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Statistics and dimension of chaos in differential delay systems

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The chaotic solution of dissipative scalar-delay-differential equations with a nonlinear feedback periodic with respect to its argument is shown to behave as a Gaussian-Markovian process in a large time scale. The short time scale is shown to be defined by the correlation time of the delayed feedback. The dimension of the chaotic attractor is shown to be approximately equal to the number of short times that are contained inside the delay.

I. INTRODUCTION

Systems driven by delayed nonlinear feedback occur in various domains: optics,[1] economy, species competition, biology, etc.[2] Here we will consider scalar dissipative systems that obey an equation of the form

$$\frac{dx}{dt}+x=kf(x(t-\tau_R)) , \tag{1}$$

where kf is the nonlinear delayed feedback with delay time τ_R and strength k. In Eq. (1) the time is scaled to the decay time of the system. Such equations describe infinite-dimensional systems in the phase space in the sense that the initial conditions must be specified on the whole interval $(0,\tau_R)$,

The chaotic regime of these deterministic systems has been studied by Farmer[3] in the case of the model of blood production due to Mackey and Glass and by Le Berre *et al.*[4] in a ring cavity optical system. These investigations were performed by using the ergodic parameters, i.e., Lyapunov exponents, entropy, and fractal dimension.[5] It has been found that the dimension of the attractors is finite and increases linearly with the delay.[3,4] Nevertheless, the temporal behavior of the solution is reminiscent of random noise, which suggested to us to use the tools of the classical theory of random signals to study this deterministic chaos.

The time scales which characterize the dynamics of $x(t)$ can indeed be deduced from the correlation functions of both $x(t)$ and $f(x(t))$. Intuitively the number of times the smallest characteristic time is contained in the delay τ_R corresponds to the "effective" number of degrees of freedom, which gives an estimate of the dimension of the attractor. We show in this paper that this conjecture is verified in a case of periodic feedback. Therefore it seems possible to link the deterministic approach with the statistical one via the attractor dimension.

The chaotic solutions of Eq. (1) are investigated for large k with two periodic feedbacks; one was chosen for pedagogical reasons and the other corresponds to an optical device. In both cases the solution was shown to display a Gaussian character for large k, and we believe that this property should be generic for any chaotic solutions of Eq. (1) with periodic feedback, and even with functions $f(x)$ oscillating over a wide range in x.

For large k, the feedback $kf(x(t))$ oscillates so fast that it behaves as a driving random force. Then on a time scale larger than the oscillation time of $f(x(t))$, $x(t)$ displays a Gaussian-Markovian behavior as if it were a solution of a Langevin equation. This result was already derived by Ikeda and Akimoto[6] from numerical computations in the case of the optical device. The order of magnitude of this oscillation time, δ, corresponds to the correlation time of $f(x(t))$ and characterizes the memory of the driving force. It is shown to be proportional to k^{-1} and is the smallest time scale in the system. Of course, this characteristic time appears in the temporal evolution of $x(t)$ which exhibits small oscillations of mean duration δ. Surprisingly it does not appear in the correlation function $\Gamma_x(t)$ of $x(t)$. This function Γ_x exhibits two regimes: It is quadratic near the origin on a time scale $t_c \approx k^{-1/2}$, and further decays exponentially as $e^{-|t|}$. This exponential decay for $t \geq t_c$ is the signature of a Markovian process. The time scale t_c corresponds to the mean duration of large fluctuations of $x(t)$. Therefore the correlation function of $x(t)$ alone gives rise to an incomplete description of the system and is sensitive to the large fluctuations of the temporal evolution only.

Another important result is the disappearance of the correlation of $x(t)$ with $x(t+\tau_R)$ in the limit of large k. For smaller k, when the process is no longer Gaussian on any time scale, the coupling between $x(t)$ and $x(t+\tau_R)$ appears in $\Gamma_x(t)$ via successive bumps located at $\tau_R, 2\tau_R, \ldots$.

All these results are derived analytically and supported by numerical simulations for the two following delayed feedbacks. The first one (A) is

$$f(x)=\mathrm{Dec}(x) , \tag{2a}$$

where Dec means the decimal part of x. This function is appropriate to illustrate the properties of the chaotic solutions which lead to a Gaussian probability distribution for large k. The second case (B) is

$$f(x)=1-R\sin(x) , \tag{2b}$$

where R is a parameter. In this case $x(t)$ is proportional to the transmitted intensity through the optical hybrid device in which chaos was observed for the first time in optical bistability systems.[7] Analytical calculations are more easily performed with the sine function. Moreover, the latter function is differentiable allowing one to compute the Lyapunov exponents by standard methods. An estimate of the dimension d of the attractor of the chaotic motion may be obtained from the Lyapunov exponents spectrum[8]

$$d=j+\sum_{i=1}^{j}\lambda_i(|\lambda_{j+1}|)^{-1} , \tag{3}$$

where j is the largest integer for which $\lambda_1+\cdots+\lambda_j\geq 0$.

We have chosen to work with the Lyapunov dimension because with the other methods already used to calculate fractals dimensions, the computation time becomes prohibitive for high dimensions.[9] Moreover, d is approximately of the order of twice the number of positive or zero Lyapunov exponents; this number is obviously an important characteristic of a chaotic solution. This number can be obtained easily and accurately even if the convergence of the Lyapunov exponents is difficult to reach.

Using the sine function, d is found to have a double interpretation. It is first approximately equal to the number of times the fluctuation time δ, exhibited by the statistical study, is contained in the delay. Another formulation is also given: d is shown to be proportional to the variance of $x(t)$, or else to the energy of the fluctuations during a time delay. This result could be more accessible from an experimental point of view.

Let us make a last remark. The adiabatic approximation of Eq. (1),

$$x_n=f(x_{n-1}) , \tag{4}$$

which is often considered valid for large τ_R,[10] will be shown to obey very different probability distributions. Therefore the processes described by the differential equation and by the difference equation [Eq. (4)], not only differ by the nature of the instabilities, the number of dimension, as already pointed out,[4] but also by their statistical properties.

The paper begins with the study of the probability distributions in the chaotic regime for both functions A and B (Sec. II). In this section are also calculated the probability distributions for the difference equation solutions. The behavior of the correlation functions of $x(t)$ and $f(x(t))$ are studied in Sec. III. Finally, the dependence of the dimension upon the parameters is described in Sec. IV.

II. PROBABILITY DISTRIBUTION

In this section we focus on the one-dimensional variable $x(t)$ which will be considered as a sample of an ergodic process. It is assumed that for any differentiable function x, the time average $\langle x\rangle=\lim[(1/T)\int_0^T dt\,X(t)]$ converges. Then it defines a probability measure invariant under time evolution and ergodic, which will be used to study the probability distribution of the variable x. Let us notice that any average for deterministic chaos is defined by the specific dynamic of the system itself, in contrast to random signals where chaos is created by an external noise.

In the first part of this section we consider the nonlinear function $f(x)=\mathrm{Dec}(x)$, because it evolves with discontinuities; for large k the function $f(x(t))$ jumps very often and practically loses its memory at each jump. This memory loss is the key of the dynamical behavior of $x(t)$, and allows a statistical investigation. We obtain qualitative expressions for all the quantities which are required to characterize the statistical properties of the chaotic solution (variance, memory time of $\mathrm{Dec}[x(t)]$, . . . ,). In Sec. II B the dynamic behavior of $x(t)$ is analyzed when driven by a sine function. The properties which make possible the loss of memory are displayed. Finally, in Sec. II C the probability distributions of the difference equations are investigated for the two cases of functions A and B.

A. Statistics of x for the Dec function

The time-independent solutions of Eqs. 1—4 with the Dec function are $x_s=n/(k-1)$, where n is an integer. They lose their linear stability when $k\geq 1$ and

$$\tau_R=(k^2-1)^{-1/2}\arccos(1/k) \tag{5}$$

so, for k larger than a few units, the condition

$$\tau_R\gg\pi/2k \tag{6}$$

ensures that the solution exhibits a fully developed chaos and will be supposed to be always realized.

