Gauge-independent Wigner functions. II. Inclusion of radiation reaction

J. Javanainen

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

S. Varró

Central Research Institute for Physics, Hungarian Academy of Sciences, H-1525 Budapest 114, Post Office Box 49, Hungary

O. T. Serimaa

Finnish State Computing Center, Box 40, SF-02101 Espoo 10, Finland

We investigate the effects of quantized radiation reaction fields on the motion of a charged particle using the gauge-independent Wigner operator (GIWO) and gauge-independent Wigner function (GIWF) introduced earlier [Phys. Rev. A 33, 2913 (1986)]. To complement the equation of motion of the GIWO, the Heisenberg equations of motion of the quantized electromagnetic fields are solved within the Markov approximation. After considering the operator orderings and orders of magnitude of the radiation reaction terms, we eliminate the quantum fields from the evolution equation of the GIWO, and obtain for the GIWF a closed equation containing relaxation terms. As an example of the formalism we derive a Fokker-Planck equation (FPE) for the GIWF of a particle in a constant magnetic field. To the order $\hbar^0$ the classical radiation damping ensues, and the first quantum correction proportional to $\hbar$ emerges as diffusion. The diffusion operator turns out to be indefinite and the FPE consequently defies our attempts at a complete analysis, but we demonstrate that at least the coherent states constructed from the Landau levels exhibit a manifestly physical time evolution under the FPE. We point out that the GIWF calculated with quantized electromagnetic fields is divergent even if the fields are in the vacuum state, and suggest that the GIWF should be associated with the particle state by ignoring the quantized fields altogether.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I) we introduced a gauge-invariant Wigner operator (GIWO) and a gauge-independent Wigner function (GIWF), and reported on an extensive study of their properties.

Our starting point was to deal with objects that pertain to the kinetic momentum operator

$$\hat{k} = \hat{p} - Q\hat{A}(\hat{r}) \tag{1.1}$$

rather than to the canonical momentum $\hat{p}$.[2] Here the vector potential A may also contain quantized degrees of freedom of the electromagnetic field,

$$\hat{A}_Q^{(+)}(\hat{r}) = -i \sum_q g(q)\hat{b}_q e^{iq\cdot\hat{r}} ; \tag{1.2}$$

$$g(q) = \left[\frac{\hbar}{2\Omega_q \epsilon_0 V} \right]^{1/2} \epsilon(q) . \tag{1.3}$$

Similarly, the positive frequency parts of the quantized electric and magnetic fields are

$$\hat{E}_Q^{(+)}(\hat{r}) = \sum_q \Omega_q g(q)\hat{b}_q e^{iq\cdot\hat{r}} , \tag{1.4}$$

$$\hat{B}_Q^{(+)}(\hat{r}) = \sum_q q \times g(q)\hat{b}_q e^{iq\cdot\hat{r}} . \tag{1.5}$$

We introduced the generating operator

$$\hat{T}(u,v) = \exp\left[\frac{i}{\hbar}(u\cdot\hat{k} + v\cdot\hat{r}) \right] , \tag{1.6}$$

and the GIWO as the Fourier transform of it,

$$\hat{W}(r,k) = \frac{1}{(2\pi\hbar)^6} \int d^3u\, d^3v$$

$$\times \exp\left[-\frac{i}{\hbar}(u\cdot k + v\cdot r) \right] \hat{T}(u,v) . \tag{1.7}$$

Under the minimal coupling Hamiltonian

$$\hat{H} = \frac{\hat{k}^2}{2M} + Q\hat{\Phi} + \hbar \sum_q \hat{b}_q^\dagger \hat{b}_q \tag{1.8}$$

the Heisenberg equation of motion for the Wigner operator was found to be

$$\left[\frac{\partial}{\partial t} + \tilde{v}\cdot\frac{\partial}{\partial r} + Q\frac{\partial}{\partial k}\cdot(\tilde{v}\times\tilde{B} + \tilde{E}_C) \right] \hat{W}(r,k,t)$$

$$+ Q\frac{\partial}{\partial k}\cdot[\tilde{E}_Q^{(-)}\hat{W}(r,k,t) + \hat{W}(r,k,t)\tilde{E}_Q^{(+)}] = 0 . \tag{1.9}$$

Here the various operators denoted by tildes are related to the electric and magnetic fields, quantized and classical;

$$\tilde{v} = v + \Delta\tilde{v} \equiv \frac{k}{M} - \frac{Q}{M}\frac{\hbar}{i}\frac{\partial}{\partial k} \times \int_{-1/2}^{1/2} d\tau\, \tau \hat{B}\left[r + i\hbar\tau\frac{\partial}{\partial k}\right],$$

(1.10a)

$$\tilde{B} = \int_{-1/2}^{1/2} d\tau\, \hat{B}\left[r + i\hbar\tau\frac{\partial}{\partial k}\right],$$

(1.10b)

$$\tilde{E}_C = \int_{-1/2}^{1/2} d\tau\, E_C\left[r + i\hbar\tau\frac{\partial}{\partial k}\right],$$

(1.10c)

$$\tilde{E}_{\varrho}^{(\pm)} = \int_{-1/2}^{1/2} d\tau\, \hat{E}_{\varrho}^{(\pm)}\left[r + i\hbar\tau\frac{\overleftarrow{\partial}}{\partial k}\right].$$

(1.10d)

The GIWF, finally, was defined as the expectation value of the GIWO. In the Heisenberg picture it is

$$W(r,k,t) = \text{Tr}[\hat{W}(r,k,t)\hat{\rho}(t_0)].$$

(1.11)

We showed that a Weyl correspondence analogous to the ordinary Wigner functions can be set up for the GIWF: Expectation values of Weyl-ordered, i.e., totally symmetrized, functions of $(\hat{r}_1,\hat{r}_2,\hat{r}_3,\hat{k}_1,\hat{k}_2,\hat{k}_3)$ can be calculated from the GIWF as if it were a classical phase-space density. Hence the GIWF can be qualitatively regarded as a phase-space distribution function for the position and kinetic momentum variables r and k. As a first example of the use of the equation of motion (1.9) and (1.10) we treated the case when all fields are classical. Then a closed equation of motion for the GIWF is obtained by just multiplying (1.9) by the initial density operator $\hat{\rho}(t_0)$ and taking the trace. In the light of the results we briefly discussed how photon recoil effects enter the dynamics.

In this paper we expand on the feature of our GIWO that it can also incorporate quantized fields. Basically, to supplement the equation of motion of the GIWO we first derive and formally solve the Heisenberg equations of motion for the electromagnetic fields. The fields are then inserted into the GIWO equation of motion, the result is multiplied by the Heisenberg-picture density operator and, after suitable approximations, a closed equation of motion is obtained for the GIWF. As a concrete example of the rather formal development we shall discuss a charged particle in a constant external magnetic field, including the action of the quantized radiation reaction field back on the particle.

Closely related Heisenberg-picture operator techniques for an electron bound in an atom are widely utilized in quantum optics,[3-6] and have also been applied to free electrons.[7] We advocate the GIWF in free-electron problems firstly because it describes the state of the electron in a compact manner facilitating classical analogs and solutions of quantum problems as $\hbar$ expansions around the classical solutions. Secondly, the gauge of the electromagnetic field does not affect the results or their interpretation, even when approximations are made.

We begin in Sec. II by introducing yet another operator $\hat{P}$ closely related to the GIWO, which we have found exceedingly helpful in problems of radiation reaction. In Sec. III we obtain expressions for the electromagnetic fields by solving their Heisenberg equations of motion under the Markov approximation. Operator orderings and orders of magnitude of the various radiation reaction effects are discussed to the extent required for the derivation of the equation of motion for the GIWF, which is finally outlined. In Sec. IV we inititate our example of the constant magnetic field by deriving for the GIWF a Fokker-Planck equation (FPE) that explicitly displays the radiative damping and quantum fluctuations up to first order in $\hbar$. The FPE turns out to be mathematically ill-behaved and so far defies our attempts at a complete analysis, but mostly by considering the coherent states built from the Landau levels we have found illustrative examples of its use. These are presented in Sec. V. Section VI contains a discussion and summary. In Appendix A we study a special commutator needed in the derivation of Sec. III. Appendix B presents a comparison between the Wigner functions when either no quantum field is present, or the quantum field is in the vacuum state.

