Dichotomous-noise-driven oscillators

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The problem of finding the probability density function of the output of an oscillator (a filter of order higher than the first) driven by dichotomous Markov noise (the random telegraph signal) is considered. No known theoretical methods are available for completely solving problems of this type. A somewhat general expression is derived for the output moments and an exact formulation for the probability density is presented in terms of Fokker-Planck—type equations. However, partly because of irregularly shaped boundaries, the Fokker-Planck equations are complicated and remain unsolved except in the first-order case. The paper concludes with some Monte Carlo results for a second-order Butterworth filter.

I. INTRODUCTION

The problem of calculating the probability density function of the output of an *RC* filter driven by a binary random process has been studied extensively over the past 25 years or so.[1–3] The much more difficult case when the *RC* filter is replaced by an oscillating higher-order filter has, to our knowledge, received no mention in the published literature, and is the subject of the present paper. For concreteness, much attention will be focused on the second-order Butterworth filter driven by the random telegraph signal. (The terms "random telegraph signal" and "dichotomous Markov noise" will be used herein interchangeably.) This example serves well to illustrate the difficulty of the general problem of higher-order filtering, and it is anticipated that the solution in the second-order case will be instrumental in providing the key to the solution in the general case.

In the *RC* filter case, the class of binary processes for which most results are available is that with intervals generated by an equilibrium renewal process.[2] The simplest of these is the random telegraph signal which has exponentially distributed intervals, and, in this case, the density of the output can be found by Fokker-Planck methods. For intervals with other distributions, integral equations can readily be derived for the output density, and can be either solved directly by numerical methods or transformed into differential equations. It is tempting to try to apply these various approaches to the higher-order, non-*RC* filter. Although some progress can be made, it does not appear that any of these previously used techniques can be directly applied to yield a complete solution. Some of the reasons why are considered in depth in this paper. It is hoped that our considerations will give some insight into the development of analytical methods for the solution of this most challenging problem.

The second-order Butterworth filter is examined in detail in Sec. II, which gives the filter differential equations, their state-space formulation, examination of the regions of state space over which the equations hold, and consideration of the output moments. Extensions of these results to the general *n*th-order filter are considered in Sec. III. After a very brief review of the applicable theory, Fokker-Planck equations for the output density are formulated in Sec. IV, and are then specialized to the first- and second-order Butterworth filters. Some Monte Carlo results are presented in Sec. V, and the final section summarizes and discusses the results.

II. SECOND-ORDER BUTTERWORTH FILTER

The only binary input that we consider here is the random telegraph signal with exponentially distributed time intervals between traversals. The random telegraph signal will be denoted by $x(t)$, and the average number of traversals of $x(t)$ per unit time by a.

A. Filter differential equations

The frequency response of the second-order Butterworth filter is

$$|H(f)|^2=\frac{1}{1+(2f/B)^4}\ , \tag{1}$$

where B is the filter 3-dB bandwidth, and the corresponding impulse response is

$$h(t)=2\beta e^{-\beta t}\sin(\beta t),\quad t>0 \tag{2}$$

in which $\beta=\pi B/2$. The second-order differential equation relating the input $x(t)$ and output $y(t)$ of the filter is then

$$\frac{d^2y(t)}{dt^2}+2\beta\frac{dy(t)}{dt}+2\beta^2y(t)=2\beta^2x(t)\ . \tag{3}$$

This differential equation can be rewritten using state-space formalism[4] as a vector differential equation. Letting $\mathbf{z}(t)$ be a column vector with components $z_1(t)=y(t)$ and $z_2(t)=\dot{y}(t)$, the vector equation is

$$\dot{\mathbf{z}}(t)=\mathbf{A}\,\mathbf{z}(t)+\mathbf{B}\,x(t)\ , \tag{4}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -2\beta^2 & -2\beta\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0\\ 2\beta^2\end{bmatrix}\ . \tag{5}$$

Although we are considering a specific second-order system, the matrix equations are valid for general nth-order systems. This is true of (4), (6), and (29).

