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Comment on first-passage times for processes driven by dichotomous fluctuations

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The recent derivation of the mean first-passage time for one-dimensional processes driven by additive dichotomous random processes [J. Masoliver, K. Lindenberg, and B. J. West, Phys. Rev. A **34**, 2351 (1986)] can be extended to situations where the noise occurs multiplicatively and nonlinearly in the stochastic differential equation. For equations with Markovian dichotomous fluctuations in particular, this result allows for a complete and general probabilistic description of the statics and dynamics of the non-Markovian solution process in terms of stationary probability distributions and first-passage time statistics.

In a recent article[1] Masoliver, Lindenberg, and West derived explicit closed expressions for the mean first-passage time of one-dimensional processes defined by stochastic differential equations of the form (adopting the notation of Ref. 1)

$$dX(t)/dt = f(X(t)) + F(t) \ , \tag{1}$$

where X is the system variable and $F(t)$ is a dichotomous random process with an exponential or rectangular temporal distribution. For example, when F takes the values $a, -b$ $(a,b>0)$, x_a^0 and x_b^0 are asymptotically stable fixed points of $f+a$ and $f-b$, and z_1 and z_2 are such that $x_b^0 \le z_2 \le x_0 \le z_1 \le x_a^0$, then the first-passage time of the process X from the point x_0 to the absorbing boundaries z_1 and z_2 is computed by carefully tracking the evolution of X for each realization of F. These trajectories can be tracked uniquely as long as there are well defined (i.e., invertible) functions ϕ_a and ϕ_b given by

$$\begin{aligned} \phi_a^{-1}(x) &= \int^x dx'[f(x')+a]^{-1} \ , \\ \phi_b^{-1}(x) &= \int^x dx'[f(x')-b]^{-1} \ . \end{aligned} \tag{2}$$

These functions are essentially the solutions to (1) while $F=a$ or $-b$, and Secs. II–IV of Ref. 1 show how these determine the statistics of the first-passage time. The existence of such functions does *not* depend on the additive nature of the driving fluctuations. The purpose of this comment is to point out that the derivations and results of Ref. 1 can be generalized in a straightforward way to apply to the stochastic differential equation

$$dX(t)/dt = f(X(t),F(t)) \ , \tag{3}$$

where in general the dichotomous process appears multiplicatively and nonlinearly.

The restrictions on the force f, the initial point x_0, and the boundaries z_1 and z_2 are simple extensions of those for the additive noise case. Without loss of generality we may require that

$$f(x,a) > 0 > f(x,-b) \tag{4}$$

in the region

$$-\infty \le x_b^0 < z_2 \le x \le z_1 < x_a^0 \le \infty \ , \tag{5}$$

where x_a^0 and x_b^0 are the asymptotically stable fixed points of $f(x,a)$ and $f(x,-b)$—if they exist. If $f(x,a)$ ($f(x,-b)$) does not have such a fixed point, then we take $x_a^0=\infty$ ($x_b^0=-\infty$). When F has either an exponential or nondegenerate rectangular temporal distribution, the restrictions (4) and (5) are just those which ensure that the first passage time of X from $x_0 \in [z_2,z_1]$ to the boundaries is finite with probability one. For a given realization of F, the trajectories of X may then be followed by using the functions

$$\begin{aligned} \phi_a^{-1}(x) &= \int^x dx'[f(x',a)]^{-1} \ , \\ \phi_b^{-1}(x) &= \int^x dx'[f(x',b)]^{-1} \ , \end{aligned} \tag{6}$$

and the derivations and formulas in Ref. 1 are generalized by making the substitutions

$$f(x)+a \to f(x,a), \ \ f(x)-b \to f(x,-b) \ . \tag{7}$$

Care must simply be taken in the case of expressions involving the derivatives of the force since we must now distinguish between $f'(x,a)=\partial f(x,a)/\partial x$ and $f'(x,-b)=\partial f(x,-b)/\partial x$.

Specifically, if $F(t)$ is a mean zero dichotomous Markov process with average residence times λ_a^{-1} and λ_b^{-1} in the states $F=a$ and $-b$, then the mean first-passage time of the process $X(t)$ from x_0 to z_1 or z_2, given that $F=a$ initially, is

$$T_1^{(a)}(x_0)=\int_{z_1}^{x_0} V^{(a)}(x)\exp[-M^{(a)}(x)]dx + C^{(a)}\int_{z_1}^{x_0}\exp[-M^{(a)}(x)]dx \ , \tag{8}$$

where

$$M^{(a)}(x)=\int^x dy[f'(y,a)/f(y,a)-\lambda_a/f(y,a)-\lambda_b/f(y,-b)] \ , \tag{9}$$

$$V^{(a)}(x)=\int^x dy\{(\lambda_a+\lambda_b)/[f(y,a)f(y,-b)]\}\exp[M^{(a)}(y)] \ , \tag{10}$$

$$C^{(a)}=\left[\exp[-M^{(a)}(z_2)]-[\lambda_a/f(z_2,a)]\int_{z_1}^{z_2}\exp[-M^{(a)}(x)]dx\right]^{-1} \times\left[-V^{(a)}(z_2)\exp[-M^{(a)}(z_2)]+f(z_2,a)^{-1}\left\{\lambda_a\int_{z_1}^{z_2}V^{(a)}(y)\exp[-M^{(a)}(y)]dy-1\right\}\right] . \tag{11}$$

Of course a similar expression holds when $F=-b$ at the start, exchanging a and $-b$, λ_a and λ_b, and z_1 and z_2. These expressions are essentially Eqs. (5.9)—(5.12) of Ref. 1 where the authors supply all the tools needed to compute all the moments of the first-passage time.

An important aspect of these results for stochastic differential equations (3) driven by the dichotomous Markov process is that we now have as much control over the non-Markovian solution process as we do over Markovian diffusion processes defined by stochastic differential equations driven by Gaussian white noise. Explicit expressions exist for the stationary probability distributions of each kind of process[2,3] describing the static probabilistic properties of the solutions. The classical expressions for the moments of the first-passage times for diffusion processes[2] are now complemented by analogous expressions for the non-Markovian processes, characterizing the dynamics of the random motion. This result is important for the analysis of physical systems under the influence of fluctuating parameters which appear nonlinearly in the dynamical equation,[4] for it is not meaningful to perform nonlinear operations on a Gaussian white noise.

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[1] J. Masoliver, K. Lindenberg, and B. J. West, Phys. Rev. A 34, 2351 (1986).

[2] Markovian diffusion processes are discussed in, for example, S. Karlin and H. M. Taylor, *A Second Course in Stochastic Processes* (Academic, New York, 1981); the correspondence with stochastic differential equations driven by Gaussian white noise is found in L. Arnold, *Stochastic Differential Equations* (Wiley, New York, 1974).

[3] Stationary probability distributions for processes defined by Eq. (3) with dichotomous Markov fluctuations are derived in W. Horsthemke and R. Lefever, *Noise Induced Transitions: Theory and Applications in Physics, Chemistry and Biology* (Springer, New York, 1984).

[4] For examples of nonlinear fluctuations in dynamical systems, see Ref. 3.