For t much larger than 1 and τ_R, the solution of Eq. (1) can be written as

$$y(t+\tau_R)=\int_0^{t+\tau_R}du\,e^{-u}\mathrm{Dec}[ky(t-u)] . \tag{7}$$

We would like to justify the large-τ_R behavior of solutions of Eq. (7): as the delay τ_R increases and if k is large enough, the solution of Eq. (7) tends to a well-defined random function with properties independent of

τ_R for generic initial data. Putting any function y into the argument of the nonlinear integral transform on the right-hand side of Eq. (7), one gets a new function y:

$$\tilde{\tilde{y}}(t)=\int_0^t du\, e^{-u}f(k\tilde{y}(t-u)) . \tag{7'}$$

Equation (7) imposes that $\tilde{\tilde{y}}(t)$ and $\tilde{y}(t+\tau_R)$ are the same function y, i.e., that they take the same value at the same time t. But as τ_R increases, this condition relates practically values of $y(t)$ at more and more distant times. Accordingly, if $y(t)$ has a random behavior, this constraint that $\tilde{y}(t)$ and $\tilde{\tilde{y}}(t)$ represent exactly the *same* function at different times becomes less and less stringent because there is less and less correlation between values of y at more and more distant times. There is also a less detailed condition imposed by Eq. (7), and Eq. (7′), i.e., that $\tilde{y}$ and $\tilde{\tilde{y}}$ represents the *same* stationary random process: in other terms, any correlation function computed with $\tilde{y}$ and $\tilde{\tilde{y}}$ should be the same, which may be mathematically translated as a condition that Eq. (7′) should define a fixed point in the space of stationary random functions. It is conceivable, however, that this fixed point is not unique, depending (among other things) on the initial data. We hope to come back to this point in future publications.

For $k\to\infty$, the functional map $y(t)\to\mathrm{Dec}[ky(t)]$ transforms a smooth positive function into a function with very fast variations between 0 and 1: the succession of "sawteeth" in Fig. (1b) illustrates this behavior of $\mathrm{Dec}[ky(t)]$ and the dashed area in Fig. (1c) of the sawteeth weighted by e^{-u} represents $y(t+\tau_R)$. Because of these fast and steep oscillations the values of $\mathrm{Dec}[ky(t)]$ are correlated almost only on a time interval of the order of the time required for $y(t)$ to change by an amount $\approx 1/k$. The mean value δ of this correlation time can be estimated $\approx 2/k$ since the area of the first sawtooth contributes to $y(t)$ by an amount $\delta/2$ that corresponds also to the difference $[y(t)-y(t-\delta)]$. More precisely let the time domain be divided in intervals $\delta_n=[t_n,t_{n+1}]$ corresponding to the widths of the sawteeth. For large k, δ_n are much smaller than unity and Eq. (7) can be written as a sum of a large number of almost independent contributions

$$y(t+\tau_R)\approx\sum_{n=0}^{\infty}e^{-t_n}Y_n(t) \tag{8}$$

with

$$Y_n(t)\approx\int_{t_n}^{t_{n+1}}du\,\mathrm{Dec}[ky(t-u)] . \tag{9}$$

Assuming the independence of the Y_n and equality of the δ_n (hypothesis H1), a slight modification of the proof of the central-limit theorem[11] shows that for large k the process is Gaussian. This modification is required because the contributions in Eq. (6) are weighted by the exponential and do not have the same variance. It is presented in Appendix A where it is also shown that both δ and σ_y^2, the variance of $y(t)$, are proportional to k^{-1},

$$\sigma_y^2=\alpha/k \tag{10}$$

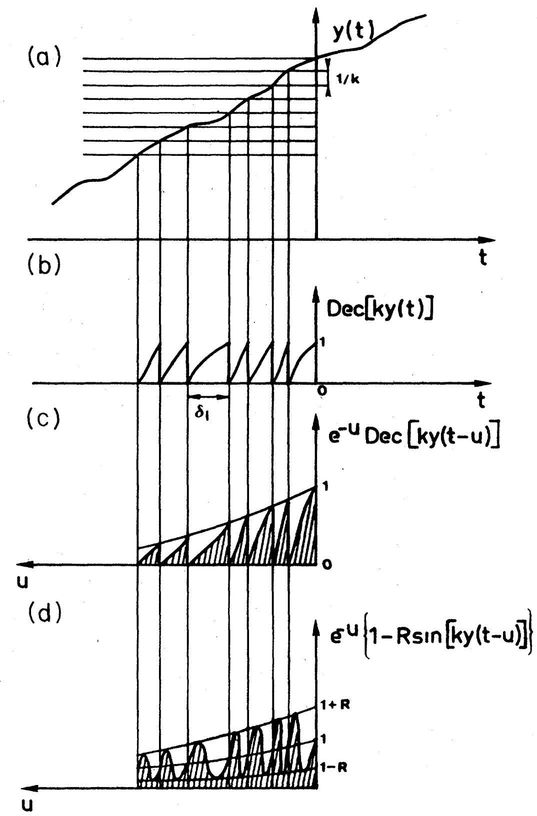

FIG. 1. The integral solution of Eq.(1) can be understood with this diagram. (a)—(c) illustrate Eq. (7). In (a) $y(t)$ is drawn with increments equal to $1/k$. (b) shows the corresponding evolution of $\mathrm{Dec}[ky(t)]$ which jumps from 1 to 0 at times t_i defined in (a). Finally, the integrand in Eq. (7) is the function drawn in (c), so the dashed area is equal to $y(t+\tau_R)$. With similar arguments, the dashed area in (d) represents $y(t+\tau_R)$ defined by Eq. (17).

and

$$\delta=\beta/k . \tag{11}$$

The magnitude of α and β are obtained via integrals involving the correlation function and the probability distribution of $\mathrm{Dec}(ky)$. If these functions are supposed to be constant, respectively, on $(0,\delta)$ and $(0,1)$ (hypotheses H2 and H3), we obtain $\alpha_{\mathrm{th}}\approx 0.07$ and $\beta_{\mathrm{th}}\approx 1.7$. Equation (11) thus confirms the heuristic argument given above to estimate δ.

These conclusions are checked by numerical simulations. In Figs. 2, the probability distribution of y evolves from non-Gaussian shapes for small value of k [Figs. 2(a) and 2(b)] to a Gaussian for $k\geq 5$ [Figs. 2(c) and 2(d)]. The quantity $R^2\sigma_y^2$, variance of the process $x(t)=ky(t)$, is plotted in Fig. 3 as a function of k. For $5\leq k\leq 20$, σ_y^2 is found to follow the relation

$$(\sigma_y^2)_{\mathrm{num}}=\frac{0.042}{k}\left[1+\frac{1}{2k}\right] \tag{12}$$

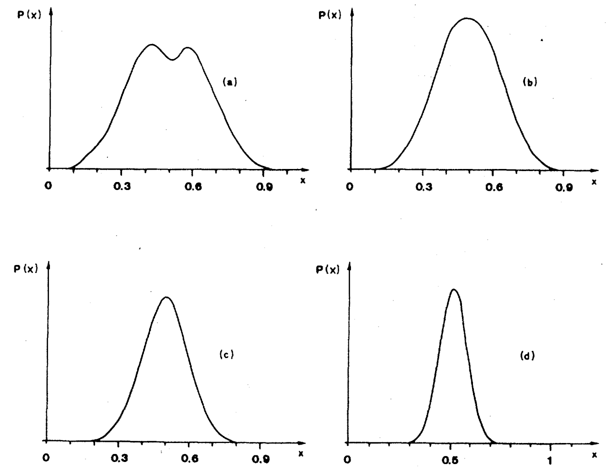

FIG. 2. Probability distributions for the solution y of Eq. (7), with (a) $k=2$, $\tau_R=10$; (b) $k=3$, $\tau_R=10$; (c) $k=5$, $\tau_R=10$; (d) $k=10$, $\tau_R=1$.

with the limit α/k for large k with $\alpha=0.042$, in agreement with Eq. (10). In Fig. 4(a) the inverse width of the correlation function $\Gamma_f(u)=\langle \mathrm{Dec}[ky(u)]\mathrm{Dec}[ky(0)]\rangle$ is plotted, as a function of k. The numerical width δ [defined as $\Gamma_f(\delta)=1/e$], tends asymptotically to

$$(\delta)_{\mathrm{num}} \to 0.6/k \ . \tag{13}$$

The numerical results in Figs. 3 and 4 confirm the prediction that the variance [Eq. (10)] and the decay time [Eq. (11)] are proportional to $1/k$. The numerical coefficients α and β are both noticeably smaller than the predicted ones. We attribute this discrepancy to the roughness of hypotheses H1 and H2: Fig. 4(b) shows that $\Gamma_f(u)$ is not constant for $0 \le u \le \delta$. Hypothesis H3 is more realistic: for the $k \ge 5$, the probability distribution of Dec(ky) is rather uniform between 0 and 1.