B. Filter output equations

The state of the system, the vector $\mathbf{z}(t)$, can be obtained as the solution to (4) in terms of the state-transition matrix $\mathbf{\Phi}(t)$ as

$$\mathbf{z}(t)=\mathbf{\Phi}(t-t_0)\mathbf{z}(t_0)+\int_{t_0}^{t}\mathbf{\Phi}(t-\tau)\mathbf{B}x(\tau)d\tau \,, \tag{6}$$

where $\mathbf{\Phi}(t)$ is the solution to the matrix equation $\dot{\mathbf{\Phi}}(t)=\mathbf{A}\mathbf{\Phi}(t)$ with initial condition $\mathbf{\Phi}(0)=\mathbf{I}$ and, for the second-order Butterworth filter, comes out to be

$$\mathbf{\Phi}(t)=e^{-\beta t}\begin{bmatrix}\cos(\beta t)+\sin(\beta t) & \beta^{-1}\sin(\beta t)\\ -2\beta\sin(\beta t) & \cos(\beta t)-\sin(\beta t)\end{bmatrix} . \tag{7}$$

The steady-state stationary solutions are then, from (6),

$$y(t)=2\beta\int_{-\infty}^{t}e^{-\beta(t-\tau)}\sin[\beta(t-\tau)]x(\tau)d\tau \tag{8}$$

and

$$\dot{y}(t)={}^2\beta^2\int_{-\infty}^{t}e^{-\beta(t-\tau)}\{\cos[\beta(t-\tau)] -\sin[\beta(t-\tau)]\}x(\tau)d\tau \,. \tag{9}$$

The output is oscillatory over intervals of constant input. Because of this, we note that the integral equation approach which formed the basis of the analyses in Refs. 1 and 2 is not tenable in the present situation, for the derivation of the integral equations requires that the inverse function y^{-1} be a single-valued function of t over intervals of constant input.

C. Output domains

Even though (8) and (9) are explicit relations for $y(t)$ and $\dot{y}(t)$ for any sample function of the input process $x(t)$, the ranges of $y(t)$ and $\dot{y}(t)$ are not immediately evident from them. Since $x(t)$ is a binary process, the integral in (8) is maximized when $x(\tau)=\text{sgn}\{\sin[\beta(t-\tau)]\}$, and we find

$$|y(t)|\le\coth(\pi/2)=1.090\,33\ldots \,. \tag{10}$$

In a similar way

$$|\dot{y}(t)|\le\sqrt{2}\beta e^{\pi/4}\text{csch}(\pi/2)=(1.347\,83\ldots)\beta \,. \tag{11}$$

However, $y(t)$ and $\dot{y}(t)$ are not independent, and the region of permissible values of the pair $(y,\dot{y})$ will not necessarily be a rectangle in the $(y,\dot{y})$ plane. Indeed, selecting $x(\tau)$ in (8) and (9) to maximize the sum $y(t)+\lambda\dot{y}(t)$ (here, λ can be viewed as a Lagrange multiplier) leads to the parametric equations

$$y=1-\text{csch}(\pi/2)e^{-\phi}(\cos\phi-\sin\phi)\,, \tag{12}$$

$$\dot{y}=2\beta\,\text{csch}(\pi/2)e^{-\phi}\cos\phi\,, \tag{13}$$

in which $-\pi/2\le\phi\le\pi/2$. The locus described by this pair is the upper half of the curve shown in Fig. 1. By symmetry, the lower half of the curve is a reflection of the upper half. The peculiar shape of the boundary region leads to certain complications in attempting to find the marginal and joint densities $p(y)$ and $p(y,\dot{y})$. These difficulties will be described later in Sec. IV.

D. Output moments

For a bounded random variable, its marginal probability density function can be determined, in principle, from knowledge of all of its moments by summing and inverting the moment generating function. Indeed, Wonham and Fuller[3] first determined the density of the output of an RC filter driven by the random telegraph signal by doing just this. However, in the case of the second-order Butterworth filter, the higher moments become increasingly complicated. Although a general expression for the moments can be written, it involves multiple sums, and its evaluation requires the use of complex arithmetic and a digital computer.