II. THE P REPRESENTATION

Before embarking on the treatment of the case with quantized radiation fields we convert the Heisenberg equation of motion of the Wigner operator, (1.9) and (1.10), into an equation for the operator

$$\hat{P}(r,u) = \int d^3k\, e^{(i/\hbar)u\cdot k}\hat{W}(r,k)$$

$$= \frac{1}{(2\pi\hbar)^3}\int d^3v\, e^{-(i/\hbar)v\cdot r}\hat{T}(u,v).$$

(2.1)

This operator is halfway between the Wigner operator $\hat{W}$, (1.7), and its generating operator $\hat{T}$, (1.6), in that the Fourier transform over u is left out. Correspondingly, derivatives acting on the k label (multiplicative factors k) of the Wigner operator transform into multiplicative factors u (derivatives with respect to u) of the operator $\hat{P}$:

$$k \leftrightarrow \frac{\hbar}{i}\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial k} \leftrightarrow -\frac{i}{\hbar}u.$$

(2.2)

The Heisenberg equation of motion for P is thus found by rewriting (1.9) and (1.10):

$$\left[\frac{\partial}{\partial t} + \tilde{v}\cdot\frac{\partial}{\partial r} - \frac{iQ}{\hbar}u\cdot(\tilde{v}\times\tilde{B} + \tilde{E}_C)\right]\hat{P}(r,u)$$

$$-\frac{iQ}{\hbar}u\cdot[\tilde{E}_{\varrho}^{(-)}\hat{P}(r,u) + \hat{P}(r,u)\tilde{E}_{\varrho}^{(+)}] = 0,$$

(2.3)

with

$$\tilde{v} = \frac{\hbar}{iM}\frac{\partial}{\partial u} + \frac{Q}{M}u \times \int_{-1/2}^{1/2} d\tau\,\tau\hat{B}(r+\tau u)$$

$$\equiv \frac{\hbar}{iM}\frac{\partial}{\partial u} + \Delta\tilde{v} , \tag{2.4a}$$

$$\tilde{B} = \int_{-1/2}^{1/2} d\tau\,\hat{B}(r+\tau u) , \tag{2.4b}$$

and similarly for the classical and quantized parts of the electric field.

By virtue of the Baker-Campbell-Hausdorff (BCH) formula (2.25b) of I, the definition (2.1) may be written

$$\hat{P}(r,u) = \frac{1}{(2\pi\hbar)^3} \int d^3v\, e^{(i/2\hbar)u\cdot\hat{p}}\exp\left[\frac{i}{\hbar}\left[v\cdot(\hat{r}-r)-Qu\cdot\int_{-1/2}^{1/2} d\tau\,\hat{A}(\hat{r}+\tau u)\right]\right]e^{(i/2\hbar)u\cdot\hat{p}} . \tag{2.5}$$

Moreover, comparison with (4.15) of I shows that the argument $\hat{r}$ of the vector potential A inside (2.5) can be replaced with the label r without changing the result. From (2.5) it is then easy to see that in the case when all fields are classical, the expectation value of the operator $\hat{P}$ is

$$P(r,u) = \mathrm{Tr}[\hat{P}(r,u)\hat{\rho}]$$

$$= \exp\left[\frac{i}{\hbar}Qu\cdot\int_{-1/2}^{1/2} d\tau\,A(r+\tau u)\right]$$

$$\times\langle r+\tfrac{1}{2}u\,|\,\hat{\rho}\,|\,r-\tfrac{1}{2}u\rangle . \tag{2.6}$$

Here the connection of the operator $\hat{P}$ to the position representation of the density operator is clearly displayed.

One reason why the P representation turns out to be very useful can be seen from an equation that may be derived just like (4.16) of I is obtained from (4.13) of I:

$$F(r)\hat{P}(r,u) = F(\hat{r}+\tfrac{1}{2}u)\hat{P}(r,u)$$

$$= \hat{P}(r,u)F(\hat{r}-\tfrac{1}{2}u)$$

$$= \hat{P}(r,u)F(r) \tag{2.7}$$

applies to an arbitrary function $F(r)$. Thus, the position argument of the fields in the equation of motion (2.3) and (2.4) can be treated as a classical label or an operator, whichever is more convenient in the problem at hand.

III. EQUATION OF MOTION OF THE WIGNER FUNCTION FOR QUANTIZED FIELDS

When only classical electromagnetic fields are taken into account, an equation of motion for the GIWF can be derived simply by multiplying the operator equation of motion of the GIWO by the density operator and tracing. However, since we have found no general expression of, say, $\mathrm{Tr}[\hat{W}(t)\hat{E}^{(+)}(t)\hat{\rho}(t_0)]$ in terms of $\mathrm{Tr}[\hat{W}(t)\hat{\rho}(t_0)]$, we have not been able to carry through the same procedure when quantized fields are present. Instead, we devise here another procedure which, under precisely stated approximations, still leads to a closed equation of motion for the GIWF.

In Sec. III A we split up the quantized field into free-field and radiation reaction components, and express the latter in terms of the kinetic momentum operator $\hat{k}$. The

next task, in Sec. III B, is to reorder the terms in the equation of motion of the GIWO in such a way that the positive (negative) frequency parts of the free-field operators stand furthest to the right (left). As a result, the free-field components do not contribute when the quantized field is initially in the vacuum state. In Sec. III C we estimate the radiation reaction terms in the equation of motion, and demonstrate that under reasonable conditions the ones entering (2.3) through the quantized electric field dominate. We have thus argued that only the radiation reaction component of the quantized electric field needs to be retained, and that it essentially is proportional to $\hat{k}$. We then complete our program by showing in Sec. III D how to express $\mathrm{Tr}[\hat{k}(t)\hat{W}(t)\rho(t_0)]$ and $\mathrm{Tr}[\hat{W}(t)\hat{k}(t)\hat{\rho}(t_0)]$ in terms of the GIWF.

A. Derivation of radiation reaction fields

In this section we shall derive an explicit expression for the quantized electromagnetic fields from the full Hamiltonian (1.8) under certain approximations, the most important one being the Markov approximation. The method, adapted from Ackerhalt and Eberly,[3] is by now a standard tool in quantum optics. Our brief exposition therefore concentrates on the aspects peculiar to an unbound particle.

We begin by writing the positive frequency parts of the quantized electric and magnetic fields in the form

$$\hat{E}^{(+)}(\hat{r}+u) = \sum_q g(q)\Omega_q e^{iq\cdot u}\hat{O}_q , \tag{3.1}$$

$$\hat{B}^{(+)}(\hat{r}+u) = \sum_q [q\times g(q)]e^{iq\cdot u}\hat{O}_q , \tag{3.2}$$

where the operators $\hat{O}_q$ are defined by

$$\hat{O}_q = e^{iq\cdot\hat{r}}\hat{b}_q . \tag{3.3}$$

To find the operator $\hat{O}_q(t)$ we start from its Heisenberg equation of motion under the Hamiltonian (1.8),

$$\dot{\hat{O}}_q = -\frac{i}{\hbar}[\hat{O}_q,\hat{H}]$$

$$= -i\Omega_q\hat{O}_q + \frac{i}{2}\left[\frac{q\cdot\hat{k}}{M}\hat{O}_q + \hat{O}_q\frac{q\cdot\hat{k}}{M}\right] - \frac{Q}{\hbar M}g(q)\cdot\hat{k} . \tag{3.4}$$

The operator $q \cdot \hat{k}/M$ in (3.4) clearly represents the Doppler shift. We ignore the Doppler shift (we comment on this approximation later), and integrate (3.4);

$$\hat{O}_q(t) = e^{-i\Omega_q(t-t_0)}\hat{O}_q(t_0)$$
$$- \frac{Q}{\hbar M}g(q)\cdot \int_{t_0}^{t} dt' e^{-i\Omega_q(t-t')}\hat{k}(t') \ . \quad (3.5)$$

The electric field at time t then becomes

$$\hat{E}^{(+)}(\hat{r}+u,t) = \hat{E}_F^{(+)}(\hat{r}+u,t) + \hat{E}_R^{(+)}(\hat{r}+u,t) \ , \quad (3.6)$$

where the "free" component is

$$\hat{E}_F^{(+)}(\hat{r}+u,t) = \sum_q g(q)\Omega_q e^{-i\Omega_q(t-t_0)}e^{iq\cdot[\hat{r}(t_0)+u]}\hat{b}_q(t_0) \ ,$$

$$\quad (3.7a)$$

and the radiation reaction field

$$\hat{E}_R^{(+)}(\hat{r}+u,t)$$
$$= -\frac{Q}{\hbar M}\sum_q \Omega_q e^{iq\cdot u}g(q)g(q)\cdot \int_{t_0}^{t} dt' e^{-i\Omega_q(t-t')}\hat{k}(t') \ .$$

$$\quad (3.7b)$$

To develop the radiation reaction field further we introduce three approximations.