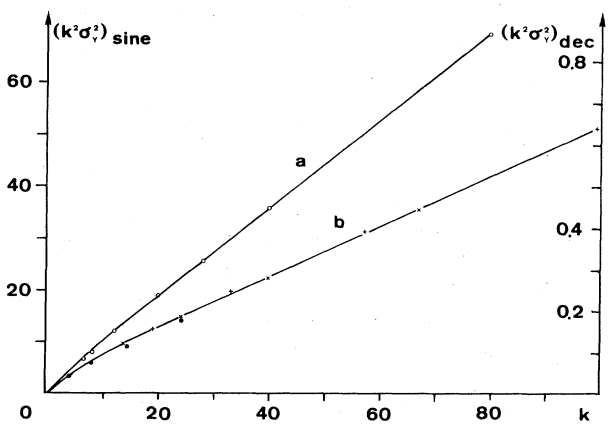

FIG. 3. Variance of $X=kY$ as a function of the parameters: curve a, $k^2\sigma_y^2$ with respect to k for the Dec function; (b) $k^2\sigma_y^2$ with respect to kR for the sine-function feedback. The three distinct notations correspond to $R=0.95$ (plus), $R=0.667$ (cross), and $R=0.4$ (dot).

Let us now predict the values of k for which the probability distribution is Gaussian. The solution $y(t)$ is a sum of an infinite number of contributions $e^{-t_n}Y_n$ but only a finite number n_{eff} of them contribute to the sum in Eq. (8) because of the exponential. The sum can be truncated at n_{eff} such that

$$t_n=\delta n_{\mathrm{eff}} \approx 3 \ , \tag{14}$$

corresponding to $e^{-3}=0.05$. Taking into account Eq. (13), one can consider $y(t)$ as a sum of

$$n_{\mathrm{eff}} \approx 5k \tag{15}$$

independent contributions. The resulting probability distribution is known to be the Student law with $(n_{\mathrm{eff}}-1)$ degrees of freedom. For $n_{\mathrm{eff}} \approx 25$ or $k \ge 5$, the Student distribution is very close to a Gaussian. This is in agreement with the distributions shown in Figs. 2, where $y(t)$ is Gaussian for

$$k \ge 5 \ . \tag{16}$$

What happens for smaller values of k? The characteristic function of y, $\varphi_y(v)$, defined in Appendix A [Eq. (A1)], contains first a quadratic term $\frac{1}{2}v^2\sigma_y^2$, and a corrective term of order $\delta v^4\sigma_y^4$ (cf. Appendix A), which affects the Gaussian function $\varphi_y(v)$ for large values of the argument v. Then the Fourier transform of $\varphi_v(v)$, which is the probability distribution for y, will be disturbed in the center of the Gaussian when k decreases from 5, as shown in Figs. 2(a) and 2(b).

In conclusion, the time evolution of $y(t)$ [the solution of Eq. (7)] is well understood in the light of the expansion in Eqs. (8) and (9), illustrated by Fig 1(c), which displays naturally the Gaussian character of the process. It follows that any differential equation of type (1) with a periodic (or simply oscillating) feedback $f(x)$ would also have a Gaussian solution for large k. This conjecture will be verified with the smooth sine-function feedback [Eq. (2b)] in Sec. II B. On the contrary, it will be proved in Sec. II C that the statistics of the difference equation solutions associated with both Dec and sine functions do not exhibit any Gaussian character. Therefore, the combination of the differential term and the oscillating properties of $f(ky)$ for large k seems to be required for delayed scalar systems in order to have Gaussian solutions.

B. Sine function

Let us consider the solution of Eq. (1) with the sine function (B):

$$y(t+\tau_R)=\int_0^{t+\tau_R} du\, e^{-u}\{1-R\sin[ky(t-u)]\} \ , \tag{17}$$

where $y(t)$ is related to $x(t)$ in Eq. (11) by the scaling $y=x/k$. The condition for $y(t)$ to be in the fully developed chaotic regime is derived in Appendix C. For k much larger than 2π, it is

$$\tau_R \gg \pi^2/4kR \ .$$

In this chaotic regime $\sin(ky)$ oscillates much more quickly than y for large k as in the case of the Dec function. Then the previous analysis may be transposed to the sine

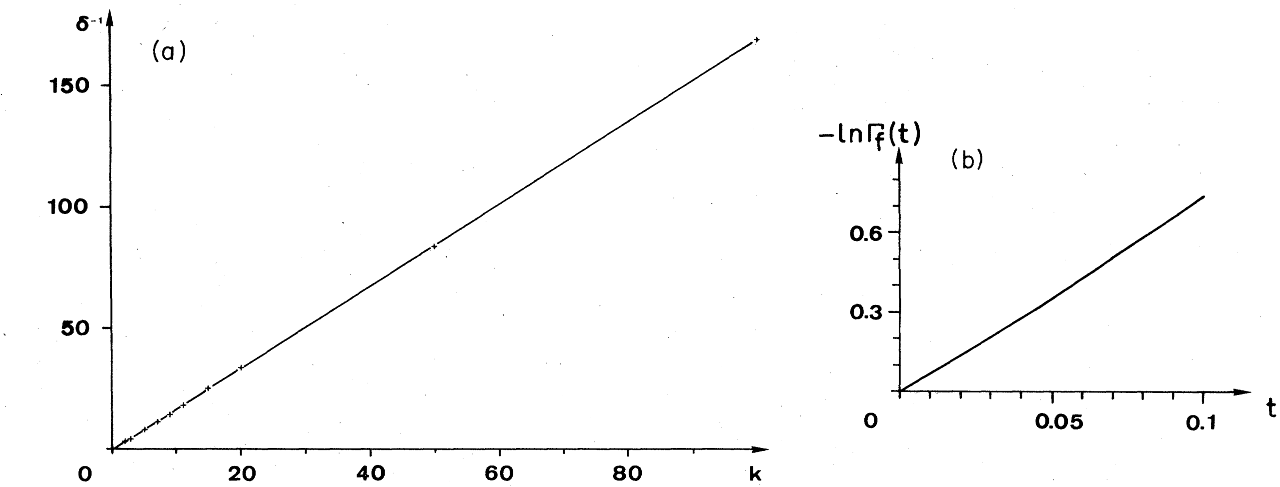

FIG. 4. Correlation time of the feedback (Dec function). The correlation function in (b) is exponentially decreasing, with a characteristic decay time δ. δ^{-1} is reported in (a) with respect to k.

function as shown in Fig. 1(c). When $y(t)$ increases until $y(t)+2\pi/k$, the function $1-R\sin[ky(t)]$ oscillates once between $1-R$ and $1+R$. It loses its memory on a time scale δ which is here the time interval for such an oscillation. With the same hypotheses as H2 and H3 above, δ is found to be proportional to $1/kR$. For this coarse estimation, the constant of proportionality is $\approx 2\pi\sqrt{2}$ (Appendix A). This qualitative analysis shows that the solution of Eq. (1) with the sine function *tends also to be Gaussian for large* k. This is confirmed by numerical simulations, as shown in Fig 5. A precise description of the statistical properties of $y(t)$ for function B will be investigated in the next section.

C. Difference equation solutions distributions

We now derive the probability distributions for the solution $\{y_n\}$ of the difference equations relative to functions (A) and (B) [Eqs. 2(a) and 2(b)]. Let $P(y)$ be the probability distribution of $\{y_n\}$. The mapping $y_{n+1}=f(y_n)$ implies the same probability distribution of the set $\{y_{n+1}\}$ and therefore the self-consistent relation[11]

$$P(y)=\sum_{\{z_i=f^{-1}(y)\}}\frac{1}{|f'(z_i)|}P(z_i)\ . \tag{18}$$

For function A, $f'=k$, Eq. (18) leads to the constant density

$$P^A(y)=1 \quad \text{for } 0\le y\le 1\ , \tag{19}$$

in agreement with Fig 6(a) obtained from 4×10^5 iterations of the difference equation with function A, listed in 100 bins.