The second moment can be obtained by squaring (8) and using $E[x(\tau_1)x(\tau_2)]=\exp(-2a\,|\tau_2-\tau_1|)$ and comes out to be ($\alpha=a/\beta$)

$$\overline{y^2}=\frac{1+\alpha}{1+2\alpha+2\alpha^2} \,. \tag{14a}$$

In a similar way, the fourth moment is (cf. Ref. 3)

$$\overline{y^4}=\frac{24}{5}\,\frac{10\alpha^4+64\alpha^3+173\alpha^2+215\alpha+75}{[1+(1+2\alpha)^2][1+(3+2\alpha)^2][9+(3+2\alpha)^2]} \,. \tag{14b}$$

The general expression can be obtained by going back to the integrals which led to (14a) and (14b) and writing the integrands in complex form. For example, instead of (14a) we get

$$\overline{y^2}=i^2(2!)\sum_{q_1,q_2=\pm1}\frac{q_1}{2\alpha+1+iq_1}\,\frac{q_2}{2+i(q_1+q_2)} \,, \tag{14c}$$

and (14b) is the same as

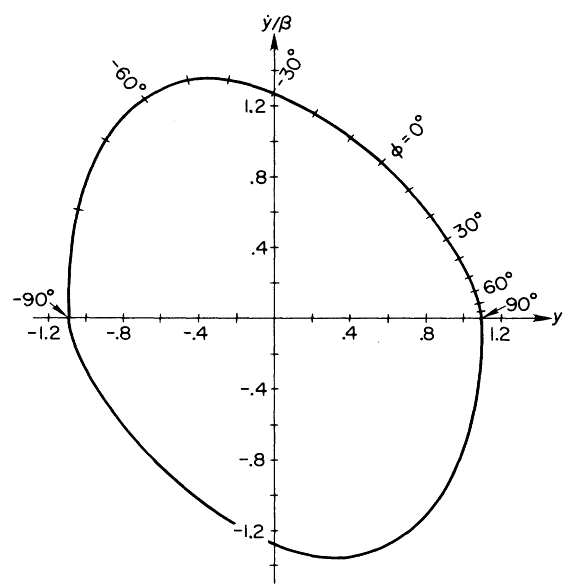

FIG. 1. The boundary of allowable values of y and $\dot{y}/\beta$. The probability density functions $p_{\pm}(y,\dot{y})$ must vanish outside the curved region.

TABLE I. Output moments for second-order Butterworth filter.

α	$\overline{y^2}$	$\overline{y^4}$	$\overline{y^6}$	$\overline{y^8}$	$\overline{y^{10}}$
$\frac{1}{5}$	0.810 81	0.787 77	0.794 92	0.816 05	0.846 85
$\frac{1}{2}$	0.600 00	0.529 41	0.511 66	0.514 68	0.529 67
1	0.400 00	0.291 58	0.251 47	0.234 33	0.228 19
2	0.230 76	0.118 98	0.080 57	0.062 67	0.052 96
5	0.098 36	0.025 64	0.009 94	0.004 85	0.002 75

$$\overline{y^4}=i^4(4!)\sum_{\text{all } q_j=\pm 1}\frac{q_1}{2\alpha+1+iq_1}\,\frac{q_2}{2+i(q_1+q_2)}\,\frac{q_3}{2\alpha+3+i(q_1+q_2+q_3)}\,\frac{q_4}{4+i(q_1+q_2+q_3+q_4)}\,, \tag{14d}$$

The general form of these last two is then

$$\overline{y^{2n}}=(-1)^n[(2n)!]\sum_{\text{all } q_j=\pm 1}\prod_{k=1}^{2n}\frac{q_k}{[1-(-1)^k]\alpha+k+iS_k}\,, \tag{14e}$$

in which the summation is a $2n$-fold summation over $q_1,q_2,\ldots,q_{2n}$ and $S_k=\sum_{m=1}^{k}q_m$. Summations of this type have been used to compute the even order moments up to $\overline{y^{14}}$, and the results of some of these computations are given in Table I.

In the case of the RC filter driven by the random telegraph signal with arbitrary interval distributions, Munford[5] has devised a recursive method for computing all of the moments. However, it does not appear that his method can be extended to the higher-order filter.

