e^{i2\rho\tau(\lambda_l^* - \lambda_n)} , \tag{6.37}$$

where the superscript "zero" indicates $N_{nl}^{(0)}$ and $F_n^{(0)}$ are evaluated using Eqs. (6.36a)–(6.36d). From Eqs. (6.30) and (6.31), we derive

$$1 - F_n^{(0)} \simeq 3 \; , \tag{6.38}$$

$$N_{nl}^{(0)} \simeq -\frac{4(\chi_n \chi_l^*)^2}{\tilde{a}^2 (\chi_n^2 - \chi_l^{*2})^2} = -\frac{4\varepsilon_n \varepsilon_l^*}{\tilde{a}^2 (\varepsilon_n - \varepsilon_l^*)^2} \; , \tag{6.39}$$

with

$$\varepsilon_n \equiv \frac{\mu_{0n}^2}{\tilde{a}^2} \frac{1}{\left[1 + \dfrac{i}{\tilde{a}} e^{-i\pi/3}\right]^2} \; . \tag{6.40}$$

In order to estimate the integral in Eq. (6.37), we use Eqs. (6.33) and (6.34) to derive

$$\lambda_n^3 - (\Delta + \varepsilon_n)\lambda_n^2 - 1 = 0 \; ,$$

where $\chi_n^2/\tilde{a}^2 \simeq \varepsilon_n$ as defined in Eq (6.40). When $\Delta + \varepsilon_n$ is small, we have the approximation

$$\lambda_n \simeq e^{2\pi i/3} + \tfrac{1}{3}(\Delta + \varepsilon_n) + \tfrac{1}{9} e^{-2\pi i/3} (\Delta + \varepsilon_n)^2 \; . \tag{6.41}$$

The exponent apearing in the integral in Eq. (6.37) is approximated by using

$$\lambda_n - \lambda_l^* \simeq i\sqrt{3} + \tfrac{1}{3}(\varepsilon_n - \varepsilon_l^*) + \tfrac{1}{9}[e^{-2\pi i/3}(\Delta + \varepsilon_n)^2 - e^{2\pi i/3}(\Delta + \varepsilon_l^*)^2] \; . \tag{6.42}$$

We let $\Delta = \Delta_s$ be the saddle point determined by

$$\frac{d}{d\Delta}(\lambda_n-\lambda_l^*)\,|_{\Delta=\Delta_s}=\tfrac{2}{9}[e^{-2\pi i/3}(\Delta+\varepsilon_n)-e^{2\pi i/3}(\Delta+\varepsilon_l^*)]=0\ , \quad (6.43)$$

explicitly

$$\Delta_s=\frac{1}{i\sqrt{3}}(e^{-2\pi i/3}\varepsilon_n-e^{2\pi i/3}\varepsilon_l^*)\ . \quad (6.44)$$

Ignoring terms quadratic in ε_n, we have

$$\lambda_n-\lambda_l^*\simeq i\sqrt{3}+\tfrac{1}{3}(\varepsilon_n-\varepsilon_l^*)-\frac{\sqrt{3}}{9}(\Delta-\Delta_s)^2\ . \quad (6.45)$$

The expression given in Eq. (6.37) for P_{nl} now reduces to

$$P_{nl}\simeq\frac{(2\rho)^2}{4\pi}mc^2\gamma_0 k_r c\tfrac{1}{9}N_{nl}^{(0)}\times\exp\left[2\rho\tau\left[\sqrt{3}-\frac{i}{3}(\varepsilon_n-\varepsilon_l^*)\right]\right]\times\int d\Delta\, e^{-(\Delta-\Delta_s)^2/2\sigma_\Delta^2}\ , \quad (6.46)$$

where

$$\sigma_\Delta=\left[\frac{3\sqrt{3}}{4\rho\tau}\right]^{1/2}\ , \quad (6.47)$$

and we have used Eq. (6.38) for $F_n^{(0)}$ and $N_{nl}^{(0)}$ is given in Eq. (6.39).

The quantity σ_Δ can be related to the bandwidth in wave number or frequency. The electric field corresponding to mode n has the form

$$A_n e^{ik_0 z-i\omega_0 t}e^{iq_{\parallel}\zeta-i\Omega_n\tau}\psi_n(\mathbf{x})\ , \quad (6.48)$$

where $\zeta=k_r(z-v_0 t)$ and $\tau=\omega_w t$. Therefore, we see that $k_z=k_0+k_r q_{\parallel}$, hence

$$\frac{\sigma_k}{k_r}=\sigma_{q_{\parallel}}=2\rho\sigma_\Delta=\left[\frac{3\sqrt{3}\rho}{\tau}\right]^{1/2}\ . \quad (6.49)$$

The Gaussian integral in Eq. (6.46) is equal to $\sqrt{2\pi}\sigma_\Delta$, so we derive the following approximation for P_{nl}:

$$P_{nl}\cong P^{(\mathrm{1D})}N_{nl}^{(0)}e^{-2i\rho\tau(\varepsilon_n-\varepsilon_l^*)/3}\ , \quad (6.50)$$

where $N_{nl}^{(0)}$ and $\varepsilon_n,\varepsilon_l$ are given by Eqs. (6.39)—(6.40), and $P^{(\mathrm{1D})}$ is the output power as calculated in the one-dimensional model,[2,3]

$$P^{(\mathrm{1D})}=\tfrac{1}{9}\rho mc^2\gamma_0\frac{c\sigma_k}{\sqrt{2\pi}}\exp(\sqrt{3}2\rho\tau)\ . \quad (6.51)$$

Equation (6.51) can be seen to be equivalent to the expression for $P^{(\mathrm{1D})}$ given in Eq. (1.3) of the Introduction, by introducing the correlation length $l_c=\sqrt{2\pi}/\sigma_k$ and the number of electrons in a correlation length $N_c=n_0\Sigma l_c$. (Σ is the cross-sectional area of the electron beam.)