For function B, the relation (18) becomes

$$P^B(y)=\frac{1}{R}\left[1-\left[\frac{y-1}{R}\right]^2\right]^{-1/2}\sum_{\{z_i=f^{-1}(y)\}}P^B(z_i)\ . \tag{20}$$

This function has two critical points $y_1^{\pm}=1\pm R$ $[f'(y_1^{\pm})=0]$. Assuming that the iterates of the critical points are not asymptotically periodic, $P^{(B)}(y)$ can be derived by iteration of Eq. (18). It is supposed that $P^B(z_i)$ is sufficiently smooth so that $\sum_{\{z_i\}}P^B(z_i)\approx 1/\pi$, and one gets

$$P_1^B(y)=\left[1-\left[\frac{y-1}{R}\right]^2\right]^{-1/2}\frac{1}{\pi R}\ . \tag{21}$$

In the vicinity of $y_1^{\pm}$, $P_1^B\approx a_1(y-y_1^{\pm})^{-1/2}$ which is integrable, and induces two new singularities at $y_2^{\pm}=f(y_1^{\pm})$ of amplitude a_2. It can be shown that the amplitudes a_j of the successive iterates of $y_1^{\pm}$ fall off roughly at the rate $\exp(-j\lambda/2)$ where λ is the Lyapunov exponent of the

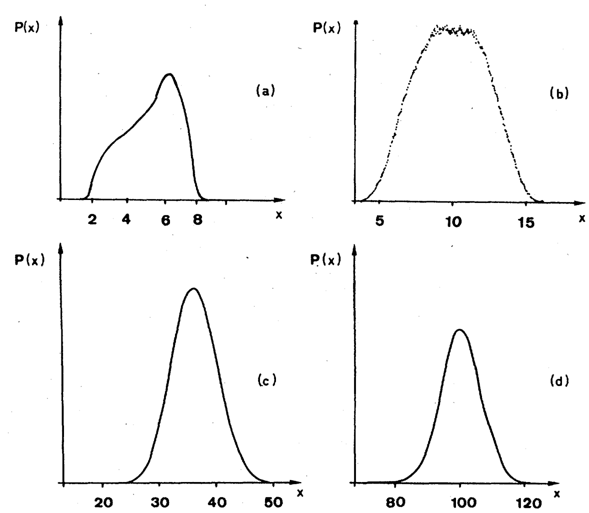

FIG. 5. Probability distributions for the solution X of Eq. (1) with sine feedback, $R=\frac{2}{3}$. (a) $k=5$, $\tau_R=10$; (b) $k=10$, $\tau_R=10$; (c) $k=36$, $\tau_R=3$; (d) $k=100$, $\tau_R=5$.

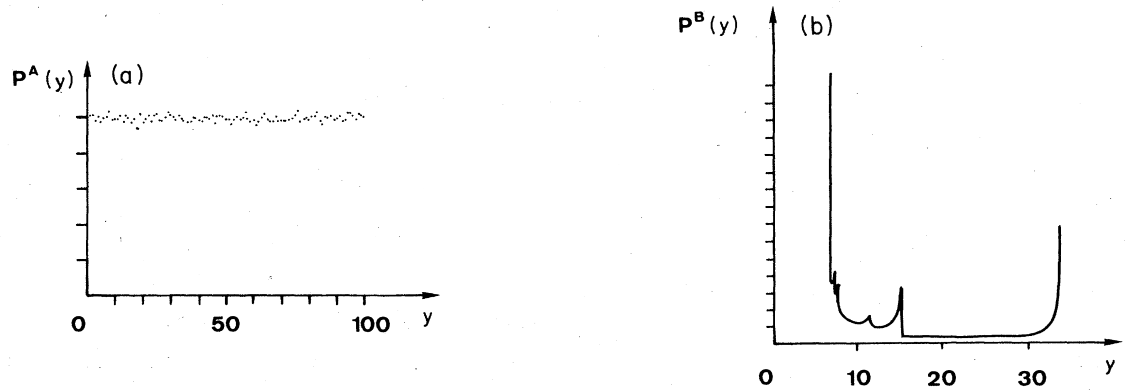

FIG. 6. Probability distribution for the solutions $\{x_n\}$ of Eq. (4). (a) With Dec function, $k=41\sqrt{3}$. The distribution is obtained from 4×10^5 iterations and 10^2 bins. (b) For the sine function, $k=20$, $R=\frac{2}{3}$, the distribution results from 6×10^4 iterations and resolving the result into 400 bins.

map, as it was already shown in the case of the logistic map.[12]

At a coarse level of resolution, the distribution behaves like a U curve, $P(y)\approx[1-(y-1)^2/R^2]^{-1/2}$, with few secondary peaks. This is confirmed by the simulations in Fig. 6(b) obtained from 6×10^4 iterations and resolving the result into 400 bins (see Ref. 12 for a discussion about the continuity of the asymptotic density). In conclusion, the probability distribution of the $\{x_n\}$, successive iterates of the difference equation Eq. (4), is radically different from the distribution of x, solution of Eq. (1), in both the function A and B cases.

III. CORRELATION FUNCTIONS AND TWO-DIMENSIONAL STATISTICS

The probability distribution for the sine case was shown to behave as a Gaussian for large k. By using the expansion of Eq. (8), the increment $y(t_1)-y(t_2)$ is shown to be Gaussian only for $t_1-t_2\gg\delta$. Therefore one can define *large* time scales, i.e., large compared to δ, and *small* time scales. The behavior of the correlation function for large time scales is studied in Sec. III A in the sine case. Some information can be obtained on the short-time dynamics because the deterministic nature of the chaotic solution allows us to derive the MacLaurin expansion of the correlation functions of both $x(t)$ and $f(x(t))$. Two "short" time scales are thus displayed, which correspond to small and large fluctuation times in the dynamical behavior of $y(t)$ (Sec. III B). Finally, the correlation of $y(t)$ and $y(t+\tau_R)$ is investigated in Sec. III C and compared with the correlation of the difference equation solutions.

A. Gaussian process on time scale much larger than δ

Let us first examine the statistics of the increments

$$\Delta y(\theta)=y(t+\tau_R)-y(t+\tau_R-\theta) \tag{22}$$

when $y(t)$ is Gaussian. We show in Appendix B that $\Delta y(\theta)$ is not Gaussian for θ of order δ, but tends to be Gaussian for large θ, i.e., $\theta\geq25\delta$. This result was already derived from numerical computations of the second- and fourth-order correlation function $\langle[y(t_1)]^n[y(t_2)]^m\rangle$ for the sine case by Ikeda and Akimoto.[6] Therefore, on a time scale larger than 25δ, the process $y(t)$ will be completely described by the first-order correlation function

$$\Gamma_y(\theta)=\langle y(t)y(t+\theta)\rangle\ , \tag{23}$$

which will now be investigated.

In this section, we shall mostly be concerned with the sine feedback (B) for which analytical calculations are tractable. Qualitative relations were given in Sec. II for σ_y^2 and δ. A more precise analysis would require the knowledge of the correlation functions $\langle f(y(t_1))f(y(t_2))\rangle$, which have been roughly described in Sec. II. The difficulty in such deterministic chaos is that an investigation of one-time statistics requires the knowledge of the two-time statistics. In principle the problem could be completely solved if $y(t)$ were a Gaussian process. In the present case $y(t)$ is *not* Gaussian on the short time scale δ that complicates the problem greatly. This difficulty can, however, be overcome. We first derive the autocorrelation of $y(t)$ for θ smaller than τ_R. Let us define the scaled correlation

$$\sigma_y^2C_y(\theta)=\Gamma_y(\theta)-\langle y\rangle^2\ . \tag{24}$$

From Eq. (17) we have

$$\sigma_y^2C_y(\theta)=\int_0^\infty\int_0^\infty e^{-u-u'}\Gamma_f(u'-u'-\theta)du\,du' \tag{25}$$

with

$$\Gamma_f(u)=R^2\{\langle\sin[ky(0)]\sin[ky(u)]\rangle-\langle\sin(ky)\rangle^2\}\ . \tag{26}$$

It will be shown that the above correlation functions are independent of the delay τ_k for large k.

For a Gaussian process with variance σ_y^2 all the mean values can be derived (see Appendix C). We obtain successively

$$\langle y\rangle=1-R\langle\sin(ky)\rangle\ , \tag{27}$$

or equivalently

$$\langle y\rangle=1-Re^{-k^2\sigma_y^2/2}\sin(k\langle y\rangle)\rightarrow1\quad\text{as } k\gg1\ , \tag{28}$$

and

$$\Gamma_f(u)=(R^2/2)e^{-k^2\sigma_y^2}[(e^{k^2\sigma_y^2C_y(u)}-1)-\cos(2k\langle y\rangle)(e^{-k^2\sigma_y^2C_y(u)}-1)] . \quad (29)$$

Equation (29) is either valid for $u=0$ [one-dimensional (1D) variable], or $u\geq 25\delta$.

Equation (29) shows that $\sin(ky)$ has a correlation time much smaller than y, as expected from the qualitative analysis of Sec. II. If $\Gamma_f(u)$ is approximated by

$$\Gamma_f(u)\simeq\eta\delta(u) , \quad (30)$$

where $\delta(x)$ is the Dirac distribution, Eq. (25) becomes

$$\sigma_y^2C_y(\theta)=(\eta/2)e^{-|\theta|} \quad (31)$$

and exhibits the Gaussian Markovian character of the process for $\delta \ll \theta \leq \tau_R$ and for large kR. Figure 7 exhibits the (numerical) curves of $\Gamma_y(\theta)$ both for Gaussian and non-Gaussian cases. The exponential behavior appears clearly for $\theta\geq 1$, i.e., the dissipation time.