III. GENERAL FILTERS

The operation of an nth order linear filter can be described by an nth order differential equation, and the state-space vector for the characterization of this system is then the vector whose components are $y(t)$ and its first $n-1$ derivatives. Letting $z_i(t)=y^{(i-1)}(t)$, $i=1,\ldots,n$, the generic state-space form of such a differential equation is

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \vdots\\ \dot z_{n-1}\\ \dot z_n\end{bmatrix}=\begin{bmatrix}z_2\\ z_3\\ \vdots\\ z_n\\ F_x(\mathbf{z})\end{bmatrix}. \tag{15}$$

For example, for the second-order Butterworth filter, $F_x(\mathbf{z})=2\beta^2x-2\beta^2z_1-2\beta z_2$. In general, for any linear filter, $F_x(\mathbf{z})$ will be a linear combination of the input and the n components of the state vector. For Butterworth filters of orders 1—3, filter differential equations, along with their impulse responses, are given in Table II. The corresponding F's can easily be read by inspection from the differential equations.

All of the analysis of the preceding section can, in principle, be repeated in the general case. However, the state transition matrix is more difficult to determine for higher-order systems, the domains of definition become oddly shaped regions described by parametric equations in n space, and the moments are even more complicated than they are in the case of the second-order Butterworth filter. Thus, even though some of the analyses can be carried through in the general case (see the Appendix for the moment formula for an arbitrary second-order filter), it is highly desirable to try to formulate the problem in such a way that some of the difficulties noted above are circumvented.

In the next section, we show how Fokker-Planck equations can be derived for the joint probability density func-

TABLE II. Butterworth filters. $|H_n(f)|^2=1/[1+(2f/B)^{2n}]$ and $\omega_0=\pi B$.

	Filter differential equation	Impulse response
$n=1$	$\frac{dy}{dt}+\omega_0 y=\omega_0 x$	$h(t)=\omega_0 e^{-\omega_0 t}$
$n=2$	$\frac{d^2y}{dt^2}+\sqrt{2}\omega_0\frac{dy}{dt}+\omega_0^2 y=\omega_0^2 x$	$h(t)=2\beta e^{-\beta t}\sin\beta t,\ \ \beta=\omega_0/\sqrt{2}$
$n=3$	$\frac{d^3y}{dt^3}+2\omega_0\frac{d^2y}{dt^2}+2\omega_0^2\frac{dy}{dt}+\omega_0^3 y=\omega_0^3 x$	$h(t)=\omega_0 e^{-\omega_0 t}+\frac{2\omega_0}{\sqrt{3}}e^{-\omega_0 t/2}\cos\left[\frac{\sqrt{3}}{2}\omega_0 t+\frac{\pi}{3}\right]$

tion of the components of the state vector in the general case. We will see that the only quantity necessary in the general formulation is the specification of the function $F_x(\mathbf{z})$, as might be expected from (15).

IV. FOKKER-PLANCK EQUATIONS

Fokker-Planck methods have been used with great success in the study of dynamical systems and transition phenomena in all branches of modern science and engineering.[6–8] Normally, all variables are either discrete or continuous; however, the situation presented by the filtering of the random telegraph signal is one of the mixed type since the input is discrete and the output is continuous. Fokker-Planck equations for this "mixed-variable case" were derived in 1967,[10] but to date have received limited attention.

A. General theory

In order to use Fokker-Planck methods for nth-order systems, it is necessary to consider the joint probability density function of all of the components of the state vector.[6] Furthermore, in the mixed-variable case, it is also necessary to append the state of the discrete process to the probability density function. Thus, we are led to consider joint probability density-probability distribution functions of the form

$$p_k(y,\dot{y},\ldots,y^{(n-1)})=p(y,\dot{y},\ldots,y^{(n-1)}\mid x=k)P\{x=k\},\quad k\in X \tag{16}$$

where X denotes the set of states of the discrete process. For the random telegraph signal, $X=\{-1,+1\}$. Only steady-state behavior will be treated, and so (16) will not be a function of time. The marginal density $p(y)$ of the output can, in principle, be determined from (16) by

$$p(y)=\int d\dot{y}\cdots\int dy^{(n-1)}\sum_{k\in X}p_k(y,\dot{y},\ldots,y^{(n-1)}) . \tag{17}$$

If only $p(y)$ is of interest, this may seem like a circuitous procedure, but to date it is the only known general approach to the problem.