The result given by (6.50) is plotted in Fig. 5 and compared with the numerical result described in Sec. VI B. From the diagram we can see that at $\tilde{a}=3.6$ the second mode is suppressed by a factor of 10. Cross terms are suppressed by 100, and output power is still near the one-dimensional result. The result also shows that when keeping ρ constant and reducing $\tilde{a}$ below 2, the output power begins to fall rapidly. When $\tilde{a}>6$, the higher modes become important. Also we can see that for large beam size the power cross terms are small because the modes are almost orthogonal. For small beam size, they are not orthogonal, but higher modes drop fast, so cross terms are always negligible.

VII. TREATMENT OF INITIAL ENERGY SPREAD IN THE ELECTRON BEAM

The discussion in Secs. II—VI has focused upon the description of amplified spontaneous emission from an electron beam which initially is monoenergetic. Now we shall show that the formalism which we have developed is easily generalized to allow the treatment of an electron beam having initial energy spread. Our results are in agreement with the recent work of Kim,[6] who has approached this problem using a method originally introduced by van Kampen.[8] In a future publication[10] we shall further generalize our work to include the effects of angular spread in the electron beam.

Let us begin by recalling the coupled Vlasov-Maxwell equations [Eqs. (2.23) and (2.24)] derived in Sec. II. The slowly varying envelope E of the radiated electric field was introduced in Eq. (2.10), and we now define the corresponding envelope F of the electron distribution by

$$f(\tau,\zeta,\mathbf{x},\gamma)=F(\tau,\zeta,\mathbf{x},\gamma)e^{i\zeta}\ . \quad (7.1)$$

Expressed in terms of E and F, the Vlasov-Maxwell equations become

$$\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}-i\nabla_\perp^2\right]E=D_1\int\frac{d\gamma}{\gamma}F\ , \quad (7.2)$$

$$\left[\frac{\partial}{\partial\tau}+\eta\left[\frac{\partial}{\partial\zeta}+i\right]\right]F=D_2\frac{1}{\gamma}\frac{\partial f_0}{\partial\gamma}E\ . \quad (7.3)$$

Here, $\eta=1-\gamma_0^2/\gamma^2$, where γ_0 is a reference energy near the center of the energy distribution, and the ensemble average of the initial distribution $f_0(\mathbf{x},\gamma)$ is independent of ζ.

We introduce Fourier transforms over ζ,

$$\tilde{E}(\tau,\mathbf{x};q_{\parallel})=\int_{-\infty}^{\infty}d\zeta\, e^{-iq_{\parallel}\zeta}E(\tau,\zeta,\mathbf{x})\ , \quad (7.4)$$

$$\tilde{F}(\tau,\mathbf{x},\gamma;q_{\parallel})=\int_{-\infty}^{\infty}d\zeta\, e^{-iq_{\parallel}\zeta}F(\tau,\zeta,\mathbf{x},\gamma)\ , \quad (7.5)$$

and Laplace transforms over τ,

$$\bar{E}(\mathbf{x};\Omega,q_{\parallel})=\int_0^{\infty}d\tau\, e^{i\Omega\tau}\tilde{E}(\tau,\mathbf{x};q_{\parallel})\ , \quad (7.6)$$

$$\bar{F}(\mathbf{x},\gamma;\Omega,q_{\parallel})=\int_0^{\infty}d\tau\, e^{i\Omega\tau}\tilde{F}(\tau,\mathbf{x},\gamma;q_{\parallel})\ , \quad (7.7)$$

where ImΩ is positive and large enough to guarantee convergence of the integral over τ. Applying the Fourier-Laplace transforms to Eqs. (7.2) and (7.3) yields

$$(-i\Omega+iq_{\parallel}-i\nabla_\perp^2)\bar{E}=D_1\int\frac{d\gamma}{\gamma}\bar{F}+\tilde{E}(\tau=0)\ , \quad (7.8)$$

$$[-i\Omega+i\eta(1+q_{\parallel})]\bar{F}=D_2\frac{1}{\gamma}\frac{\partial f_0}{\partial\gamma}\bar{E}+\tilde{F}(\tau=0)\ , \quad (7.9)$$

Using Eq. (7.9) to express $\bar{F}$ in terms of $\bar{E}$ and $\tilde{F}(\tau=0)$, and substituting the result into Eq. (7.8), we derive the following equation determining $\bar{E}$:

$$[\Omega-q_{\parallel}+\nabla_{\perp}^2+U(\mathbf{x};\Omega,q_{\parallel})]\bar{E}(\mathbf{x};\Omega,q_{\parallel})=S(\mathbf{x};\Omega,q_{\parallel}) , \tag{7.10}$$

where

$$U(\mathbf{x};\Omega,q_{\parallel})=D_1D_2\int\frac{d\gamma}{\gamma^2}\frac{\partial f_0(\mathbf{x},\gamma)/\partial\gamma}{\Omega-\eta(\gamma)(1+q_{\parallel})} \tag{7.11}$$

and

$$S(\mathbf{x};\Omega,q_{\parallel})=i\tilde{E}(\tau=0,\mathbf{x};q_{\parallel}) -D_1\int\frac{d\gamma}{\gamma}\frac{\tilde{F}(\tau=0,\mathbf{x},\gamma;q_{\parallel})}{\Omega-\eta(\gamma)(1+q_{\parallel})} . \tag{7.12}$$

Since Eq. (7.10) has the form of an inhomogeneous two-dimensional Schrödinger equation, the techniques developed in Sec. IV and V to treat the monoenergetic electron beam are applicable to the warm beam. The inhomogeneous term $S(\mathbf{x};\Omega,q_{\parallel})$ is a known function, determined by the initial values of the electric field and electron distribution at $\tau=0$. The potential $U(\mathbf{x};\Omega,q_{\parallel})$ is also a known function, expressed in terms of the distribution $f_0(\mathbf{x},\gamma)$. In the special case when f_0 factorizes,

$$f_0(\mathbf{x},\gamma)=u(x)h(\gamma) , \tag{7.13}$$

the potential U has the form

$$U(\mathbf{x};\Omega,q_{\parallel})=u(x)D_1D_2\int\frac{d\gamma}{\gamma^2}\frac{h'(\gamma)}{\Omega-\eta(\gamma)(1+q_{\parallel})} . \tag{7.14}$$

The height of the potential well is proportional to the dispersion integral, and the shape is given by the transverse profile $u(x)$ of the electron beam.