To go further we need the value of the variance. However, the exact derivation of σ_y^2 is very difficult, because it implies knowledge of $C_f(u)$ on a range of order δ. We propose three ways to bypass this difficulty. They anticipate results which will be obtained further and are given in Appendix C.

The first method uses the expansion of $\Gamma_f(u)$ valid for small u, and it leads to

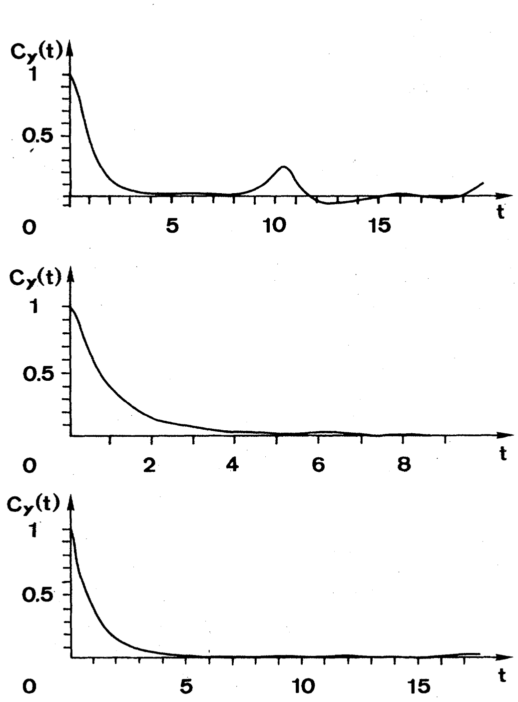

FIG. 7. Normalized correlation function of $y(t)$ [Eq. (24)] for the sine feedback $R=\frac{2}{3}$. (a) $k=10$, $\tau_R=10$; (b) $k=20$, $\tau_R=5$; (c) $k=36$, $\tau_R=3$.

$$(\sigma_y^2)_{\text{th}}=0.67R/k . \quad (32)$$

The two other methods start from Eq. (29) with two distinct expressions for $C_y(\theta)$. They give, respectively, $\sigma_y^2=0.7R/k$ and $\sigma_y^2=0.9R/k$.

The numerical variance is shown in Fig. 3(b). It is found to obey the *following* law

$$(\sigma_y^2)_{\text{num}}=0.5R/k+3/k^2 \quad (33)$$

in the whole range of Gaussian statistics. In Fig. 3(b), the variance of $x=ky$ is plotted as a function of kR for $5\leq k\leq 100$ and three values of R.

In conclusion, the numerical results confirm the qualitative predictions in Eq. (10). Moreover, they agree with the theoretical variance in Eq. (32) with a discrepancy of 30%. A discussion on the different methods used to calculate σ_y^2 is given in Appendix C. The above derivation is a typical example of the interconnection of 1D and 2D statistics in problem of deterministic chaos, since to calculate the variance, the expansion of $\Gamma_y(u)$ is needed. Nevertheless, for clarity this derivation has been given in Sec. III C.

B. Short time scales

The MacLaurin expansion of the even function $C_y(\theta)$ is

$$C_y(\theta)=1+C_y^{(2)}(0)\theta^2/2+C_y^{(4)}(0)\theta^4/4!+\cdots . \quad (34)$$

The $(n+m)$th-order derivation of $\Gamma_y(\theta)=\sigma_yC_y(\theta)$ [Eq. (23)] may be expressed in terms of the cross-correlation of the nth and mth-order derivatives of $y(t)$,

$$(-1)^m\frac{d^{n+m}}{d\theta^{n+m}}\Gamma_y(\theta)=\langle y^{(n)}(t)y^{(m)}(t+\theta)\rangle . \quad (35)$$

Therefore $C_y^{(2)}(0)$ appears in the auto correlation of the expression

$$[y(t)-1]+\frac{d}{dt}y(t)=-R\sin[ky(t-\tau_R)] , \quad (36)$$

which gives

$$\sigma_y^2[C_y(0)-C_y^{(2)}(0)]=R^2\langle\sin^2(ky)\rangle . \quad (37)$$

The right-hand side of Eq. (37) is $\Gamma_f(0)$, which was derived in Eq. (29). For large k, $\Gamma_f(0)\rightarrow R^2/2$. Therefore $C_y^{(2)}(0)\rightarrow -R^2/2\sigma_y^2$ and

$$C_y(\theta)\approx 1-(R^2/2\sigma_y^2)\theta^2/2 \text{ near } \theta=0 . \quad (38)$$

Equation (38) shows that for small θ, $C_y(\theta)$ is a quadratic function of θ, evolving with a characteristic time

$$t_c=2\sigma_y/R=(2/kR)^{1/2} . \quad (39)$$

The range of θ for such a behavior can be estimated to be of order $t_c/2$ which corresponds to $C_y(\theta)$ having decreased from 1 to 0.75.

The calculated correlation functions $C_y(\theta)$ are shown in Fig. 7 for $R=\frac{2}{3}$ and $k=10,20,36$. For the last two values, the process is Gaussian and a quadratic behavior of $C_y(\theta)$ is predicted for $\theta\leq 0.2$ and 0.15, respectively. This is in agreement with Figs. 7(b) and 7(c). Let us notice that in the case of Fig. 7(a), even if the process is not

Gaussian, this quadratic behavior is observed for times $t \leq t_c$ [Eq. (39)].

In Sec. II A, we have analyzed the process $y(t)$ in terms of the three time scales $\delta, 1, \tau_R$. While the present analysis of $C_y(\theta)$ does not display the short time δ, it exhibits a new time t_c, proportional to $k^{-1/2}$. Note that t_c is larger than δ [Eq. (11)]. A glance at the temporal evolution of $y(t)$ can help to understand this result: the solution of $y(t)$ evolves with very small fluctuations on a short time scale, superimposed to larger fluctuations which are on a time scale t_c. The fastest fluctuations have so small an amplitude that they have no signature on $C_y(\theta)$. We shall see in Sec. IV that, in fact they have a fundamental role in the dynamical definition of the deterministic chaos.

A more precise estimation of δ may be obtained by the expansion of $\Gamma_f(\theta)$ in the vicinity of $\theta=0$. $\Gamma_f(\theta)$ appears in the autocorrelation of boths sides of Eq. (36),

$$\Gamma_f(\theta)=\sigma_y^2[C_y(\theta)-C_y^{(2)}(\theta)] \; . \tag{40}$$

A MacLaurin expansion of $C_y(\theta)$ and $C_y^{(2)}(\theta)$ near $\theta=0$ gives

$$\begin{aligned}\sigma_y^{-2}\Gamma_f(\theta)=&[1-C_y^{(2)}(0)]\\&-[C_y^{(4)}(0)-C_y^{(2)}(0)]\theta^2/2\\&-[C_y^{(6)}(0)-C_y^{(4)}(0)]\theta^4/4!+\cdots \; .\end{aligned} \tag{40'}$$

The derivatives $C_y^{(2k)}(0)$ are calculated by successive derivations of Eq. (36) (see Appendix D). In the limit of large k, Eq. (40) becomes

$$\Gamma_f(\theta)\frac{2}{R^2}=1-\left[\frac{kR\theta}{2}\right]^2+\frac{1}{3}\left[\frac{kR\theta}{2}\right]^4+\cdots \; . \tag{41}$$

In Eq. (41) the memory time is $2/kR$; it is nothing but the time width of an oscillation of $\sin[ky(t)]$,

$$\delta=2/kR \; , \tag{42}$$

which was introduced to display the Gaussian character of $y(t)$ (see Fig. 1). Let us now discuss different consequences of Eqs. (41) and (42).

First, the value of δ in Eq. (42) allows us to predict that $y(t)$ is Gaussian for

$$kR>15 \; . \tag{43}$$

This relation is indeed equivalent to the condition $3/\delta>25$ stated in Sec. II. It agrees with the probability distribution shown in Figs. 5(c) and 5(d) and with other simulations obtained with $R=0.4$ and 0.95 that are not presented here.

Secondly, the expansion of $\Gamma_f(\theta)$ near $u=0$ used in Appendix C to estimate the variance leads to a law of variation [Eq. (32)] of σ_y^2 with kR which agrees rather well with the numerical law [Eq. (33)].