Fokker-Planck—type equations for densities like $p_k(y,\dot{y},\ldots,y^{(n-1)})$ were derived in [Ref. 9, Appendix A, and Ref. 10, Eq. (35)]. Again, using the state variable notation with $z_i=y^{(i-1)}$, $p_k(y,\dot{y},\ldots,y^{(n-1)})=p_k(\mathbf{z})$, these equations in the n-dimensional case are

$$0=\sum_{M}\left\{\prod_{i=1}^{n}\frac{(-1)^{m_i}}{m_i!}\frac{\partial^{m_i}}{\partial z_i^{m_i}}\right\}[A_{\mathbf{m}k}p_k(\mathbf{z})]+\sum_{i\in X}a_{ki}p_i(\mathbf{z}) , \tag{18}$$

where $\mathbf{m}=(m_1,\ldots,m_n)$, the summation is over all integers m_i such that $M\geq 1$, where $M=\sum_{i=1}^{n}m_i$, and the conditional moments are

$$a_{ki}=\lim_{\Delta\to 0}\left\{\frac{1}{\Delta}[P\{x(t+\Delta)=k\mid \mathbf{z}(t),x(t)=i\}-\delta_{ik}]\right\} \tag{19}$$

and

$$A_{\mathbf{m}k}=\lim_{\Delta\to 0}\left\{\frac{1}{\Delta}E\left[\prod_{i=1}^{n}\{z_i(t+\Delta)-z_i(t)\}^{m_i}\mid \mathbf{z}(t),x(t)=x(t+\Delta)=k\right]\right\} . \tag{20}$$

In using these, the conditional moments are first calculated from the physics of the problem; i.e., the filter differential equation and the statistical description of the input process. Although the limits of the sums in (18) are infinity, in most cases of practical interest they will be *two* or less (see Ref. 7, Sec. 4.5, and Ref. 8, Sec. 4.3) since the A_{mk}'s will vanish if any of the m_i's is larger than *two*.

B. The conditional moments

For the random-telegraph-signal input, the state variables in the conditioning in the definition of the a_{ki}'s can be dropped, and we have

$$a_{ki}=\lim_{\Delta\to 0}\left\{\frac{1}{\Delta}[P\{x(t+\Delta)=k\mid x(t)=i\}-\delta_{ik}]\right\}=\begin{cases}-a, & \text{if } i=k ,\\ a, & \text{if } i\neq k ,\end{cases} \tag{21}$$

where, again, a is the average number of traversals of the random telegraph per unit time.

The remaining conditional moments are evaluated by use of the system equations (15). The only nonzero $A_{\mathbf{m}k}$'s will be those which result from the first power of one of the state variables and the zeroth powers of all of the others, so that only first partial derivatives with respect to each of the state variables will be present in (18). These

A_{mk}'s are readily shown to be

$$A_{mk}=\begin{cases}0 \text{ if } M>1, \\ z_{i+1} \text{ if } M=1 \text{ and } m_i=1, \quad i\neq n, \\ F_k(\mathbf{z}) \text{ if } M=1 \text{ and } m_n=1 .\end{cases} \tag{22}$$

C. The Fokker-Planck equations

When the moments (21) and (22) are used in (18), the Fokker-Planck equations for the density of the state variables in the case of the general linear filter are given by the pair

$$\sum_{i=1}^{n-1} z_{i+1}\frac{\partial p_\pm}{\partial z_i}+\frac{\partial}{\partial z_n}(F_\pm p_\pm)+ap_\pm=ap_\mp , \tag{23}$$

where $F_\pm=F_k(\mathbf{z})$ with $k=\pm1$, and $p_\pm=p_k(\mathbf{z})$ with $k=\pm1$. Specific examples of the use of these equations will be given in the following.