Since $U(\mathbf{x};\Omega,q_{\parallel})$ and $S(\mathbf{x};\Omega,q_{\parallel})$ are known functions, the inhomogeneous equation (7.10) can be solved using the Green's function $G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})$, which we define by

$$[\Omega-q_{\parallel}+\nabla_{\perp}^2+U(\mathbf{x};\Omega,q_{\parallel})]G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})=\delta(\mathbf{x}-\mathbf{x}') . \tag{7.15}$$

We determine this Green's function by following the approach developed in Sec. IV. This can be done since Eq. (7.15) has the same form as Eq. (4.3), except that the potential U now includes the effects of energy spread. It is easy to see that in the case of a monoenergetic electron beam, with

$$f_0=u(x)\delta(\gamma-\gamma_0) , \tag{7.16}$$

the potential U reduces to

$$U=\frac{\alpha}{\Omega^2}(\Omega-q_{\parallel}-1)u(x) \tag{7.17}$$

where $\alpha=(2\rho)^3=D_1D_2/\gamma_0^3$. In this case, Eq. (7.15) is identical to Eq. (4.3)

Following the technique of Sec. IV, we express the Green's function $G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})$ in terms of the eigenfunctions $\Phi_n(x)$ and eigenvalues Λ_n of the associated homogeneous eigenvalue problem

$$[\Lambda_n+\nabla_{\perp}^2+U(\mathbf{x};\Omega,q_{\parallel})]\Phi_n(\mathbf{x})=0 . \tag{7.18}$$

Using the orthonormality [Eq. (4.5)] and completeness [Eq. (4.6)] of the eigenfunctions, we find

$$G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})=\sum_n\frac{\Phi_n(\mathbf{x};\Omega,q_{\parallel})\Phi_n(\mathbf{x}';\Omega,q_{\parallel})}{\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})} . \tag{7.19}$$

Once $G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})$ is determined, $\bar{E}(\mathbf{x};\Omega,q_{\parallel})$ can be found from

$$\bar{E}(\mathbf{x};\Omega,q_{\parallel})=\int d^2x'\,G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})S(\mathbf{x}';\Omega,q_{\parallel}) . \tag{7.20}$$

The inverse Fourier-Laplace transform can now be employed to obtain $E(\tau,\zeta,\mathbf{x})$ in the form

$$E(\tau,\zeta,\mathbf{x})=\int\frac{dq_{\parallel}}{2\pi}e^{iq_{\parallel}\zeta}\int_{-\infty+is}^{\infty+is}\frac{d\Omega}{2\pi}e^{-i\Omega\tau}\sum_n\frac{\Phi_n(\mathbf{x};\Omega,q_{\parallel})}{\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})}\int d^2x'\,S(\mathbf{x}';\Omega,q_{\parallel})\Phi_n(\mathbf{x}';\Omega,q_{\parallel}) . \tag{7.21}$$

As in Sec. IV, we assume that the leading behavior for large τ is described by the poles in the complex Ω plane corresponding to the solution of

$$\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})=0 . \tag{7.22}$$

We keep only those solutions with $\mathrm{Im}\Omega_n>0$, and find the asymptotic representation for $E(\tau,\zeta,\mathbf{x})$:

$$E(\tau,\zeta,\mathbf{x})=\int\frac{dq_{\parallel}}{2\pi i}e^{iq_{\parallel}\zeta}\sum_n\frac{e^{-i\Omega_n(q_{\parallel})\tau}\psi_n(\mathbf{x};q_{\parallel})}{1-\left[\dfrac{\partial\Lambda_n}{\partial\Omega}\right]_{\Omega=\Omega_n(q_{\parallel})}}\int d^2x'\,S(\mathbf{x}';\Omega_n(q_{\parallel}),q_{\parallel})\psi_n(\mathbf{x}';q_{\parallel}) , \tag{7.23}$$

where

$$\psi_n(\mathbf{x};q_{\parallel})=\Phi_n(\mathbf{x};\Omega_n(q_{\parallel}),q_{\parallel}) . \tag{7.24}$$

Equation (7.23) can be seen to be in agreement with the work of Kim,[6] by expressing the derivative $\partial\Lambda_n/\partial\Omega$ as follows. Using Eq. (7.18) and the normalization condition, $\int d^2x\,\Phi_n^2=1$, we find

$$\Lambda_n=-\int d^2x\,\Phi_n(\nabla_{\perp}^2+U)\Phi_n \tag{7.25}$$

and

$$\frac{\partial \Lambda_n}{\partial \Omega} = -\int d^2x\, \Phi_n^2 \frac{\partial U}{\partial \Omega} , \tag{7.26}$$

hence

$$\left[\frac{\partial \Lambda_n}{\partial \Omega}\right]_{\Omega=\Omega_n(q_{\|})} = -\int d^2x\, \psi_n^2 \left[\frac{\partial U}{\partial \Omega}\right]_{\Omega=\Omega_n(q_{\|})} . \tag{7.27}$$

The use of Eq. (7.27) in Eq. (7.23) yields Kim's result.[6]

Let us now use Eq. (7.12) to explicitly evaluate $S(\mathbf{x};\Omega_n(q_{\|}),q_{\|})$. Since we are considering amplified spontaneous radiation, we assume the electric field vanishes at $\tau=0$. We express $\widetilde{F}(\tau=0)$ as

$$\begin{aligned}\widetilde{F}(\tau=0,\mathbf{x}',\gamma;q_{\|}) &= \int d\zeta'\, e^{-iq_{\|}\zeta'} F(\tau=0,\zeta',\mathbf{x}';\gamma) \\ &= \int d\zeta'\, e^{-i(1+q_{\|})\zeta'} f(\tau=0,\zeta',\mathbf{x}'\gamma) .\end{aligned} \tag{7.28}$$