(1) The Markov approximation. Since the spectrum of the electromagnetic field is broad, (3.7b) conveys oscillations at wildly varying frequencies. We assume that the oscillations sum up destructively in such a way that only the times $t' \approx t$ contribute to the integral. In particular, the lower limit of integration t_0 may be replaced with $-\infty$.

(2) The Born approximation. Instead of the full time-dependent solution for $\hat{k}(t)$, we employ in (3.7b) the backward solution [with the final condition $\hat{k}(t') = \hat{k}(t)$ at $t' = t$] of the equation of motion for $\hat{k}(t)$ where only the classical fields are retained. This replacement ought to apply during the short time interval before $t' = t$ that is sampled in (3.7b).

(3) The secular approximation. We assume that, at least for the times t' immediately preceding t, the solution to the equation of motion with only the classical fields is of the form

$$\hat{k}(t') = \sum_\alpha [\hat{k}_\alpha^{(+)}(t)e^{i\omega_\alpha(t-t')} + \text{H.c.}] \ . \quad (3.8)$$

In other words, we demand that it is possible to break up $\hat{k}(t')$ into components oscillating at (a limited number of well-separated) frequencies ω_α. A particle must experience an acceleration in order to radiate; we assume that the acceleration is due to a superposition of oscillatory motions.

With (3.8), (3.7b) contains integrals of the form

$$\int_{-\infty}^{t} dt' e^{-i\Omega_q(t-t')}e^{\pm i\omega_\alpha(t-t')}$$

$$= \pi\delta(\Omega_q \mp \omega_\alpha) - \text{P}\left[\frac{1}{\Omega_q \mp \omega_\alpha}\right] \ . \quad (3.9)$$

When the sum over the photon modes in (3.7b) is carried out, the principal-value part of (3.9) results in a divergence. On account of the fact that this divergence reflects the shape of the spectrum of the electromagnetic field rather than the frequencies ω_α, we assume that it can be removed by a suitable renormalization of the mass and charge of the particle. Only a small frequency-dependent mass shift is expected to remain, which we henceforth ignore. We thus neglect the whole principal-value integral, and obtain from (3.7b)

$$\hat{E}_R^{(+)}(\hat{r}+u,t)$$
$$= -\frac{\pi Q}{\hbar M}\sum_{q,\alpha}\Omega_q e^{-iq\cdot u}\delta(\Omega_q - \omega_\alpha)g(q)g(q)\cdot \hat{k}_\alpha^{(+)}(t) \ .$$

$$\quad (3.10)$$

We then carry out the polarization sum implicit in the sum over q, and replace the sum over the photon modes with an integral. The radiation reaction field, in component form, finally reads

$$\hat{E}_R^{(+)}(\hat{r}+u,t)_i = -\sum_{j,\alpha}\frac{Q\omega_\alpha^2}{6\pi\epsilon_0 Mc^3}W_{ij}^\alpha(u)\hat{k}_\alpha^{(+)}(t)_j \ , \quad (3.11a)$$

where the tensor $W_{ij}^\alpha(u)$ is

$$W_{ij}^\alpha(u) = \frac{3}{8\pi}\int d^2\Omega(\delta_{ij} - n_i n_j)e^{i(\omega_\alpha/c)(n\cdot u)} \ . \quad (3.11b)$$

The integral runs over the unit sphere, and n is the corresponding dummy vector.

As ω_α/c is the wave number of a photon with frequency ω_α, we conclude that the u dependence of the tensor W corresponds to photon recoil effects. When either $c \to \infty$ or $u \to 0$, $W_{ij}^\alpha(u) \to \delta_{ij}$. We loosely formulate the conclusion that in the absence of recoil effects the force due to the component of the radiation field at the frequency ω_α,

$$Q\hat{E}_{R,\alpha}^{(\pm)} = -\frac{Q^2\omega_\alpha^2}{6\pi\epsilon_0 Mc^3}\hat{k}_\alpha^{(\pm)} \ , \quad (3.12)$$

opposes the motion at the same frequency.

The positive (and negative) frequency part of the magnetic field can be calculated in the same way. The free-field component, of course, is

$$\hat{B}_F^{(+)}(\hat{r}+u,t) = \sum_q [q \times g(q)]e^{-i\Omega_q(t-t_0)}$$
$$\times e^{iq\cdot[\hat{r}(t_0)+u]}\hat{b}_q(t_0) \ , \quad (3.13)$$

and the radiation reaction field, in component form, reads

$$\hat{B}_R^{(+)}(\hat{r}+u,t)_i = -\sum_{\alpha,j}\frac{Q\omega_\alpha^2}{6\pi\epsilon_0 Mc^4}T_{ij}^\alpha(u)\hat{k}_\alpha^{(+)}(t)_j \ , \quad (3.14a)$$

where the tensor T is defined by

$$T_{ij}^\alpha(u) = \frac{3}{8\pi}\sum_k \epsilon_{ijk}\int d^2\Omega \, n_k e^{i(\omega_\alpha/c)(n\cdot u)} \ . \quad (3.14b)$$

When recoil effects are neglected ($u = 0$), the magnetic component of the radiation reaction field is simultaneously forced to zero (within our premises, e.g., nonrelativistic

motion of the particle).

We conclude this section with a discussion of some of the approximations. First, it can be seen that the δ function in (3.10) corresponds to momentum conservation by stating that the radiation field only contains wave vectors lying on the sphere $|q| = \omega_\alpha/c$. However, even if the particle is excited by an external field at the frequency ω_α, due to the Doppler shift the scattered field may contain other frequencies than ω_α. If the motion of the particle were treated classically, the Doppler shift would result in a slight deformation of the momentum sphere inside the q sum in (3.10), and the radiation reaction field would acquire relativistic v/c corrections. It is such corrections we have neglected going from (3.4) to (3.5).

The Born approximation is justified as long as the radiation reaction field does not modify the motion of the particle appreciably in the characteristic times determined by ω_α^{-1}.

Finally, the secular approximation is valid if the external classical fields make the particle oscillate at discrete frequencies. As a result, "hard" photons at these frequencies are sent out. The cyclotron motion of a particle in a magnetic field and the forced oscillations of a particle in a monochromatic radiation field provide two examples. At first sight it may seem that invoking characteristic frequencies imposed by classical fields sharply distinguishes the present situation from the case of electrons bound in an atom. However, the characteristic frequencies of an atom also result from electromagnetic interactions, and to zeroth order the associated fields are usually taken to be classical. A true counterexample to the secular approximation is the scattering of the particle from a center of force. Then "soft" photons with a continuous range of frequencies are emitted.[8]

B. Ordering of field operators

In the sequel we always assume that the particle is primarily driven by fully classical fields, i.e., that at the ini-

tial time t_0 the quantized field is empty. Accordingly, the field operators and the density operator satisfy

$$\hat{\rho}(t_0)\hat{b}^\dagger(t_0) = \hat{b}(t_0)\hat{\rho}(t_0) = 0 . \tag{3.15}$$

If the quantum field terms in the equation of motion of the GIWO, (1.9) or (2.3), can be brought to such a form that all positive frequency parts stand to the right and negative frequency parts to the left, the free-field terms do not contribute when (1.9) or (2.3) is multiplied by the density operator and the trace is taken. We study the quantum field contributions to the equation of motion term by term from this standpoint using the P representation.[9] We avoid premature splitting of the fields into free and radiation reaction components in order to rely on the boson commutators which are necessarily preserved in the unitary time evolution.[10]

The terms involving quantized fields in (2.3) are the one due to the quantized electric field,

$$\hat{T}_E(r,u) = -\frac{iQ}{\hbar}u\cdot[\tilde{E}_Q^{(-)}(r,u)\hat{P}(r,u) + \hat{P}(r,u)\tilde{E}_Q^{(+)}(r,u)] , \tag{3.16a}$$

the one corresponding to the magnetic component of the Lorentz force

$$\hat{T}_B(r,u) = -\frac{Q}{M}u\cdot\left[\frac{\partial}{\partial u}\times[\tilde{B}^{(+)}(r,u)+\tilde{B}^{(-)}(r,u)]\right]\hat{P}(r,u), \tag{3.16b}$$

the quantum Δv correction to the convective derivative

$$\hat{T}_{\Delta v}(r,u) = [\Delta\tilde{v}^{(-)}(r,u) + \Delta\tilde{v}^{(+)}(r,u)]\cdot\frac{\partial}{\partial r}\hat{P}(r,u) , \tag{3.16c}$$

and the term reflecting the Δv correction to the magnetic component of the Lorentz force,

$$\hat{T}_{\Delta v\times B}(r,u) = -\frac{iQ}{\hbar}u\cdot\{[\Delta\tilde{v}^{(+)}(r,u) + \Delta\tilde{v}^{(-)}(r,u)]\times[\tilde{B}^{(+)}(r,u)+\tilde{B}^{(-)}(r,u)]\}\hat{P}(r,u) . \tag{3.16d}$$

The term $\hat{T}_E$ already is in the desired order, but none of the magnetic field contributions has the positive frequency parts to the right. We demonstrate the procedures needed to move them to the proper position, using $\hat{T}_B$ as the detailed example.