Third, the width of $\Gamma_f(u)$ also agrees with the simulation. In Fig 8, the correlation functions $\Gamma_f(u)$ are shown for k increasing from 6 to 100. The curves are bell-shaped near $u=0$, with a half-width defined at $\Gamma_f(u)=1/e$ of order $2/kR$ in agreement with Eq. (42). Let us point out that for any similar *differentiable* feedback, the correlation Γ_f must exhibit a quadratic (or quartic, etc.) expansion near the origin [Eq. (41)]. In the case of the nondifferentiable feedback function A, the correlation behaves as $e^{-|t|/\delta}$ which is not differentiable at the origin: see Fig. 4(b), and compare with Fig. 8.

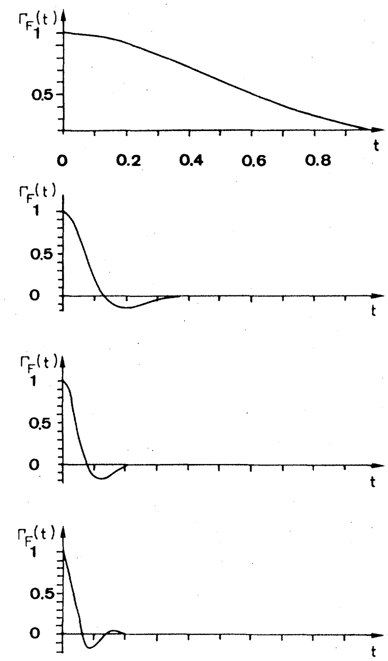

FIG. 8. Normalized correlation function $\Gamma_f(u)/\Gamma_f(0)$ [Eq. (26)] for the sine feedback, $R=\frac{2}{3}$. (a) $k=6$, $\tau_R=8.7$; (b) $k=36$, $\tau_R=5$; (c) $k=60$, $\tau_R=5$; (d) $k=100$, $\tau_R=5$.

Let us now investigate the long-time-scale correlations.

C. Correlation at τ_R

The behavior of the correlation function for $y(t)$ has been described for small θ and also for $\delta \ll \theta \ll \tau_R$: after a short quadratic decrease on a time scale of order t_c, $C_y(\theta)$ decays exponentially as $e^{-|\theta|}$. What happens for $\theta=\tau_R$? The quantity $C_y(\tau_R)$ is derived from $y(t+\tau_R)$ defined in Eq. (17),

$$\sigma_y^2 C_y(\tau_R)=-R\int_0^\infty du\, e^{-u}\langle y(t)\sin[ky(t-u)]\rangle \; . \tag{44}$$

The integrand in Eq. (44) can be calculated from the 2D characteristic function [Eq. (C4) in Appendix C]

$$\langle y(t)e^{iky(t-u)}\rangle$$
$$=-i\frac{d}{dv}\langle \exp[ivy(t)+iky(t-u)]\rangle\,|_{v=0} \; . \tag{45}$$

In the limit of large kR,

$$C_y(\tau_R)=-Rke^{-k^2\sigma_y^2/2}\cos(k)\int_0^\infty du\, e^{-u}C_y(u) \tag{46}$$

or

$$C_y(\tau_R)=-(Rk/2)e^{-k^2\sigma_y^2/2}\cos k \; . \tag{47}$$

This expression shows that the correlation between $y(t)$

and $y(t+\tau_R)$ is independent of τ_R and vanishes for large k. More precisely, for $k=20$, $R=\frac{2}{3}$, it gives $C_y(\tau_R)=-0.01$, in agreement with numerical results in Figs. 7(b) and 7(c). This result is not surprising if we return to our qualitative analysis in Sec. II, schematized in Fig. 1(c) or Eq. (8): $y(t)$ is in fact correlated with $n_{\rm eff}\simeq\frac{3}{2}kR$ individual oscillations of $\sin[ky(t-\tau_R)]$. As kR increases, the correlation between $y(t)$ and a particular Y_n decreases.

For kR smaller than 15, Eq. (47) is no longer valid; however, it predicts secondary maxima, as observed numerically. The heights of the first maximum are -0.3 and $+0.2$ for $k=5$ and 10 ($R=\frac{2}{3}$), respectively, whereas Eq. (47) predicts about half of these values, with the correct sign [cf. Fig. 7(a)]. We can also derive $C_y(2\tau_R), C_y(3\tau_R), \ldots$. Successive maxima appear when the process is no longer Gaussian ($kR<15$).

Let us also point out that the heights of the secondary maxima of $C_y(n\tau_R)$ have no relation with the correlation of the successive iterates of the difference equation [Eq. (4)]

$$\gamma_n=(1/\sigma_{y_i}^2)(\langle y_i y_{n+i}\rangle-\langle y_i\rangle^2) . \qquad (48)$$

For example, in the case $R=\frac{2}{3}$, $k=10$, we obtain

$$\gamma_n=-0.25,-0.14,+0.06,-0.02 \quad (n=1,2,3,4) ,$$

while Fig. 7(a) gives $C_y(n\tau_R)=+0.2,+0.2,+0.1,+0.1$, $(n=1,2,3,4)$. This result illustrates the fact that $y(t)$ and $\{y_n\}$ have completely *different memory processes*; this is true either for small values of $kR\le 15$, or for large kR.

In conclusion, the 2D statistical study has displayed the rather complex temporal behavior of the solution of the delay-differential equation. Four increasing time scales have been found when $kR\gg 1$ and $\tau_R\gg 1$,

$$\delta=1/\sigma_x^2\simeq 2/kR,\quad t_c=1/\sigma_x\simeq[2/kR]^{1/2},1,\tau_R , \qquad (49)$$

which will be now investigated from a deterministic point of view.

IV. DIMENSION OF THE CHAOS

It was recently shown that the dimension of the chaos increases approximately linearly with the delay τ_R,

$$d=\gamma\tau_R , \qquad (50)$$

in two particular cases of delay-differential equations.[3,4] The same behavior is found in the case of Eq. (1) with the sine feedback.

The constant γ^{-1} can be seen as the effective response time of the system so that the dimension would have the simple interpretation of the effective number of degrees of freedom within a τ_R, as conjectured in Ref. 4. This conjecture implies that γ^{-1} should be of the order of the shortest oscillation time of the dynamics, i.e.,

$$d\simeq\tau_R/\delta . \qquad (51)$$

We have numerically studied the dimension of the chaotic attractor as a function of k and R and γ^{-1} has been estimated with the help of the Lyapunov dimension. Since we are concerned only with a connection between the dimension and the effective response time of the system, the fractal part of Eq. (3) will be discarded. Therefore d will be the maximum number of characteristic exponents with a positive sum. As already pointed out in the Introduction, it is practically equal to twice the number of positive exponents. The law of variation of d/τ_R is plotted in Fig. 9 as a function of kR for different R. It gives rise to the approximate relation

$$d/\tau_R\simeq 0.4kR , \qquad (52)$$

which confirms quite well the conjecture [Eq. (51)].

The dimension can be also expressed in terms of the normalized variance, $\sigma^2=\sigma_x^2/\langle x\rangle^2=\sigma_y^2/\langle y\rangle^2$, as

$$d\simeq k^2\sigma^2\tau_R . \qquad (53)$$

This relation could be fortuitous, but it would be interesting to investigate it further, because $\sigma^2\tau_R$ represents the mean energy of the fluctuations of $y(t)$ (in the sense of signal theory) during a time τ_R. If such a relation between the dimension and the variance could be established for any periodic feedback, it would be very useful for experimentalists.

V. PERSPECTIVES

In conclusion, the statistical study of dynamical system which obeys a differential equation, with a linear dissipative term and nonlinear periodic delayed force, has displayed a rich temporal behavior with four increasing time scales. It also leads to a physical interpretation of the Lyapunov dimension.

It would be interesting to know how this character is modified when the dissipation is no longer linear. For example, what happens to the solution $x(t)$ which obeys

$$\frac{dx}{dt}+dV/dx=f(x(t-\tau_R)) ,$$

if $V(x)$ is a double-well potential? Will the chaotic solutions oscillate between two distincts mean values of x such that

$$\langle dV/dx\rangle=\langle f(x)\rangle \; ?$$

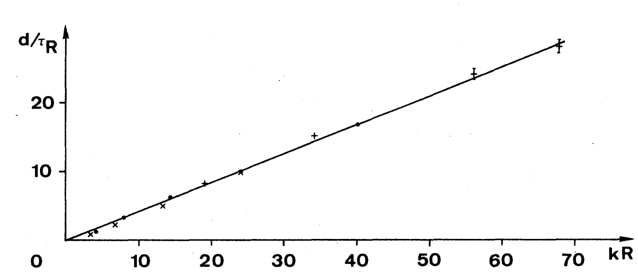

FIG. 9. Lyapunov dimension d of the chaotic attractor [Eq. (3)] for the sine-function feedback. The ratio d/τ_R is plotted for the three values of $R=0.95$, $\frac{2}{3}$, 0.4, (with their respective notations being the plus, cross, and dot), as a function of the parameter kR.