Substituting Eq. (7.28) into the expression for $S(\mathbf{x}';\Omega_n,q_{\|})$ given in Eq. (7.12), and using the result in Eq. (7.23), we find

$$E(\tau,\zeta,\mathbf{x}) = \int d\zeta'\, d^2x'\, d\gamma\, j_0(\zeta',\mathbf{x}',\gamma) g(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}',\gamma) , \tag{7.29}$$

where we have defined

$$g(\tau,\zeta,\mathbf{x},\mathbf{x}',\gamma) = \int \frac{dq_{\|}}{2\pi} e^{iq_{\|}\zeta} \sum_n G_n(\tau,\gamma;q_{\|})\psi_n(\mathbf{x};q_{\|})\psi_n(\mathbf{x}';q_{\|}) , \tag{7.30}$$

$$G_n(\tau,\gamma;q_{\|}) = \frac{e^{-i\Omega_n(q_{\|})\tau}}{(-i)[\Omega_n(q_{\|})-\eta(\gamma)(1+q_{\|})]\left[1-\left[\dfrac{\partial \Lambda_n}{\partial \Omega}\right]_{\Omega=\Omega_n(q_{\|})}\right]} , \tag{7.31}$$

and

$$j_0(\zeta,\mathbf{x},\gamma) = \frac{D_1}{\gamma} e^{-i\zeta} f(\tau=0,\zeta,\mathbf{x},\gamma) . \tag{7.32}$$

The treatment of shot noise given earlier for a monochromatic electron beam, in Sec. V, can now be generalized to include energy spread and energy shot noise. The initial electron distribution is taken to be

$$f(\tau=0) = \frac{1}{n_0} \sum_i \delta(z-z_i)\delta(\mathbf{x}-\mathbf{x}_i)\delta(\gamma-\gamma_i) . \tag{7.33}$$

As in Sec. V, the output power is determined by taking the ensemble average $\langle EE^* \rangle$. The only difference in the derivation from that given in Sec. V, is that when we replace the sum over individual electrons i by an integral, as in Eq. (5.9), we now must include an integration over energy γ:

$$\sum_i \to \int dz\, d^2r\, d\gamma\, n_0 f_0(\mathbf{r},\gamma) = \frac{1}{b} \int d\zeta\, d^2x\, d\gamma\, n_0 f_0(\mathbf{x},\gamma) . \tag{7.34}$$

When the ensemble average of the intiial distribution f_0 factorizes according to Eq. (7.13), the result for the output power corresponding to Eq. (5.15) is

$$P = \sum_{n,l} P_{nl} , \tag{7.35a}$$

$$\begin{aligned}P_{nl} = \tfrac{1}{2} mc^2\gamma_0(2\rho)^3 k_r c \int \frac{dq_{\|}}{2\pi} \int d^2x\, \psi_n(\mathbf{x};q_{\|})\psi_l^*(\mathbf{x};q_{\|}) \\ \times \int d^2x'\, u(x')\psi_n(\mathbf{x}';q_{\|})\psi_l^*(\mathbf{x}';q_{\|}) \int d\gamma \frac{\gamma_0^2}{\gamma^2} h(\gamma) G_n(\tau,\gamma;q_{\|}) G_l^*(\tau,\gamma;q_{\|}) .\end{aligned} \tag{7.35b}$$

VIII. CONCLUDING REMARKS

In this paper, we have presented a theoretical description of amplified spontaneous emission in a long wiggler magnet. In the special case when the electron beam is initially monoenergetic, the envelope of the radiated electric field is determined by the differential equation of Eq. (2.39):

$$\frac{\partial^2}{\partial\tau^2}\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}-i\nabla_\perp^2\right]E-\alpha\left[\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}+i\right](uE)=0\ .$$

When the initial energy spread is nonvanishing, the dependence of the envelope of the field on the coordinates $\tau,\zeta,\mathbf{x}$ is no longer determined by a partial differential equation. To treat the case of nonvanishing energy spread, one introduces a Laplace transform over the τ dependence, and a Fourier transform over the ζ dependence. The Fourier-Laplace transform $\bar{E}(\mathbf{x};\Omega,q_{\parallel})$ of the envelope of the radiated electric field is determined by the partial differential equation of Eq. (7.10):

$$[\Omega-q_{\parallel}+\nabla_\perp^2+U(\mathbf{x};\Omega,q_{\parallel})]\bar{E}(\mathbf{x};\Omega,q_{\parallel})=S(\mathbf{x};\Omega,q_{\parallel})\ .$$

This inhomogeneous two-dimensional Schrödinger equation can be solved by introducing the Green's function $G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})$ via Eq. (7.15):

$$[\Omega-q_{\parallel}+\nabla_\perp^2+U(\mathbf{x};\Omega,q_{\parallel})]G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})=\delta(\mathbf{x}-\mathbf{x}')\ .$$

This Green's function is determined by expanding it in terms of the orthonormal eigenfunctions of the homogeneous Schrödinger equation with non-self-adjoint Hamiltonian [Eq. (7.18)]:

$$[\Lambda_n+\nabla_\perp^2+U(\mathbf{x};\Omega,q_{\parallel})]\Phi_n(\mathbf{x})=0\ .$$

The Green's function is then given by Eq. (7.19):

$$G(\mathbf{x},\mathbf{x}';\Omega,q_{\parallel})=\sum_n\frac{\Phi_n(\mathbf{x};\Omega,q_{\parallel})\Phi_n(\mathbf{x}';\Omega,q_{\parallel})}{\Omega-q_{\parallel}-\Lambda_n(\Omega,q_{\parallel})}\ .$$

The dependence of the electric field on the coordinates τ and ζ is recovered by carrying out the inverse Fourier-Laplace transformation, as discussed in Sec. VII.