Thus, let us study the commutator $[\tilde{B}^{(+)},\hat{P}]$. Using the form (2.5) for $\hat{P}$ and (2.4b) for $\tilde{B}$ it can be seen immediately that the commutator vanishes if

$$\hat{C} = \left[\int_{-1/2}^{1/2}d\tau\,\hat{B}^{(+)}(r+\tau u)_i, \int_{-1/2}^{1/2}d\tau\,\hat{A}^{(-)}(r+\tau u)_j\right] = 0 . \tag{3.17}$$

From the explicit expressions of the quantum fields (1.2) and (1.5) we obtain

$$\hat{C} = \sum_q [q\times g(q)]_i g(q)_j \left[f\left[\frac{q\cdot u}{2}\right]\right]^2 , \tag{3.18a}$$

$$f(x) = \frac{\sin x}{x} . \tag{3.18b}$$

But the function of q inside the sum changes sign when q is inverted, hence the sum vanishes. This is basically the method that can be employed to prove the commutators

$$[\tilde{B}^{(\pm)},\hat{P}] = 0, \quad [\Delta\tilde{v}^{(\pm)},\hat{P}] = 0, \quad [\Delta\tilde{v}^{(\pm)},\tilde{B}^{(\mp)}] = 0 . \tag{3.19}$$

Hence, $\Delta\tilde{v}^{(+)}$ and $\tilde{B}^{(+)}$ can be moved freely to the right of $\hat{P}$ in $\hat{T}_B$ and $\hat{T}_{\Delta v}$.

The remaining $\hat{T}_{\Delta v \times B}$ term may be developed to the form

$$\hat{T}_{\Delta v \times B} = -\frac{iQ}{\hbar} u \cdot [(\Delta\tilde{v}^{(-)} \times \tilde{B}^{(-)})\hat{P}$$

$$+ \Delta\tilde{v}^{(-)}(\hat{P}\tilde{B}^{(+)})$$

$$- \tilde{B}^{(-)} \times (\hat{P}\Delta\tilde{v}^{(+)})$$

$$+ \hat{P}(\Delta\tilde{v}^{(+)} \times \tilde{B}^{(+)})] . \qquad (3.20)$$

Unfortunately, the first and fourth terms in this expansion introduce another problem. To see this, display in $\Delta\tilde{v}^{(+)}$ and $\tilde{B}^{(+)}$ the free-field and radiation reaction components separately:

$$\Delta\tilde{v}^{(+)} \times \tilde{B}^{(+)} = (\Delta\tilde{v}_F^{(+)} + \Delta\tilde{v}_R^{(+)}) \times (\tilde{B}_F^{(+)} + \tilde{B}_R^{(+)})$$

$$= \Delta\tilde{v}_R^{(+)} \times (\tilde{B}_F^{(+)} + \tilde{B}_R^{(+)})$$

$$+ \Delta\tilde{v}_F^{(+)} \times \tilde{B}_F^{(+)} + \Delta\tilde{v}_F^{(+)} \times \tilde{B}_R^{(+)} . $$

$$(3.21)$$

In the last term the free-field component does not appear furthest to the right. To study the resulting circumstance in more detail we first restore the operator argument $\hat{r}$ in $\Delta\tilde{v}^{(+)}$ and $\tilde{B}^{(+)}$ using (2.7),

$$\hat{P}(r,u)\Delta\tilde{v}_F^{(+)}(r,u) \times \tilde{B}_R^{(+)}(r,u)$$

$$= \hat{P}(r,u)\Delta\tilde{v}_F^{(+)}(\hat{r} - \tfrac{1}{2}u, u) \times \tilde{B}_R^{(+)}(\hat{r} - \tfrac{1}{2}u, u) . \quad (3.22)$$

From (2.4a) and (3.13) we obtain

$$\Delta\tilde{v}_F^{(+)}(\hat{r} - \tfrac{1}{2}u, u; t) = \sum_q \mathscr{A}_q(u) e^{iq\cdot\hat{r}(t_0)} \hat{b}_q(t_0) , \quad (3.23)$$

and from (2.4b) and (3.14) it follows that

$$\tilde{B}_R^{(+)}(\hat{r} - \tfrac{1}{2}u, u; t) = \sum_\alpha \mathscr{B}^\alpha(u)\hat{k}_\alpha^{(+)}(t) . \quad (3.24)$$

Here $\mathscr{A}_q(u)$ is a vector and $\mathscr{B}^\alpha(u)$ a tensor, whose precise forms are irrelevant to the argument.

The main problem with (3.22) is that Heisenberg operators with different time arguments appear in $\Delta\tilde{v}_F^{(+)}$ and $\tilde{B}_R^{(+)}$. We have not been able to prove that these operators commute in general. Nevertheless, in Appendix A we demonstrate that if the Markov approximation is made and the amplitude of the forced motion of the particle is small compared to the wavelength of the driving classical field, then

$$[\hat{b}_q(t_0), \hat{k}_\alpha^{(+)}(t)] = 0 . \qquad (3.25)$$

In view of (3.22)—(3.24) this justifies moving all field operators $\hat{b}_q(t_0)$ furthest to the right, so the troublesome terms in (3.22) give zero when it multiplies a density operator satisfying (3.15). As our intention is to derive an equation of motion for the GIWF for precisely such density operators, we may in practice write

$$\hat{T}_{\Delta v \times B} = -\frac{iQ}{\hbar} u \cdot [(\Delta\tilde{v}_R^{(-)} \times \tilde{B}_R^{(-)})\hat{P}$$

$$+ \Delta\tilde{v}_R^{(-)}(\hat{P}\tilde{B}_R^{(+)})$$

$$- \tilde{B}_R^{(-)} \times (\hat{P}\Delta\tilde{v}_R^{(+)})$$

$$+ \hat{P}(\Delta\tilde{v}_R^{(+)} \times \tilde{B}_R^{(+)})] . \qquad (3.26)$$

In conclusion, in the equation of motion of the GIWO, (2.3) and (2.4), we may split the field operators into positive and negative frequency parts, and move the former freely to the right of the operator P. Moreover, in theories where (3.25) is satisfied, all free-field terms may be neglected after this ordering, provided the equation of motion is used in conjunction with a density operator with the property (3.15).

C. Estimates of the radiation reaction terms

Our next task is to estimate the contributions from the radiation reaction fields to the equation of motion of the GIWO, and isolate the leading ones. We fix a characteristic frequency of the secular motion ω, and denote the typical scale of variations of the GIWO with k and r by Δk and Δr. Correspondingly, we use in (2.3) or (3.16) the estimates

$$u \sim \frac{\hbar}{\Delta k}, \quad r \sim \Delta r, \quad \frac{\partial}{\partial u} \sim \frac{\Delta k}{\hbar}, \quad \frac{\partial}{\partial r} \sim \frac{1}{\Delta r} . \quad (3.27)$$

We shall always assume that the width of the k distribution greatly exceeds the recoil momentum of the emitted photons,

$$\Delta k \gg \hbar\omega/c , \qquad (3.28a)$$

or

$$\omega u/c \ll 1 . \qquad (3.28b)$$

Let E be the scale of $\tilde{E}_R^{(+)}$. In view of (3.28b), we obtain from (3.11) and (3.14) the estimate

$$\tilde{B}_R^{(+)} \sim \frac{E}{c}\frac{\omega}{c}u \sim E\frac{\hbar\omega}{\Delta k c^2} . \qquad (3.29)$$

Similarly, the quantum correction to the velocity $\Delta\tilde{v}$ in (2.4a) is roughly

$$\Delta\tilde{v}^{(+)} \sim \frac{\hbar Q}{M\Delta k} \cdot \frac{\omega\tilde{B}_R^{(+)}}{c} \cdot \frac{\hbar}{\Delta k} \sim \frac{\hbar^3 \omega^2 QE}{M(\Delta k)^3 c^3} . \quad (3.30)$$