D. First-order Butterworth filter

The first-order Butterworth filter is merely the RC filter, and in this case the density $p(y)$ is well known.[3] However, since $p(y)$ is ordinarily not obtained by the method developed herein, it will be useful to consider this case as an illustration. The Fokker-Planck equations follow by using the differential equation from Table II with $F_x(\mathbf{z})=\omega_0 x-\omega_0 y$; viz., with $\alpha=a/\omega_0$,

$$\frac{d}{dy}[(1-y)p_+]+\alpha p_+=\alpha p_- , \tag{24a}$$

$$-\frac{d}{dy}[(1+y)p_-]+\alpha p_-=\alpha p_+ . \tag{24b}$$

Defining $p=p_++p_-$ and $q=p_+-p_-$, adding and subtracting (24a) and (24b), and eliminating q between them leads to

$$\frac{d^2}{dy^2}[(1-y^2)p]+2\alpha\frac{d}{dy}(yp)=0 . \tag{25}$$

For the "boundary conditions" $p(y)=p(-y)$ and that $p(y)$ must integrate to *one*, this has the solution

$$p(y)=\frac{(1-y^2)^{\alpha-1}}{2^{2\alpha-1}B(\alpha,\alpha)} , \tag{26}$$

in which $B(\ )$ denotes the beta function.[3]

E. Second-order Butterworth filter

From Table II we now have $F_x(\mathbf{z})=\omega_0^2x-\omega_0^2y-\sqrt{2}\omega_0\dot{y}$, and (23) gives the pair

$$\dot{y}\frac{\partial p_+}{\partial y}+\frac{\partial}{\partial \dot{y}}[(\omega_0^2-\omega_0^2y-\sqrt{2}\omega_0\dot{y})p_+]+ap_+=ap_- , \tag{27a}$$

$$\dot{y}\frac{\partial p_-}{\partial y}-\frac{\partial}{\partial \dot{y}}[(\omega_0^2+\omega_0^2y+\sqrt{2}\omega_0\dot{y})p_-]+ap_-=ap_+ . \tag{27b}$$

The region of validity of these equations is the interior of the curve shown in Fig. 1. The only apparent "boundary conditions" for the solution of (27a) and (27b) are that $p_\pm(y,\dot{y})$ each integrate to $\frac{1}{2}$, and the symmetry relation

$$p_+(y,\dot{y})=p_-(-y,-\dot{y}) . \tag{28}$$

It is reasonable to expect, from intuition, that $p_+(y,\dot{y})$ should vanish on the boundary in the third quadrant, and that $p_-(y,\dot{y})$ should vanish on the boundary in the first quadrant. However, at present these are conjectures whose validity remains to be established. Even if they were true, they might be difficult to apply in view of the parametric representation of the boundary.

Although the Fokker-Planck equations have been relatively easy to obtain, their solution in all but the first-order case appears formidable. Even if they could be solved, integration of the joint density to get the marginal density, as illustrated in (17), would be complicated because of the parametric representation of the boundary. Consequently, the limits in the integrals of (17) would not be expressible as explicit functions.

TABLE III. Monte Carlo results for second-order Butterworth filter.

			$P\{y\leq Y\}$		
Y	$\alpha=\frac{1}{5}$	$\alpha=\frac{1}{2}$	$\alpha=1$	$\alpha=2$	$\alpha=5$
0	0.500	0.496	0.497	0.501	0.499
0.1	0.513	0.526	0.543	0.575	0.622
0.2	0.530	0.555	0.587	0.641	0.704
0.3	0.543	0.585	0.633	0.699	0.799
0.4	0.557	0.615	0.679	0.757	0.881
0.5	0.575	0.646	0.723	0.812	0.937
0.6	0.589	0.678	0.769	0.866	0.972
0.7	0.608	0.711	0.816	0.912	0.989
0.8	0.628	0.748	0.858	0.950	0.997
0.9	0.656	0.789	0.900	0.981	1.000
0.95	0.670	0.826	0.927	0.988	1.000
1.00	0.798	0.871	0.944	0.995	1.000
1.05	0.937	0.953	0.974	1.000	1.000

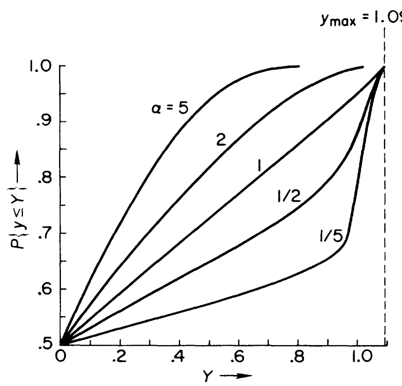

FIG. 2. Monte Carlo results for the output distribution of a second-order Butterworth filter driven by the random telegraph signal.