It is our belief that the essential physics of the amplified spontaneous emission process is elucidated by study of the monoenergetic electron beam. The detailed numerical calculations and the analytical approximation of Sec. VI were carried out for a monoenergetic electron beam, with the goal of determining the region of validity of the one-dimensional calculation.[2,3] In order to carry out the design of a single-pass free-electron laser for the production of high peak power pulses of short-wavelength radiation, the inclusion of energy spread and angular spread in the electron beam is essential. Numerical calculations including these effects are now under study.

ACKNOWLEDGMENTS

Our interest in amplified spontaneous emission was originally stimulated by C. Pellegrini. We have learned much from discussions with A. Gover, K. J. Kim, G. T. Moore, J. Murphy, and J. M. Wang. This work was performed under the auspices of the U.S. Department of Energy.

APPENDIX A: MOORE'S NOTATION

To aid the reader, we provide a correspondence between our notation and that of Moore.[5] Refer back to the mode Eq. (3.2). Let us neglect the small term $\Omega-q_{\parallel}$ relative to unity in the coefficient of $u(x)$. Moore uses the original dimensioned coordinate $\mathbf{r}$, rather than the dimensionless $\mathbf{x}=\sqrt{2k_0k_w}\mathbf{r}$, and hence the dimensioned Laplacian $\nabla_T^2=2k_0k_w\nabla_\perp^2$. His growth rate parameter β and detuning μ_0 are related to ours via

$$\beta=-ik_w(\Omega-q_{\parallel})\ , \tag{A1}$$

$$\mu_0=-k_wq_{\parallel}\ . \tag{A2}$$

Employing these relations, we can rewrite Eq. (3.2) as

$$\left[\beta+\frac{1}{2ik_0}\nabla_T^2-(2\rho)^3k_w^3\frac{i}{(\beta+i\mu_0)^2}u(\mathbf{r})\right]\psi=0\ . \tag{A3}$$

Moore's form of the mode equation is obtained by introducing the pumping parameter,

$$C_0=(2\rho)^3k_w^3\Sigma\ , \tag{A4}$$

where $\Sigma=\pi r_0^2$ is the electron-beam cross section, and the normalized transverse profile

$$\tilde{u}=u/\Sigma,\quad \int\tilde{u}\,d^2r=1\ . \tag{A5}$$

Using Eqs. (A4) and (A5) in Eq. (A3) yields Moore's equation:

$$\left[\beta+\frac{1}{2ik_0}\nabla_T^2-C_0\tilde{u}(\mathbf{r})\frac{i}{(\beta+i\mu_0)^2}\right]\psi=0\ . \tag{A6}$$

A key parameter in Moore's analysis is the characteristic transverse dimension a_c defined by

$$a_c^{-4}=8k_0^3C_0/\pi\ . \tag{A7}$$

Inserting the expression for C_0 given in (A4) into Eq. (A7) yields

$$a_c^{-4}=(2\rho 2k_0k_w)^3r_0^2\ . \tag{A8}$$

Moore defines a scaled electron beam radius $\hat{a}$ by

$$\hat{a}=r_0/a_c \tag{A9}$$

and from Eq. (A8) it follows that

$$\hat{a}^2=(2\rho 2k_0k_w)^{3/2}r_0^3=\tilde{a}^3\ , \tag{A10}$$

where $\tilde{a}$ is the scaled electron beam radius we defined in the Introduction in Eq. (1.1).

Moore introduces the characteristic length l_c by

$$l_c=k_0a_c^2\ , \tag{A11}$$

and writes the gain in the form

$$G=G_0e^{\hat{g}L/l_c} \tag{A12}$$

where L is the length of the wiggler and $\hat{g}$ is the scaled gain. The gain length $l_G=l_c/\hat{g}$. In Eq. (1.4) of the Introduction, we noted that Moore's result for the gain in the limit of small electron-beam size could be written in the form

$$k_0r_{\rm em}^2\simeq l_G\ . \tag{A13}$$

To see this, we rewrite (A13) as

$$k_0r_{\rm em}^2\cong\frac{l_c}{\hat{g}}=\frac{k_0a_c^2}{\hat{g}}\ , \tag{A14}$$

hence

$$r_{\rm em} \simeq \hat{g}^{-1/2} a_c \ , \tag{A15}$$

which is the result derived by Moore.

APPENDIX B: GREEN'S THEOREM AND THE INITIAL VALUE PROBLEM

The Green's function $g = g(\tau-\tau', \zeta-\zeta', \mathbf{x}, \mathbf{x}')$ satisfies the equations:

$$\frac{\partial^2}{\partial \tau^2}\left[\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \zeta}-i\nabla_\perp^2\right]g - \alpha u(\mathbf{x})\left[\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \zeta}+i\right]g = \delta(\tau-\tau')\delta(\zeta-\zeta')\delta(\mathbf{x}-\mathbf{x}')$$

and

$$\frac{\partial^2}{\partial \tau'^2}\left[-\frac{\partial}{\partial \tau'}-\frac{\partial}{\partial \zeta'}-i\nabla_\perp'^2\right]g - \alpha u(\mathbf{x}')\left[-\frac{\partial}{\partial \tau'}-\frac{\partial}{\partial \zeta'}+i\right]g = \delta(\tau-\tau')\delta(\zeta-\zeta')\delta(\mathbf{x}-\mathbf{x}') \ .$$

Let $E = E(\tau', \zeta', \mathbf{x}')$ be a solution of

$$\frac{\partial^2}{\partial \tau'^2}\left[\frac{\partial}{\partial \tau'}+\frac{\partial}{\partial \zeta'}-i\nabla_\perp'^2\right]E - \alpha u(\mathbf{x}')\left[\frac{\partial}{\partial \tau'}+\frac{\partial}{\partial \zeta'}+i\right]E = 0 \ .$$