Here $(\omega/c)\tilde{B}_R^{(+)}$ stands for $(\partial/\partial r)\hat{B}_R^{(+)}(r)$, obviously representing the lowest nonvanishing contribution to the integral in (2.4a). We may now write down order-of-magnitude expressions for the radiation reaction terms (3.16):

$$\hat{T}_E \sim \frac{Qu}{\hbar}\hat{E}_R^{(+)}\hat{P} \sim \frac{QE}{\Delta k}\hat{P} , \qquad (3.31a)$$

$$\hat{T}_B \sim \frac{Q}{M}u\frac{\partial}{\partial u}\tilde{B}_R^{(+)}\hat{P} \sim \frac{Q}{M}\tilde{B}_R^{(+)}\hat{P} \sim \frac{\hbar\omega}{Mc^2}\hat{T}_E , \quad (3.31b)$$

$$\hat{T}_{\Delta v} \sim \Delta \tilde{v}^{(+)} \cdot \frac{\partial}{\partial r} \hat{P} \sim \frac{\hbar \omega}{Mc^2} \cdot \frac{\hbar \omega}{c \, \Delta k} \cdot \frac{\hbar}{\Delta k \, \Delta r} \hat{T}_E$$

$$\sim \frac{\hbar \omega}{c \, \Delta k} \frac{\hbar}{\Delta k \, \Delta r} \hat{T}_B , \qquad (3.31c)$$

$$\hat{T}_{\Delta v \times B} \sim \frac{M \, \Delta \tilde{v}^{(+)}}{\Delta k} \hat{T}_B$$

$$\sim \left[\frac{\hbar \omega}{\Delta kc} \right]^2 \left[\frac{\hbar \omega}{Mc^2} \right] \left[\frac{QEc/\omega}{(\Delta k)^2/M} \right] \hat{T}_B$$

$$\sim \alpha \left[\frac{\hbar \omega}{\Delta kc} \right]^3 \left[\frac{\hbar \omega}{Mc^2} \right] \hat{T}_B . \qquad (3.31d)$$

In (3.31d) we have used (3.11) to express E in terms of $k \sim \Delta k$; $\alpha = Q^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant if $Q = e$.

The magnetic field term $\hat{T}_B$ is smaller than the electric field term $\hat{T}_E$ by the factor $\hbar \omega / Mc^2$. Next, by quantum mechanics $\Delta k \, \Delta r \geq \hbar$, and as a result of (3.28a) $\hat{T}_{\Delta v} \ll \hat{T}_B$.

Finally, $\hat{T}_{\Delta v \times B} \ll \hat{T}_B$. If the parameters $\hbar \omega / Mc^2$ and $\hbar \omega / c \, \Delta k$ are both much smaller than unity, $\hat{T}_E$ is the largest radiation reaction contribution to the equation of motion (2.3). In the sequel we always assume this, and retain *only* $\hat{T}_E$.

D. Elimination of radiation reaction fields

The dominating radiation reaction term in the equation of motion of the GIWO is the one depending on the electric field,

$$\hat{T}_E = Q \frac{\partial}{\partial k} \cdot (\tilde{E}_R^{(-)} \hat{W} + \hat{W} \tilde{E}_R^{(+)}) .$$

By virtue of (3.11), the field $\tilde{E}_R^{(+)}$ ($\tilde{E}_R^{(-)}$) may be expressed in terms of the positive (negative) frequency components $\hat{k}_\alpha^{(+)}$ ($\hat{k}_\alpha^{(-)}$) of $\hat{k}$. It thus pays to study the products $\hat{k}_i \hat{P}$ and $\hat{P} \hat{k}_i$.

From Eqs. (2.26a) and (4.30b) of I, it follows immediately that

$$\hat{k}_i \hat{T}(u,v) = e^{(i/2\hbar)u \cdot v} e^{(i/\hbar)v \cdot \hat{r}} \left[v_i + \frac{\hbar}{i} \frac{\partial}{\partial u_i} - Q \int_0^1 d\tau (1-\tau)[u \times \hat{B}(\hat{r} + \tau u)]_i \right] e^{(i/\hbar)u \cdot \hat{k}} . \qquad (3.32)$$

By partial integrations the result transforms into the P representation as

$$\hat{k}_i \hat{P}(r,u) = \left[\frac{\hbar}{i} \frac{\partial}{\partial u_i} - \frac{\hbar}{2i} \frac{\partial}{\partial r_i} - Q \int_0^1 d\tau (1-\tau)[u \times \hat{B}(\hat{r} + \tau u)]_i \right] \hat{P}(r,u) . \qquad (3.33)$$

Similarly,

$$\hat{P}(r,u) \hat{k}_i = \left[\frac{\hbar}{i} \frac{\partial}{\partial u_i} + \frac{\hbar}{2i} \frac{\partial}{\partial r_i} + Q \int_0^1 d\tau (1-\tau)[u \times \hat{B}(\hat{r} + \tau u)]_i \right] \hat{P}(r,u) . \qquad (3.34)$$

Since in $\hat{B}$ the quantized fields are also included, a procedure similar to that in Secs. III B and III C is executed next. First, by a similar argument as before, the positive frequency part of $\hat{B}$ may be moved to the right of $\hat{P}$, allowing us to neglect the free-field component. Second, the leading terms in (3.33) are of the order of Δk, $\hbar / \Delta r$, and whatever the integral may give for the classical part of the magnetic field. To the order of magnitude, the ratio of the integral of the radiation reaction field $\hat{B}_R$ to the first term inside the large parentheses in (3.33) can be estimated to be

$$\alpha \frac{\hbar \omega}{Mc^2} \cdot \left[\frac{\hbar \omega}{c \, \Delta k} \right]^2 .$$

For consistency, all radiation-reaction contributions to the integrals in (3.33) and (3.34) must thus be neglected.

Assume now that it is possible to express $\hat{k}_\alpha^{(\pm)}(t)$, and hence $\hat{E}_R^{(\pm)}(t)$, in terms of the Cartesian components $\hat{k}_i(t)$. The only remaining radiation reaction terms in the equation of motion (2.3) are of the form of the right-hand sides of Eqs. (3.33) and (3.34). When the equation of motion (2.3) is subsequently multiplied by a density opera-

tor satisfying (3.15), the free-field component of $\hat{B}$ as in (3.33) and (3.34) drops out and the radiation reaction component of $\hat{B}$ is negligible. In addition, (2.7) can be used to convert the operator argument $\hat{r}$ of the classical part of $\hat{B}$ to the label r. Consequently, the whole radiation reaction is accounted for by an operator that only acts on the labels of the operator $\hat{P}$. When finally the trace is taken, a closed equation of motion obtains for $P(r,u)$, and hence for the GIWF $W(r,k)$. We have formally completed the objective of Sec. III.

IV. EXAMPLE: FOKKER-PLANCK EQUATION IN A CONSTANT MAGNETIC FIELD

To give a concrete example of the formalism and methods introduced in the preceding sections we shall treat in detail the motion of a charged particle in a constant magnetic field. We write the field in the form $B = be^3$, with e^i denoting the unit vector in the direction $i = 1$, 2, or 3, and to avoid notational complications assume that $b, Q > 0$. Then the cyclotron frequency $\omega_c = Qb/M$ is positive. When only the electric component of the radiation reaction field is kept, the equation of

motion of the GIWO reads

$$\left[\frac{\partial}{\partial t} + \frac{k}{M} \cdot \frac{\partial}{\partial r} + \omega_c (k \times e^3) \cdot \frac{\partial}{\partial k} \right] \hat{W}$$

$$+ Q \frac{\partial}{\partial k} \cdot (\widetilde{E}_R^{(-)} \hat{W} + \hat{W} \widetilde{E}_R^{(+)}) = 0 . \quad (4.1)$$

The task is to develop further the last two terms in (4.1).

When only the constant magnetic field is present, the Heisenberg equation of motion for $\hat{k}(t')$ [e.g., from (A1a)] reads

$$\frac{d}{dt'} \hat{k}(t') = \omega_c \hat{k}(t') \times e^3 . \quad (4.2)$$

The solution to this equation with the final condition $\hat{k}(t') = \hat{k}(t)$ at $t' = t$ can be written

$$\hat{k}(t') = \hat{k}_3(t) e^3 + \{ \tfrac{1}{2} [\hat{k}_1(t) + i \hat{k}_2(t)]$$

$$\times (e^1 - ie^2) e^{i\omega_c(t-t')} + \text{H.c.} \} . \quad (4.3)$$

A comparison with (3.8) shows that the system has two characteristic frequencies, 0 and ω_c. The former corresponds to free motion in the direction 3, and no radiation is associated with it. The only relevant positive frequency component of $\hat{k}$ is thus

$$\hat{k}^{(+)} = \tfrac{1}{2} (\hat{k}_1 + i\hat{k}_2)(e^1 - ie^2) . \quad (4.4)$$

Notice that $\hat{k}^{(+)}$ is a simple linear combination of the Cartesian components $\hat{k}_1$ and $\hat{k}_2$.