APPENDIX A

We present here a slight modification of the central-limit theorem for a sum of an infinity of independent random variables with decreasing variance.

Let us suppose first (hypothesis H1), that the widths of the intervals (t_n, t_{n+1}) defined in Eqs. (7) and (8) are constant. In Eq. (8), e^{-t_n} will be then replaced by $e^{-n\delta}$ where δ is the mean value of the intervals.

Let us introduce the characteristic function of $y-\langle y\rangle=\tilde{y}$

$$\varphi_{\tilde{y}}(v)=\langle e^{iv\tilde{y}}\rangle \; . \tag{A1}$$

Since the measure is ergodic, $\langle y\rangle=\langle Y_n\rangle=\frac{1}{2}$. By taking into account the independence of the $\tilde{Y}_n=Y_n-\langle y\rangle$, the characteristic function becomes

$$\varphi_{\tilde{y}}(v)=\prod_{n=0}^{\infty}\langle \exp(iv\delta e^{-n\delta}\tilde{Y}_n)\rangle \; . \tag{A2}$$

For $v\delta \ll 1$, the expansion of each $\langle\ \rangle$ in powers of $w=v\delta e^{-n\delta}$ leads to

$$\ln\varphi_{\tilde{y}}(v)=-\sum_{n=0}^{\infty}\left[\frac{1}{2}w_n^2\langle \tilde{Y}_n^2\rangle-\frac{1}{4!}w_n^4\langle \tilde{Y}_n^4\rangle+\cdots\right] \tag{A3}$$

with

$$\langle \tilde{Y}_n^2\rangle=\frac{1}{\delta^2}\left\langle \int_{\delta_n}\int_{\delta_n} du\, du'\{\mathrm{Dec}[ky(t-u)]\mathrm{Dec}[ky(t-u')]-\langle y\rangle^2\}\right\rangle \; . \tag{A4}$$

This last value is (cf. H1) of order

$$\frac{1}{\delta^2}\int_{\delta}\int_{\delta} du\, du'\langle \mathrm{Dec}[ky(0)]\mathrm{Dec}[ky(u'-u)]\rangle-\tfrac{1}{4} \; . \tag{A5}$$

We suppose now (hypothesis H2) that the correlation function of Dec (ky) is constant over the whole interval δ and vanishes for a time larger than δ, and (hypothesis H3) that the probability distribution of Dec (ky) is uniform in (0,1). It results that $\langle [\mathrm{Dec}(ky)]^2\rangle=\int_0^1 dz\, z^2=\frac{1}{3}$, and one gets

$$\langle Y_n^2\rangle=\tfrac{1}{12} \; . \tag{A6}$$

Define

$$\sigma_y^2=(\delta/2)\langle Y_n^2\rangle=\delta/24 \; . \tag{A7}$$

Then Eq. (A3) becomes

$$\ln\varphi_{\tilde{y}}(v)=-v^2\sigma_y^2\delta\sum_{n=0}^{\infty}e^{-2n\delta}+v^4\delta^4\gamma\sum_{0}^{\infty}e^{-4n\delta}+\cdots \; , \tag{A8}$$

where $\gamma=\frac{1}{8}(\langle \tilde{Y}_n^4\rangle/3-\langle \tilde{Y}_n^2\rangle^2)$ may be calculated with hypotheses H2 and H3. One gets $\gamma=3.4\times10^{-4}$. Equation (A8) becomes

$$\varphi_{\tilde{y}}(v)=\exp\left[-\frac{v^2}{2}\sigma_y^2+\frac{v^4\sigma_y^4}{4}\times 0.2\delta\right] \tag{A9}$$

and shows that $y(t)$ behaves as a Gaussian variable with variance σ_y^2 for large k.

The width of the sawteeth may be evaluated by the following argument. For a small interval δ_t, the increment of $y(t)$ is

$$y(t+\delta_t)-y(t) \simeq -\delta_t y(t)+e^{-\delta_t}\int_0^{\delta_t}e^{v}\mathrm{Dec}[ky(t-\tau_R-v)]dv \; . \tag{A10}$$

For an increment equal to $1/k$, one gets

$$k^{-2}=\langle [y(t+\delta_t)-y(t)]^2\rangle=\langle\delta^2\rangle(\sigma_y^2+\tfrac{1}{4})-\tfrac{1}{4}\langle\delta_t^2\rangle+\left\langle \int_{\delta_t}\int_{\delta_t}\{\mathrm{Dec}[ky(0)]\mathrm{Dec}[ky(u'-u)]du\, du'\}\right\rangle \; . \tag{A11}$$

The last term in Eq. (A11) is equal to $\langle\delta^2\rangle/3$ (cf. H1 and H2), and Eq. (A11) becomes

$$k^{-2}=\langle\delta^2\rangle/3 \tag{A12}$$

or else

$$\sigma_y^2=0.07/k \; . \tag{A13}$$

For the map A, the width of an oscillation in Fig. 1(c) may be estimated with the same argument as above. For large k, $\langle y(t)\rangle\rightarrow 1$, $\langle \sin(ky)\rangle\rightarrow 0$, and $\langle \sin^2(ky)\rangle\rightarrow\frac{1}{2}$. An increment of $y(t)$ over a small interval δ_t is

$$y(t+\delta_t)-y(t)=-\delta_t[y(t)-1]-\int_{\delta_t}R\sin[ky(t-\tau_R-v)]dv \tag{A14}$$

When the increment is equal to $2\pi/k$,

$$(2\pi/k)^2=\langle\delta^2\rangle\sigma_y^2+\langle\delta^2\rangle R^2/2 \; . \tag{A15}$$

In the frame of hypotheses like H2 and H3 this leads to

$$\langle\delta^2\rangle_{\text{sine map}}\simeq(2\pi\sqrt{2}/kR)^2 \; . \tag{A16}$$

APPENDIX B

Let us consider an increment $\Delta y(\theta)$ of the process $y(t)$ on a time interval $\theta=m\delta$. From Eqs. (8) and (13) $\Delta y(\theta)$ can be written as

$$\Delta y(\theta)=\sum_{i=1}^{\infty}e^{-u_i}Y_i-\sum_{j=m}^{\infty}e^{-u_j}Y_j \tag{B1}$$

with Y_i, Y_j defined in Eq. (9). With hypothesis H1 used in Appendix A, Eq. (B1) becomes

$$\Delta y(\theta)=\sum_{i=1}^{m} e^{-i\delta}Y_i+\sum_{j=m}^{\infty}(e^{-(m+j)\delta}-e^{-j\delta})Y_j \;. \tag{B2}$$

For small values of m, of the order of a few units, we have

$$\Delta y(\theta)=\sum_{i=1}^{m} e^{-i\delta}Y_i-m\delta\sum_{j=m}^{\infty}e^{-j\delta}Y_j \;, \tag{B3}$$

which reduces to the first sum for large k, or small δ. $\Delta y(\theta)$ is therefore Gaussian for $\theta\simeq 25\delta$, since $m\simeq 25$ contributions are required as seen in Sec. II.

APPENDIX C

The fixed points of Eq. (36) are given by the implicit relation $y^{(s)}-1=-R\sin(ky^{(s)})$. The linear stability analysis of a small deviation Δy from $y^{(s)}$

$$y=y^{(s)}+\Delta y\exp(\lambda t)$$

gives

$$\lambda+1=f'(y^{(s)})\exp(-\lambda\tau_R)\;, \tag{C1}$$

with

$$f'=df/dy \;.$$

The solution $y^{(s)}$ is stable for small τ_R and loses its stability at $\tau_R^{(s)}$ such that Re $\lambda=0$ in Eq. (C1), i.e.,

$$\cos[f'(y^{(s)})^2-1]^{1/2}\tau_R^{(s)}=1/|f'(y^{(s)})| \;. \tag{C2}$$

For the sine map, the derivative f' depends on the stationary point $y^{(s)}$. For large k, $|f'|$ is mostly of order $2kR/\pi$ and the condition for chaos is

$$\tau_R \gg \pi^2/4kR \;,\quad \text{or } \tau_R\gg\delta \;. \tag{C3}$$

Let us now consider the chaotic regime, with τ_R of the order of a few units, and $kR\geq 15$. The correlation of the driving force $f[y(t)]=-R\sin[ky(t)]$, may be calculated with the help of the 2D characteristic function

$$\varphi_{Y_0,Y_u}(v,w)=\langle e^{i(vY_0+wY_u)}\rangle e^{-i(v+w)\langle y\rangle} \;. \tag{C4}$$

From Eq. (C4) we have

$$\langle\cos\{k[y(t_1)+y(t_2)]\}\rangle=\mathrm{Re}[\varphi_{Y_{t_1},Y_{t_2}}(k,k)e^{2ik\langle y\rangle}] \;, \tag{C5}$$

$$\langle\cos\{k[y(t_1)-y(t_2)]\}\rangle=\mathrm{Re}[\varphi_{Y_{t_1},Y_{t_2}}(k,-k)] \;. \tag{C6}$$

Equations (C5) and (C6) lead to

$$\Gamma_f(u)=\frac{R^2}{2}[\varphi(k,-k)-\cos(2k\langle y\rangle)\varphi_{Y_{t_1},Y_{t_2}}(k,k)] \;. \tag{C7}$$

In the Gaussian case, the 2D characteristic function in Eq. (C4) is

$$\varphi_{Y_0,Y_u}=\exp\{-\tfrac{1}{2}\sigma_y^2[v^2+w^2+2vwC_y(u)]\} \;, \tag{C8}$$

where $C_y(u)$ is the normalized correlation of y defined in Eq. (24). The correlation of f is therefore given by Eq. (29).