We introduce the notation:

$$E_0 = E(\tau'=0, \zeta', \mathbf{x}') \ ,$$

$$\dot{E}_0 = \left[\frac{\partial}{\partial \tau'}E(\tau', \zeta', \mathbf{x}')\right]_{\tau'=0} \ ,$$

$$g_0 = g(\tau, \zeta-\zeta', \mathbf{x}, \mathbf{x}') \ ,$$

$$\dot{g}_0 = \frac{\partial g_0}{\partial \tau} = -\left.\frac{\partial g}{\partial \tau'}\right|_{\tau'=0} \ ,$$

$$K(f) = \int d^2x'\, d\zeta' \int_0^\infty d\tau' f \ ,$$

$$L(f) = \int d^2x'\, d\zeta' f \ .$$

Employing integration by parts and noting that the Green's function g vanishes for $\tau' \to \infty$ and $\zeta' \to \pm\infty$, due to causality, one derives

$$K\left[E\frac{\partial^3 g}{\partial \tau'^3}+g\frac{\partial^3 E}{\partial \tau'^3}\right] = -L(E_0\ddot{g}_0+\dot{E}_0\dot{g}_0+\ddot{E}_0 g_0) \ ,$$

$$K\left[E\frac{\partial^3 g}{\partial \tau'^2 \partial \zeta'}+g\frac{\partial^3 E}{\partial \tau'^2 \partial \zeta'}\right] = -L\left[\frac{\partial E_0}{\partial \zeta'}\dot{g}_0+\frac{\partial \dot{E}_0}{\partial \zeta'}g_0\right] \ ,$$

$$K\left[E\frac{\partial^2}{\partial \tau'^2}\nabla_\perp'^2 g - g\frac{\partial^2}{\partial \tau'^2}\nabla_\perp'^2 E\right] = L(\dot{g}_0\nabla_\perp'^2 E_0 + g_0\nabla_\perp'^2\dot{E}_0) \ ,$$

$$K\left[E\frac{\partial g}{\partial \tau'}+g\frac{\partial E}{\partial \tau'}\right] = -L(E_0 g_0) \ ,$$

$$K\left[E\frac{\partial g}{\partial \zeta'}+g\frac{\partial E}{\partial \zeta'}\right] = 0 \ .$$

The initial value problem can now be solved in terms of the Green's function by using Green's theorem:

$$\begin{aligned} E(\tau, \zeta, \mathbf{x}) &= \int d^2x'\, d\zeta' \int_0^\infty d\tau'\, E(\tau', \zeta', \mathbf{x}')\delta(\tau-\tau')\delta(\zeta-\zeta')\delta(\mathbf{x}-\mathbf{x}') \\ &= K\left\{E\left[-\frac{\partial^3 g}{\partial \tau'^3}-\frac{\partial^3 g}{\partial \tau'^2 \partial \zeta'}-\frac{\partial^2}{\partial \tau'^2}i\nabla_\perp'^2 g+\alpha u(\mathbf{x}')\left[\frac{\partial g}{\partial \tau'}+\frac{\partial g}{\partial \zeta'}-ig\right]\right]\right\} \\ &= L\left[E_0\ddot{g}_0+\left[\dot{E}_0+\frac{\partial E_0}{\partial \zeta'}-i\nabla_\perp'^2 E_0\right]\dot{g}_0+\left[\ddot{E}_0+\frac{\partial \dot{E}_0}{\partial \zeta'}-i\nabla_\perp'^2\dot{E}_0-\alpha u E_0\right]g_0\right] \ . \end{aligned} \tag{B1}$$

The right-hand side of Eq. (B1) can be simplified, because of the wave equation [Eq. (2.35)]:

$$\left[\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \zeta}-i\nabla_\perp^2\right]E = J \ ,$$

$$J = D_1 e^{-i\zeta}\int \frac{d\gamma}{\gamma} f \ .$$

Recall that the constant D_1 was defined in Eq. (2.25). It follows that

$$\dot{E}_0+\frac{\partial E_0}{\partial \zeta}-i\nabla_\perp^2 E_0 = J_0 \ , \tag{B2}$$

with

$$J_0 = D_1 e^{-i\zeta}\int \frac{d\gamma}{\gamma} f(\tau=0) \ . \tag{B3}$$

We also see that

$$\ddot{E}_0+\frac{\partial \dot{E}_0}{\partial \zeta}-i\nabla_\perp^2\dot{E}_0 = \left.\frac{\partial J}{\partial \tau}\right|_{\tau=0+} \ , \tag{B4}$$

where

$$\left.\frac{\partial J}{\partial \tau}\right|_{\tau=0+} = D_1 e^{-i\zeta}\int \frac{d\gamma}{\gamma}\left.\frac{\partial f}{\partial \tau}\right|_{\tau=0+} \ . \tag{B5}$$

In Eqs. (B4) and (B5) we employ a one-sided derivative since we consider the interaction of the electron beam with the radiation field to begin at $\tau=0$. Using the linearized Vlasov equation (2.24), we find

$$\left.\frac{\partial f}{\partial \tau}\right|_{\tau=0+} = -\left[1-\frac{\gamma_0^2}{\gamma^2}\right]\left.\frac{\partial f}{\partial \zeta}\right|_{\tau=0} + D_2 E_0 e^{i\zeta} u(\mathbf{x})\frac{1}{\gamma}\delta'(\gamma-\gamma_0) , \quad \text{(B6)}$$

and inserting (B6) into Eq. (B5) yields

$$\left.\frac{\partial J}{\partial \tau}\right|_{\tau=0+} = \dot{J}_0 + \alpha u E_0 , \quad \text{(B7)}$$

with

$$\dot{J}_0 = -D_1 e^{-i\zeta}\int \frac{d\gamma}{\gamma}\left[1-\frac{\gamma_0^2}{\gamma^2}\right]\left.\frac{\partial f}{\partial \zeta}\right|_{\tau=0} . \quad \text{(B8)}$$

Finally, from Eqs. (B4) and (B7), we conclude

$$\ddot{E}_0 + \frac{\partial \dot{E}_0}{\partial \zeta} - i\nabla_\perp^2 \dot{E}_0 - \alpha u E_0 = \dot{J}_0 . \quad \text{(B9)}$$