We now focus on the radiation reaction terms, in the P representation. Using (2.7) we first obtain from (3.16a)

$$\hat{T}_E(r,u) = -\frac{iQ}{\hbar} u \cdot \int_0^1 d\tau [\hat{E}_R^{(-)}(\hat{r} + \tau u) \hat{P}(r,u)$$

$$+ \hat{P}(r,u) \hat{E}_R^{(+)}(\hat{r} - \tau u)] . \quad (4.5)$$

From (3.11) we find that

$$\int_0^1 d\tau E_R^{(\pm)}(\hat{r} \mp \tau u)_i = -\frac{Q\omega_c^2}{6\pi\epsilon_0 M c^3} \sum_j T_{ij}(u) \hat{k}_j^{(\pm)} , \quad (4.6a)$$

with

$$T_{ij}(u) = \frac{3}{8\pi} \int_0^1 d\tau \int d^2\Omega (\delta_{ij} - n_i n_j) e^{i\tau(\omega_c/c)(n \cdot u)} . \quad (4.6b)$$

By (4.4), $\hat{k}^{(\pm)}$ may be expressed in terms of $\hat{k}_1$ and $\hat{k}_2$, and using (3.33) and (3.34), $\hat{k}_i \hat{P}$ and $\hat{P} \hat{k}_i$ may be obtained in terms of P. Thus the whole radiation reaction term $\hat{T}_E$ can be expressed in terms of $\hat{P}$. First, when the magnetic radiation reaction field is neglected, (3.33) and (3.34) give

$$\hat{k}_i \hat{P}(r,u) = \left[\frac{\hbar}{i} \frac{\partial}{\partial u_i} - \left[\frac{\partial}{\partial r_i} + \frac{iQ}{\hbar}(u \times B)_i \right] \right] \hat{P}(r,u) ,$$

$$\hat{P}(r,u) \hat{k}_i = \left[\frac{\hbar}{i} \frac{\partial}{\partial u_i} + \left[\frac{\partial}{\partial r_i} + \frac{iQ}{\hbar}(u \times B)_i \right] \right] \hat{P}(r,u) , \quad (4.7)$$

and after some straightforward algebra Eqs. (4.4)–(4.7) result in

$$\hat{T}_E(r,u) = \Gamma \frac{i}{\hbar} \sum_{i=1}^{3} \sum_{j=1}^{2} u_i T_{ij}(u) \left\{ \frac{\hbar}{i} \frac{\partial}{\partial u_j} - \frac{\hbar}{2} \left[e^3 \times \left[\frac{\partial}{\partial r} + \frac{iQ}{\hbar}(u \times B) \right] \right] \right]_j \right\} . \quad (4.8)$$

Here we have introduced the rate

$$\Gamma = \frac{Q^2 \omega_c^2}{6\pi\epsilon_0 M c^3} . \quad (4.9)$$

It can be seen from (4.6b) that in the variable u the characteristic scale of the functions $T_{ij}(u)$ is c/ω_c. When (4.8) is transformed back to k space, these functions give integral transformations sampling the k dependence of $\hat{W}(r,k)$ over a span $\hbar\omega_c/c$. As this is precisely the recoil momentum associated to photons with the cycloctron frequency ω_c, we again conclude that the dependence on u of $T_{ij}(u)$ conveys the recoil effects. For simplicity we from now on neglect the recoil, and use in (4.8) $T_{ij}(0) = \delta_{ij}$ instead of $T_{ij}(u)$. The result, when transformed to k space, reads

$$\hat{T}_E(r,k) = -\Gamma \sum_{i=1}^{2} \frac{\partial}{\partial k_i} \left\{ k_i - \frac{\hbar}{2} \left[e^3 \times \left[\frac{\partial}{\partial r} + QB \times \frac{\partial}{\partial k} \right] \right]_i \right\} \hat{W}(r,k) . \quad (4.10)$$

We now know the explicit form of Eq. (4.1). It remains to multiply the equation with a $\hat{\rho}(t_0)$ satisfying (3.15) and take the trace. The final equation of motion for the GIWF reads

$$\left\{ \frac{\partial}{\partial t} + \frac{k}{M} \cdot \frac{\partial}{\partial r} + \omega_c \left[k_2 \frac{\partial}{\partial k_1} - k_1 \frac{\partial}{\partial k_2} \right] - \Gamma \left[\frac{\partial}{\partial k_1} k_1 + \frac{\partial}{\partial k_2} k_2 \right] \right.$$

$$\left. - \frac{\Gamma\hbar}{2} \left[\frac{\partial^2}{\partial k_1 \partial r_2} - \frac{\partial^2}{\partial k_2 \partial r_1} + M\omega_c \left[\frac{\partial^2}{\partial k_1^2} + \frac{\partial^2}{\partial k_2^2} \right] \right] \right\} W(r,k,t) = 0 . \quad (4.11)$$

The first two terms in the FPE (4.11) constitute the ordinary convective derivative of a free particle. The following term in large parentheses describes the Lorentz force due to the magnetic field, and the ensuing cyclotron motion. The remaining terms are proportional to Γ, the well-known damping coefficient for cyclotron motion. Since Γ does not depend on $\hbar$, the friction term in the second pair of large parentheses in (4.11) is classical in origin. If $\hbar$ were zero, the motion of the particle in the 1-2 plane would eventually be damped out. However, the first quantum corrections to classical dynamics ($\propto \hbar$) come out as diffusion, hence quantum fluctuations keep $\Delta k_{1,2}$ nonzero.

V. SOLUTIONS OF THE FOKKER-PLANCK EQUATION

The second-order derivatives in (4.11) define a quadratic form of the vector

$$\frac{\partial}{\partial x} = \left[\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}, \frac{\partial}{\partial k_1}, \frac{\partial}{\partial k_2} \right]$$

as

$$\sum_{i,j=1}^{2} D_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} .$$

It is easy to see that the matrix D_{ij} has two positive and two negative eigenvalues. The diffusion operator in (4.11) is indefinite, and the equation itself is mathematically ill-behaved.

Although mathematical problems can thus be anticipated, let us still initiate the conventional treatment of the diffusion equation. First, the motion of the particle in the direction 3 may be separated from the FPE, and henceforth we only consider the spatial directions 1 and 2. Next, to shorten the ensuing expressions we adopt the dimensionless variables

$$\tau = t\omega_c, \quad \xi = (M\omega_c/\hbar)^{1/2}r, \quad \kappa = (\hbar M\omega_c)^{-1/2}k , \tag{5.1}$$

effectively corresponding to the system of units with $\omega_c = M = \hbar = 1$. We therefore speak of such units below. The FPE (4.11) becomes

$$\left\{ \frac{\partial}{\partial \tau} + \kappa \cdot \frac{\partial}{\partial \xi} + \left[\kappa_2 \frac{\partial}{\partial \kappa_1} - \kappa_1 \frac{\partial}{\partial \kappa_2} \right] - \gamma \left[\frac{\partial}{\partial \kappa} \cdot \kappa + \frac{1}{2} \left[\frac{\partial^2}{\partial \kappa_1 \partial \xi_2} - \frac{\partial^2}{\partial \kappa_2 \partial \xi_1} + \frac{\partial^2}{\partial \kappa_1^2} + \frac{\partial^2}{\partial \kappa_2^2} \right] \right] \right\} W(\xi,\kappa,\tau) = 0 . \tag{5.2}$$

The only parameter of the problem turns out to be $\gamma = \Gamma/\omega_c$. Finally, we define the Fourier transform with respect to the variables ξ, κ as

$$\widetilde{F}(R,K) = \int d^2\kappa \, d^2\xi \, e^{i(\kappa \cdot R + \xi \cdot K)} F(\xi,\kappa), \tag{5.3a}$$

$$F(\xi,\kappa) = \frac{1}{(2\pi)^4} \int d^2R \, d^2K \, e^{-i(\kappa \cdot R + \xi \cdot K)} \widetilde{F}(R,K) . \tag{5.3b}$$

It is easy to derive from (5.2) the equation of motion for the Fourier transform $\widetilde{W}$ of the GIWF W. We divide the ensuing equation by $\widetilde{W}$, and obtain for the new unknown function

$$f(R,K) = \ln \widetilde{W}(R,K) \tag{5.4}$$

the equation

$$\left[\frac{\partial}{\partial \tau} - K \cdot \frac{\partial}{\partial R} - \left[R_1 \frac{\partial}{\partial R_2} - R_2 \frac{\partial}{\partial R_1} \right] + \gamma R \cdot \frac{\partial}{\partial R} \right] f(R,K,\tau) = -\frac{\gamma}{2}(R_1 K_2 - R_2 K_1 + R_1^2 + R_2^2). \tag{5.5}$$