V. SOME MONTE CARLO RESULTS

On any interval over which the input is constant, say (t_0,t), (6) gives

$$\mathbf{z}(t)=\mathbf{\Phi}(t-t_0)\mathbf{z}(t_0)+\left[\int_0^{t-t_0}\mathbf{\Phi}(\tau)d\tau\right]\mathbf{B}x(t_0+)$$
$$=\mathbf{\Phi}(t-t_0)\mathbf{z}(t_0)+\mathbf{A}^{-1}[\mathbf{\Phi}(t-t_0)-\mathbf{I}]\mathbf{B}x(t_0+)\ . \tag{29}$$

For the second-order Butterworth filter, this equation can be used to simulate $y(t)$ and $\dot{y}(t)$ at points between the traversals of $x(t)$, and from $y(t)$, the probability distribution $P\{y\le Y\}$ estimated as the percentage of time $y(t)$ is below the level Y. Results of the simulations for this case are listed in Table III and are plotted in Fig. 2 for several values of the parameter $\alpha=a/\beta$. The corresponding probability density function $p(y)$ appears to be roughly uniform for α near 1, convex upward for $\alpha<1$, and convex downward (going to zero at $y=\pm y_{\max}$) for $\alpha>1$. This behavior is similar to that in the case of first-order RC filtering; however, the form $p(y)=c(1-y^2/y_{\max}^2)^{\alpha-1}$ for the density is precluded since this form is not consistent with the moments (14a) and (14b).

VI. SUMMARY AND CONCLUSIONS

We have considered in some detail the problem of finding probability densities after higher-order filtering of the random telegraph signal. In order to be concrete, and also since this is one of the simplest higher-order filters, much of our attention was directed to the second-order Butterworth filter. In the higher-order cases, classical methods lead to the consideration of the state vector of the system and the joint probability density function of all of the components of the state vector. Although we were able to formulate Fokker-Planck equations for the general filter, the equations are coupled partial-differential equations in n variables (for the nth-order filter) and are difficult to solve. The solution is further complicated by the irregular shape of the boundary region and its implicit characterization through parametric equations. Furthermore, suitable boundary conditions for the solution are as yet unknown. Analytical determination of the marginal density of the output from all of its moments appears to be intractable because of the complexity of the higher moments.

This is the type of problem that the neophyte would expect to find all worked out in some standard text on probability and random processes. It is hard to imagine a random process as well behaved and as easily characterizable as the random telegraph signal, and intuition strongly suggests that the problem should be tractable. Perhaps this, and the fact that so little progress has been made to date in getting an analytic solution, make the problem more fascinating and intriguing. It is hoped that the Monte Carlo results will provide some insight and guidance for future studies.

ACKNOWLEDGMENTS

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APPENDIX: MOMENTS FOR ARBITRARY SECOND-ORDER SYSTEMS

Equation (14e) can be generalized for an arbitrary second-order filter whose impulse response is the sum of two complex exponentials; viz.,

$$h(t)=Ae^{-iBt}+A^*e^{-iB^*t}\ . \tag{A1}$$

Then, the moments are

$$\overline{y^{2n}}=[(2n)!]\sum_{\text{all }q_j=\pm1}\prod_{k=1}^{2n}\frac{A_r+iq_kA_i}{[1-(-1)^k]a+k\beta_r+i\beta_iS_k}\ , \tag{A2}$$

in which we have expressed A and B in terms of their real and imaginary parts as $A=A_r+iA_i$ and $B=\beta_r+i\beta_i$. When $A_r=0$ and $A_i=\beta_r=\beta_i=\beta$, this reduces to (14e).

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