The desired expression for the field $E(\tau,\zeta,\mathbf{x})$ in terms of the initial conditions is derived by using Eqs. (B2) and (B9) in Eq. (B1):

$$\begin{aligned} E(\tau,\zeta,\mathbf{x}) &= \int d^2x'\, d\zeta' [E_0(\zeta',\mathbf{x}')\ddot{g}(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}') \\ &\quad + J_0(\zeta',\mathbf{x}')\dot{g}(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}') \\ &\quad + \dot{J}_0(\zeta',\mathbf{x}') g(\tau,\zeta-\zeta',\mathbf{x},\mathbf{x}')] . \end{aligned} \quad \text{(B10)}$$

Here, $E_0 = E(\tau=0,\zeta',x')$ is the initial electric field, and J_0 as defined in Eq. (B3), describes the initial spatial bunching of the electron beam. The quantity $\dot{J}_0$ [Eq. (B8)] is a measure of the initial energy modulation existing at $\tau=0$. Our motivation for using the notation $\dot{J}_0$ can be seen as follows: Suppose the initial state $f(\tau=0)$ resulted from an evolution during $\tau<0$ described by the unperturbed Vlasov equation,

$$\frac{\partial f}{\partial \tau} + \left[1-\frac{\gamma_0^2}{\gamma^2}\right]\frac{\partial f}{\partial \zeta} = 0 \quad (\tau<0) ,$$

then it would follow that

$$\left.\frac{\partial f}{\partial \tau}\right|_{\tau=0-} = -\left[1-\frac{\gamma_0^2}{\gamma^2}\right]\left.\frac{\partial f}{\partial \zeta}\right|_{\tau=0} . \quad \text{(B11)}$$

In this case, $\dot{J}_0$ as defined in Eq. (B8) is given by

$$\dot{J}_0 = D_1 e^{-i\zeta}\int \frac{d\gamma}{\gamma}\left.\frac{\partial f}{\partial \tau}\right|_{\tau=0-} = \left.\frac{\partial J}{\partial \tau}\right|_{\tau=0-} .$$

APPENDIX C: NORMALIZATION AND OVERLAP INTEGRALS

Consider the radial wave function

$$R_n(x) = \begin{cases} C_n J_0\left[\chi_n \frac{x}{a}\right], & x \le a \\ D_n H_0^{(1)}\left[\phi_n \frac{x}{a}\right], & x \ge a \end{cases}$$

$$C_n J_0(\chi_n) = D_n H_0^{(1)}(\phi_n) ,$$

$$\chi_n \frac{J_0'(\chi_n)}{J_0(\chi_n)} = \phi_n \frac{H_0^{(1)\prime}(\phi_n)}{H_0^{(1)}(\phi_n)} \equiv \xi_n ,$$

$$\phi_n^2 - \chi_n^2 = \frac{\tilde{a}^2}{\lambda_n^2}\left[1-2\rho\frac{\phi_n^2}{\tilde{a}^2}\right] \simeq \frac{\tilde{a}^2}{\lambda_n^2} .$$

(1) Normalization integral [Eq. (6.29)]:

$$\int_0^\infty R_n^2 x\, dx = \frac{a^2}{2}R_n^2(a)\xi_n^2\left[\frac{1}{\chi_n^2}-\frac{1}{\phi_n^2}\right] . \quad \text{(C1)}$$

This follows from the formulas

$$\int_0^u [J_0(x)]^2 x\, dx = \frac{u^2}{2}\{[J_0'(u)]^2 + [J_0(u)]^2\} ,$$

$$\int_u^\infty [H_0^{(1)}(x)]^2 x\, dx = -\frac{u^2}{2}\{[H_0^{(1)\prime}(u)]^2 + [H_0^{(1)}(u)]^2\} .$$

Let $J_0 \equiv J_0(\chi_n)$ and $H_0 \equiv H_0^{(1)}(\phi_n)$. Then

$$\begin{aligned} \int_0^\infty R_n^2 x\, dx &= C_n^2 \int_0^a \left[J_0\left[\chi\frac{x}{a}\right]\right]^2 x\, dx + D_n^2 \int_a^\infty \left[H_0^{(1)}\left[\phi_n\frac{x}{a}\right]\right]^2 x\, dx \\ &= \frac{a^2}{2}C_n^2(J_0'^2+J_0^2) - \frac{a^2}{2}D_n^2(H_0'^2+H_0^2) \\ &= \frac{a^2}{2}C_n^2\left[J_0'^2+J_0^2-\frac{J_0^2}{H_0^2}(H_0'^2+H_0^2)\right] \\ &= \frac{a^2}{2}C_n^2 J_0'^2\left[1-\left[\frac{J_0}{J_0'}\right]^2\left[\frac{H_0'}{H_0}\right]^2\right] \\ &= \frac{a^2}{2}C_n^2 J_0^2 \xi_n^2\left[\frac{1}{\chi_n^2}-\frac{1}{\phi_n^2}\right] . \end{aligned}$$

(2) Overlap integrals:

$$\int_0^a R_n R_l^* x\, dx = a^2 R_n(a) R_l^*(a)\frac{\xi_l^* - \xi_n}{\chi_n^2 - \chi_l^{*2}} , \quad \text{(C2)}$$

$$\int_a^\infty R_n R_l^* x\, dx = -a^2 R_n(a) R_l^*(a)\frac{\xi_l^* - \xi_n}{\phi_n^2 - \phi_l^{*2}} , \quad \text{(C3)}$$

$$\int_0^\infty R_n R_l^* x\, dx = a^2 R_n(a) R_l^*(a)(\xi_l^* - \xi_n)\left[\frac{1}{\chi_n^2-\chi_l^{*2}} - \frac{1}{\phi_n^2-\phi_l^{*2}}\right] . \quad \text{(C4)}$$