We look for solutions in the form

$$f = A_{11}R_1^2 + A_{12}R_1R_2 + A_{22}R_2^2 + B_1R_1 + B_2R_2 + C , \tag{5.6}$$

where the coefficients A, B, and C may be functions of τ and K, but not of R. The function f is a solution to (5.5) if the coefficients satisfy

$$\frac{\partial}{\partial \tau} A_{11} - A_{12} + 2\gamma A_{11} = -\frac{\gamma}{2} , \tag{5.7a}$$

$$\frac{\partial}{\partial \tau} A_{12} - 2A_{22} + 2A_{11} + 2\gamma A_{12} = 0 , \tag{5.7b}$$

$$\frac{\partial}{\partial \tau} A_{22} + A_{12} + 2\gamma A_{22} = -\frac{\gamma}{2} , \tag{5.7c}$$

$$\frac{\partial}{\partial \tau} B_1 - 2K_1 A_{11} - K_2 A_{12} - B_2 + \gamma B_1 = -\frac{\gamma}{2}K_2 , \tag{5.7d}$$

$$\frac{\partial}{\partial \tau} B_2 - K_1 A_{12} - 2K_2 A_{22} + B_1 + \gamma B_2 = \frac{\gamma}{2}K_1 , \tag{5.7e}$$

$$\frac{\partial}{\partial \tau} C - K_1 B_1 - K_2 B_2 = 0 . \tag{5.7f}$$

The set (5.7) is conveniently analyzed using the Laplace transformation for the time variable τ. It turns out that the coefficients $A_{11}(\tau), \ldots$ evolve from their initial values $A_{11}(\tau=0) = A_{11}^0, \ldots$ smoothly to the stationary values $A_{11}(\tau=\infty) = A_{11}^\infty, \ldots$, which are

$$A_{11}^\infty = A_{22}^\infty = -\tfrac{1}{4}, \quad A_{12}^\infty = 0 ;$$

(5.8a)

$$B_1^\infty = -K_2/2, \quad B_2^\infty = K_1/2 ;$$

(5.8b)

$$C^\infty = C^0 - \frac{K_1^2 + K_2^2}{4(1+\gamma^2)} + [-B_1^0 K_2 + B_2^0 K_1 + \gamma(B_1^0 K_1 + B_2^0 K_2)]/(1+\gamma^2)$$

$$+ \{K_1^2(A_{22}^0 + \gamma A_{12}^0 + \gamma^2 A_{11}^0) + K_2^2(A_{11}^0 - \gamma A_{12}^0 + \gamma^2 A_{22}^0) + K_1 K_2[(\gamma^2 - 1)A_{12}^0 + 2\gamma(A_{22}^0 - A_{11}^0)]\} / (1+\gamma^2)^2 .$$

(5.8c)

The appearance of the initial values in (5.8c) indicates that Eqs. (5.7) do not have a unique stationary solution. Consequently, the FPE (5.2) [and (4.11)] does not have a unique stationary solution either. In fact, because of the translational invariance of (5.2) in the variable ξ, if $W^\infty(\xi,\kappa)$ is a stationary solution then so is $W^\infty(\xi - \xi^0, \kappa)$ for an arbitrary ξ^0.

In what follows we basically try to solve Eq. (5.2), with an emphasis on finding a stationary solution. In Sec. V A we use a δ function in both ξ and κ as the initial GIWF W^0. It will be seen that the solution, the propagator of the FPE, does not exist. We are thus deprived of the most important tool in the analysis of Fokker-Planck equations, and the questions of whether the time evolution ever leads to mathematically well-defined and physically meaningful stationary solutions of (5.2) are brought to the foreground. In an attempt to study the existence problems we in Sec. V B replace the decomposition of phase-space distributions into δ functions (which is the idea behind the propagator) by coherent-state representations of the density operator. We are able to demonstrate that a class of physical initial Wigner functions leads to a mathematically well-defined stationary GIWF, but a proof for all Wigner functions is still lacking. In Sec. V C we point out that during the whole time evolution a coherent state remains a coherent state, thus giving one example where the GIWF always corresponds to a physical density operator. Section V D summarizes both the results of our analysis of the FPE, and the remaining obvious problems.

A. Nonexistence of the propagator

By definition, the propagator $\mathscr{G}(\xi,\kappa,\infty;\xi^0,\kappa^0,0)$ from $\tau=0$ to $\tau=\infty$ of (5.2) is the solution at $\tau=\infty$ with the initial condition

$$W^0(\xi,\kappa) = \delta(\xi - \xi^0)\delta(\kappa - \kappa^0) .$$

(5.9)

Taking the logarithm of the Fourier transform $\widetilde{W}^0$ we find that the initial function f^0 corresponds to (5.6) with

$$A_{11}^0 = A_{12}^0 = A_{22}^0 = 0 ,$$

(5.10a)

$$B_1^0 = i\kappa_1^0, \quad B_2^0 = i\kappa_2^0 ,$$

(5.10b)

$$C^0 = iK \cdot \xi^0 .$$

(5.10c)

Inserting these into (5.8) we obtain the Fourier transform of the propagator

$$\mathscr{G}(R,K,\infty;\xi^0,\kappa^0,0) = \exp\left[\sum_{i,j=1}^4 x_i \mathscr{D}_{ij} x_j + \mathscr{L} \cdot x\right] ,$$

(5.11a)

where we have defined the vector

$$x = (R_1, K_2, K_1, R_2) ,$$

(5.11b)

the matrix of the quadratic form is

$$\mathscr{D} = \begin{bmatrix} -\tfrac{1}{4} & -\tfrac{1}{4} & 0 & 0 \\ -\tfrac{1}{4} & \dfrac{-1}{4(1+\gamma^2)} & 0 & 0 \\ 0 & 0 & \dfrac{-1}{4(1+\gamma^2)} & \tfrac{1}{4} \\ 0 & 0 & \tfrac{1}{4} & -\tfrac{1}{4} \end{bmatrix} ,$$

(5.11c)

and the linear term is determined by

$$\mathscr{L} = \left[0, i\left(\xi_2^0 + \frac{\gamma\kappa_2^0 - \kappa_1^0}{1+\gamma^2}\right), i\left(\xi_1^0 + \frac{\gamma\kappa_1^0 + \kappa_2^0}{1+\gamma^2}\right), 0\right] .$$

(5.11d)

Since for $\gamma > 0$ the matrix (5.11c) has two positive and two negative eigenvalues, the quadratic form in (5.11a) is indefinite. In some directions in $\mathbb{R}^4$ the function $\mathscr{G}$ grows like e^{x^2} and its Fourier transform, i.e., the propagator $\mathscr{G}$, does not exist in the class of tempered distributions.

Suppose now the initial distribution $W^0(\xi,\kappa)$ is a Gaussian function of ξ,κ, but with a finite nonzero width in all directions of the 4-plane (ξ,κ). Then its Fourier transform $\widetilde{W}^0(R,K)$ is also a Gaussian function whose logarithm is of the desired form (5.6). Since the mapping of the coefficients of $\ln(\widetilde{W})$, (5.8), is continuous, obviously a narrow enough initial state $W^0(\xi,\kappa)$ again evolves into a highly singular object $W^\infty(\xi,\kappa)$. Moreover, the evolution of the coefficients $A_{11}(\tau),\dots$ is smooth, so the singularity already sets in at some finite time τ_s. When the initial distribution is narrow enough, the distribution starts shrinking in certain directions in the (ξ,κ) plane, becomes a δ function in some direction at time τ_s, and then evolves further an object whose mathematical nature is unclear. We stress that the singularity may emerge at some finite time, and hence is completely different from the singular evolution from $\tau=0$ to $\tau=\infty$ of the solutions to the ordinary diffusion equation. In the latter case the propagator does not exist because the distribution simply keeps on spreading *ad infinitum*. The singularity of (5.2) is more like the one encountered in an attempt to integrate the diffusion equation backward in time.

B. Coherent-state representations

Because the propagator of (5.2) does not exist, we seek to find solutions by expressing the initial state in terms of coherent states introduced in the manner presented in Ref. 11. We point out already at the outset that, rather than assuming a quantized field in the vacuum state, below we shall use as an initial state the GIWF $W^0(r,k)$ computed from the density matrix of the particle $\hat{\rho}_P^0$ as if *no* quantized fields were present. In Appendix B we discuss the spurious divergencies of the GIWF in the presence of the vacuum field that forced us to take this route, and offer a qualitative justification for it.

We choose a gauge where the vector potential reads

$$A(r) = \tfrac{1}{2}(B \times r) , \qquad (5.12a)$$

or in the units with $\omega_c = M = \hbar = 1$,

$$A(\xi) = \frac{1}{2Q}(e^3 \times \xi) . \qquad (5.12b)$$

In this gauge the Schrödinger equation for the particle in the constant magnetic field can be interpreted as the Schrödinger equation for two coupled harmonic oscillators in the directions 1 and 2.