Let us point out that the expression of $\Gamma_f(u)$ in Eq. (29) is valid either for $u=0$, or $u\gg\delta$, i.e., on the far edges of the function. This restriction is especially questionable for carrying on the calculation of

$$\sigma_y^2=\eta/2=\int_0^\infty\Gamma_f(u)du \;. \tag{C9}$$

We shall bypass this difficulty by using the three following methods, and comparing their results. The first method uses the series expansion of $\Gamma_f(u)$ valid for small u [Eq. (41)]. The two other methods suppose that the Gaussian statistics are valid all the time [Eq. (29)], and approximate $C_y(u)$ either by $e^{-|u|}$ or by $1-(u/t_c)^2$ [Eqs. (31) and (38), respectively].

(a) When $\Gamma_f(u)$ is given by its expansion in Eq. (41), with $0<u<\delta$, we obtain

$$\eta=R^2\int_0^{2/kR}\left[1-\left[\frac{kR\theta}{2}\right]^2+\frac{1}{3}\left[\frac{kR\theta}{2}\right]^4\right]d\theta$$
$$=\frac{4R}{3k}\;, \tag{C10}$$

which leads to

$$(\sigma_y^2)^a=0.67R/k \;. \tag{C11}$$

(b) For a Markov process with exponential correlation [Eq. (31)], the constant η can be calculated exactly. It involves exponential integrals functions. However, we obtain the same result if we approximate the exponential by $C_y(\theta)\simeq 1-|\theta|$ because $k^2\sigma_y^2\gg 1$. Equations (29) and (C9) lead to

$$\eta=R^2\int_0^\infty e^{-k^2\sigma_y^2\theta}d\theta \tag{C12}$$

or else

$$\eta=R^2(k^2\sigma_y^2)^{-1} \;. \tag{C13}$$

Equations (31) and (C13) give

$$(\sigma_y^4)^b=R^2/(2k^2)\;, \tag{C14}$$

or else Eq. (32).

(c) Let us still suppose that $y(t)$ is Gaussian even on the small time scale δ, and now approximate $C_y(u)$ by its value near $u=0$ [Eq. (38)]. It gives

$$\eta=R^2\int_0^\infty\exp(-kR\theta/2)^2d\theta \;, \tag{C15}$$

where we can integrate from 0 to $+\infty$ because $t_c\gg\delta$. Therefore

$$\eta=\sqrt{\pi}R/k \;. \tag{C16}$$

Equations (31) and (C16) give

$$(\sigma_y^2)^c=\frac{\sqrt{\pi}}{2}R/k \;. \tag{C17}$$

The last two derivations in (b) and (c) give, respectively,

$$(\sigma_y^2)^b=0.7R/k \;,$$
$$(\sigma_y^2)^c=0.9R/k \;, \tag{C18}$$

The second method would have been better because

$C_y(u)$ is correctly described in the whole range $\theta \ll t_c$, large enough to calculate η. As a matter of fact, these two methods overestimate the factor α in Eq. (10). We conclude that $\Gamma_f(u)$ is not correctly described by Eq. (29), for u of order δ. This conclusion is confirmed by Fig 8: neither Eq. (C12) nor Eq. (C15) take account of the oscillating part of $C_f(u)$ which reduces the area in the calculation of η.

Finally, the best way to calculate the variance is to use the expansion of $\Gamma_f(u)$ for small u, which is in powers of $kRu/2$, i.e., in powers of $(\sigma_x^2 u)$. This is an example of the strong connection between 1D and 2D statistics in this problem.

APPENDIX D

By derivation of Eq. (36), we obtain successively

$$\dot{y}_t + \ddot{y}_t = -Rk\dot{y}_R \cos(ky_R) \; , \tag{D1}$$

$$\ddot{y}_t + \dddot{y}_t = R(k\dot{y}_R)^2 \sin(ky_R) - Rk\ddot{y}_R \cos(ky_R) \; , \tag{D2}$$

where the indices t and R refer, respectively, to time t and $t-\tau_R$.

Taking into account Eq. (35), we can express the autocorrelation of Eqs. (40), (D1), and (D2) for $\theta=0$; it gives

$$\sigma_y^2[1-C_y^{(2)}(0)] = R^2\langle \sin^2(ky)\rangle \; , \tag{D3}$$

$$\sigma_y^2[C_y^{(4)}(0)-C_y^{(2)}(0)] = (Rk)^2\langle \dot{y}_t^{\,2}\cos^2(ky_t)\rangle \; , \tag{D4}$$

$$\begin{aligned} -\sigma_y^2[C_y^{(6)}(0)-C_y^{(4)}(0)] &= R^2k^4\langle \dot{y}_R^{\,4}\sin^2(ky_R)\rangle \\ &\quad -2(k^3R^2)\langle \dot{y}_R^{\,2}\sin(ky_R)\cos(ky_R)\rangle \\ &\quad +R^2k^2\langle \ddot{y}_R^{\,2}\cos^2(ky_R)\rangle \; . \end{aligned} \tag{D5}$$

From Appendix C we know that

$$\langle \cos(2ky)\rangle = \cos(2k)\exp(-2k^2\sigma_y^2) \; , \tag{D6}$$

$$\langle \sin(2ky)\rangle = \sin(2k)\exp(-2k^2\sigma_y^2) \; . \tag{D7}$$

By successive derivations with respect to k, we obtain

$$\langle y\cos(2ky)\rangle = e^{-2k^2\sigma^2}[\cos(2k)-2k\sigma_y^2\sin(2k)] \; , \tag{D8}$$

which exponentially vanishes for large k. In the same way $\langle y^n\cos(mky)\rangle$ and $\langle y^n\sin(mky)\rangle$ also vanish. Therefore

$$\sigma_y^2[1-C_y^{(2)}(0)] \to R^2/2 \tag{D9}$$

and

$$\begin{aligned} \langle \dot{y}^2\cos(2ky)\rangle &= \langle \dot{y}^2/2\rangle \\ &= \langle [R\sin(ky_R)+(y-1)_t]^2\rangle \; . \end{aligned} \tag{D10}$$

We know that $y(t)$ and $y(t+\tau_R)$ are independent for large k [Eq. (47)], and this gives

$$\langle \dot{y}^2\rangle = \frac{R^2}{2} + \sigma_y^2 \; , \tag{D11}$$

or else, with Eqs. (D4) and (D10),

$$\sigma_y^2[C_y^{(4)}(0)-C_y^{(2)}(0)] \to R^4k^2/4 \; . \tag{D12}$$

Now

$$\sigma_y^2[C_y^{(6)}(0)-C_y^{(4)}(0)] = \frac{R^2k^4}{2}\langle \dot{y}^4\rangle + \frac{R^2k^2}{2}\langle \ddot{y}^2\rangle \tag{D13}$$

with

$$\langle \dot{y}^4\rangle \to \frac{3R^4}{8} + 3\sigma_y^4 + 3R^2\sigma_y^2 \; , \tag{D14}$$

$$\langle \ddot{y}^2\rangle = \langle \dot{y}^2\rangle \left[1+\frac{k^2}{2}R^6\right] . \tag{D15}$$

Finally,

$$\sigma_y^2[C_y^{(6)}(0)-C_y^{(4)}(0)] \to -\tfrac{1}{4}k^4R^6 \; , \tag{D16}$$

and Eqs. (D9), (D12), and (D16) lead to Eq. (41).

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