Equation (C2) is derived from

$$\int_0^a R_l^* \left[\frac{1}{x}\frac{\partial}{\partial x}\left[x\frac{\partial R_n}{\partial x}\right]+\frac{\chi_n^2}{a^2}R_n\right]x\,dx=0\ ,$$

$$\int_0^a R_n \left[\frac{1}{x}\frac{\partial}{\partial x}\left[x\frac{\partial R_l^*}{\partial x}\right]+\frac{\chi_l^{*2}}{a^2}R_l^*\right]x\,dx=0\ ,$$

from which one easily sees that

$$\frac{1}{a^2}(\chi_n^2-\chi_l^{*2})\int_0^a R_nR_l^*x\,dx = aR_nR_l^*\left[\frac{R_l^{*\prime}}{R_l^*}-\frac{R_n'}{R_n}\right]\Bigg|_{x=a}\ ,$$

which becomes the desired result once one notes that

$$\frac{R_n'}{R_n}\Bigg|_{x=a}=\frac{\xi_n}{a}\ .$$

Equation (C3) is derived in an analogous manner, and (C4) follows immediately as the sum of (C2) and (C3).

(3) $N_{nl}=\int_0^\infty R_nR_l^*2\pi x\,dx \int_0^a R_nR_l^*2\pi x\,dx$ [Eq. (6.26)]. The wave function is normalized by

$$\int_0^\infty R_n^2 2\pi x\,dx=1\ ,$$

so using Eq. (C1) we find

$$\pi a^2R_n^2(a)\xi_n^2\left[\frac{1}{\chi_n^2}-\frac{1}{\phi_n^2}\right]=1\ . \tag{C5}$$

Now N_{nl} can be evaluated from Eqs. (C2), (C4), and (C5),

$$N_{nl}=\frac{4(\xi_l^*-\xi_n)^2}{\xi_l^{*2}\xi_n^2}\frac{(\phi_n\phi_l^*)^2(\chi_n\chi_l^*)^2}{\tilde{a}^2(\chi_n^2-\chi_l^{*2})^2(\phi_n^2-\phi_l^{*2})}(\lambda_l^{*2}-\lambda_n^2)\ , \tag{C6}$$

where we have used $\phi_n^2-\chi_n^2\simeq\tilde{a}^2/\lambda_n^2$.

APPENDIX D: EVALUATION OF $\partial\Lambda(\Omega,q_{\parallel})/\partial\Omega$

We consider the eigenvalue problem of Eq. (4.4):

$$[\nabla_\perp^2+\Lambda+Vu(x)]\Phi(\mathbf{x})=0\ , \tag{D1}$$

$$V=\frac{(2\rho)^3}{\Omega^2}(\Omega-q_{\parallel}-1)\ . \tag{D2}$$

In the special case when $u(x)$ is the step function profile of Eq. (6.4).

$$\Phi=R(x)e^{im\theta}\ , \tag{D3}$$

$$R(x)=\begin{cases} J_m\left[\chi\dfrac{x}{a}\right], & x\le a \\ H_m^{(1)}\left[\phi\dfrac{x}{a}\right], & x\ge a\ . \end{cases} \tag{D4}$$

The eigenvalue Λ is determined by

$$\Lambda=\frac{\phi^2}{a^2}\ , \tag{D5}$$

$$\phi^2-\chi^2=-a^2V\ , \tag{D6}$$

$$\chi\frac{J_m'(\chi)}{J_m(\chi)}=\phi\frac{H_m^{(1)\prime}(\phi)}{H_m^{(1)}(\phi)}\equiv\xi_m\ . \tag{D7}$$

Differentiating Eqs. (D6) and (D7) yields

$$d\chi=\frac{\phi}{\chi}d\phi+\frac{a^2}{2\chi}dV\ , \tag{D8}$$

$$(J'H+\chi J''H-\phi H'J')d\chi=(H'J+\phi H''J-\chi J'H')d\phi\ , \tag{D9}$$

where we use the shorthand notation $J=J_m(\chi)$ and $H=H_m^{(1)}(\phi)$. Using Eqs. (D8) and (D9) together with (D4), we derive

$$\frac{d\Lambda}{dV}=\frac{2\phi}{a^2}\frac{d\phi}{dV}=\frac{1}{\dfrac{\chi(H'J+\phi H''J-\chi J'H')}{\phi(J'H+\chi J''H-\phi J'H')}-1}\ . \tag{D10}$$

From Bessel's equation, it follows that

$$\phi H''+H'=-\phi\left[1-\frac{m^2}{\phi^2}\right]H\ ,$$

$$\chi J''+J'=-\chi\left[1-\frac{m^2}{\chi^2}\right]J\ ,$$

so Eq. (D10) can be written in the simpler form

$$\frac{d\Lambda}{dV}=\frac{\phi^2}{\chi^2-\phi^2}\left[1+\frac{\chi^2}{\xi^2-m^2}\right]\ . \tag{D11}$$

We now specialize to the axially symmetric modes ($m=0$), for which

$$\frac{d\Lambda}{dV}=\frac{\phi^2}{\chi^2-\phi^2}\left[1+\frac{\chi^2}{\xi^2}\right]\ . \tag{D12}$$

Differentiating Eq. (D2), we find

$$\frac{\partial V}{\partial\Omega}=(2\rho)^3\frac{2(1+q_{\parallel})-\Omega}{\Omega^3}=\frac{2+4\rho\Delta-2\rho\lambda}{\lambda^3}\ , \tag{D13}$$

where $q_{\parallel}=2\rho\Delta$ and $\Omega=2\rho\lambda$. We suppose $|2\rho\Delta|\ll1$ and $|2\rho\lambda|\ll1$, so

$$\frac{\partial V}{\partial\Omega}\simeq\frac{2}{\lambda^3}\ . \tag{D14}$$

Combining Eqs. (D12) and (D14), it follows that

$$\frac{\partial\Lambda}{\partial\Omega}=\frac{\partial\Lambda}{\partial V}\frac{\partial V}{\partial\Omega}=\frac{\phi^2}{\chi^2-\phi^2}\left[1+\frac{\chi^2}{\xi^2}\right]\frac{2}{\lambda^3}\ . \tag{D15}$$

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