Let us for the moment forget about the coupling. A convenient basis in the Hilbert space of the particle states is provided by the two-dimensional harmonic oscillator states $|n_1 n_2\rangle$ with the wave functions

$$\Psi_{n_1 n_2}(\xi_1, \xi_2) = \langle \xi_1 \xi_2 | n_1 n_2 \rangle$$

$$= (\pi 2^{n_1 + n_2} n_1! n_2!)^{-1/2} e^{-(\xi_1^2 + \xi_2^2)/2}$$

$$\times H_{n_1}(\xi_1) H_{n_2}(\xi_2) . \qquad (5.13)$$

Here H_n are the Hermite polynomials.[12] The coherent states and the corresponding wave functions are then defined as

$$|\alpha\beta\rangle = e^{-(|\alpha|^2 + |\beta|^2)/4}$$

$$\times \sum_{n_1, n_2 = 0}^{\infty} \frac{\left[\dfrac{\alpha}{\sqrt{2}}\right]^{n_1} \left[\dfrac{\beta}{\sqrt{2}}\right]^{n_2}}{\sqrt{n_1! n_2!}} |n_1 n_2\rangle , \qquad (5.14a)$$

$$\Psi_{\alpha\beta}(\xi_1, \xi_2) = \langle \xi_1 \xi_2 | \alpha\beta \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \exp\{ -\tfrac{1}{4}[\xi_1^2 + \xi_2^2 - 2(\alpha\xi_1 + \beta\xi_2)$$

$$+ \alpha_r^2 + \beta_r^2 + i(\alpha_r \alpha_i + \beta_r \beta_i)]\} . \qquad (5.14b)$$

$\alpha = \alpha_r + i\alpha_i$ and $\beta = \beta_r + i\beta_i$ $(\alpha_r, \alpha_i, \beta_r, \beta_i \in \mathbb{R})$ are arbitrary complex numbers.

The coherent states are centered around the points

$$\eta_1 = \langle \hat{\xi}_1 \rangle = \alpha_r, \quad \eta_2 = \langle \hat{\xi}_2 \rangle = \beta_r ; \qquad (5.15a)$$

$$\nu_1 = \langle \hat{\kappa}_1 \rangle = \tfrac{1}{2}(\alpha_i + \beta_r), \quad \nu_2 = \langle \hat{\kappa}_2 \rangle = \tfrac{1}{2}(-\alpha_r + \beta_i) . \qquad (5.15b)$$

They are minimum-uncertainty wave packets in that they minimize the uncertainty products for the operator pairs $\hat{r}_1, \hat{p}_1$ and $\hat{r}_2, \hat{p}_2$. When the coupling of the harmonic oscillators is taken into account, a coherent state evolves in time. Its form as a coherent state is preserved, and its center point follows the classical trajectory of the charged particle in the magnetic field. What happens when also the radiation reaction field is present is the subject of Sec. V C.

Coherent states are normalized but not orthogonal. They form a complete, in fact an overcomplete set. One may therefore envisage representations of the initial density operator of the particle $\hat{\rho}_P^0$ in the form

$$\hat{\rho}_P^0 = \int d\mu(\alpha, \beta, \alpha', \beta') | \alpha\beta \rangle \langle \alpha'\beta' | \rho^0(\alpha\beta; \alpha'\beta') , \qquad (5.16)$$

where μ is a measure on $\mathbb{C}^4$ (or $\mathbb{R}^8$) or on some subset (say, a line in $\mathbb{C}^4$) thereof. Representations of this kind are studied in Refs. 13 and 14 for a single-mode theory where the coherent states are labeled by one complex number. However, the existence and uniqueness theorems of these representations heavily rely on the theory of complex functions, and a generalization of the proofs to the present case would require the theory of analytic functions of several complex variables. We have not gone into this, but merely give one example of (5.16). Namely,

$$\mathbb{1} = \frac{1}{2\pi^2} \int d^2\alpha \, d^2\beta \, | \alpha\beta \rangle \langle \alpha\beta | \qquad (5.17)$$

is a valid resolution of unity. A representation of the form (5.16) follows where

$$d\mu = d^2\alpha \, d^2\beta \, d^2\alpha' \, d^2\beta' \qquad (5.18a)$$

is the Lebesgue measure on $\mathbb{R}^8$, and

$$\rho^0(\alpha\beta; \alpha'\beta') = \frac{1}{4\pi^4} \langle \alpha\beta | \hat{\rho}_P^0 | \alpha'\beta' \rangle . \qquad (5.18b)$$

Whatever the representation used in (5.16) is, to find the density operator after the time evolution from $\tau = 0$ to $\tau = \infty$ it suffices to find the "GIWF" $W_{\alpha\beta; \alpha'\beta'}^{\infty}(\xi, \kappa)$ that has evolved from the initial "GIWF" $W_{\alpha\beta; \alpha'\beta'}^0(\xi, \kappa)$ corresponding to the dyad $|\alpha\beta\rangle\langle\alpha'\beta'|$. Then

$$W^{\infty}(\xi, \kappa) = \int d\mu \, W_{\alpha\beta; \alpha'\beta'}^{\infty}(\xi, \kappa) \rho^0(\alpha\beta; \alpha'\beta') . \qquad (5.19)$$

We call $W_{\alpha\beta; \alpha'\beta'}(\xi, \kappa)$ the coherence kernel. In the next two subsections we address the two obvious questions.

(i) Is $W_{\alpha\beta; \alpha'\beta'}^{\infty}(\xi, \kappa)$ well defined?

(ii) When the representation in (5.16) and (5.18) is used, does the integral in (5.19) converge; i.e., is $W^{\infty}(\xi, \kappa)$ well defined?

1. Evolution of the coherence kernel

The "GIWF" corresponding to the dyad $|\alpha\beta\rangle\langle\alpha'\beta'|$ is obtained from (3.2) of I and (5.14):

$$W^0_{\alpha\beta;\alpha'\beta'}(\xi,\kappa)=\frac{1}{4\pi^2}\int d^2u\,e^{iu\cdot[\kappa+i(\xi_1e^2-\xi_2e^1)]}\langle\xi-\tfrac{1}{2}u\mid\alpha\beta\rangle\langle\alpha'\beta'\mid\xi+\tfrac{1}{2}u\rangle$$

$$=\frac{1}{\pi^2}\exp\{-[(\xi_1-\xi_1^0)^2+(\xi_2-\xi_2^0)^2+2(\kappa_1-\kappa_1^0)^2+(\kappa_2-\kappa_2^0)^2+2(\kappa_2-\kappa_2^0)(\xi_1-\xi_1^0)-2(\kappa_1-\kappa_1^0)(\xi_2-\xi_2^0)-C]\}\ ,$$

(5.20)

where we have defined

$$\xi_1^0=\tfrac{1}{2}(\alpha+\alpha'^*),\quad \xi_2^0=\tfrac{1}{2}(\beta+\beta'^*)\ ,$$ (5.21a)

$$\kappa_1^0=\tfrac{1}{4}[\beta+\beta'^*-i(\alpha-\alpha'^*)],\quad \kappa_2^0=\tfrac{1}{4}[-(\alpha+\alpha'^*)-i(\beta-\beta'^*)]\ ,$$ (5.21b)

$$C=-\tfrac{1}{8}(\mid\alpha\mid^2+\mid\alpha'\mid^2-2\alpha\alpha'^*+\mid\beta\mid^2+\mid\beta'\mid^2-2\beta\beta'^*)\ .$$ (5.21c)

Since this "GIWF" is a Gaussian function of the variables ξ,κ, a procedure exactly like the attempted derivation of the propagator can be carried out right away. This time, however, $\tilde{W}^\infty_{\alpha\beta;\alpha'\beta'}(R,K)$ is a Gaussian function that does have the Fourier transform back to the variables ξ,κ. We obtain

$$W^\infty_{\alpha\beta,\alpha'\beta'}(\xi,\kappa)=\frac{1}{\pi^2}\exp\{-[(\xi_1-\xi_1^\infty)^2+(\xi_2-\xi_2^\infty)^2+2\kappa_1^2+2\kappa_2^2+2\kappa_2(\xi_1-\xi_1^\infty)-2\kappa_1(\xi_2-\xi_2^\infty)-C]\}\ ,$$ (5.22)

where $\xi_1^\infty,\xi_2^\infty$ are

$$\xi_1^\infty=\tfrac{1}{2}(\alpha+\alpha')-\frac{\alpha+\alpha'-i(\beta-\beta')+\gamma[\beta+\beta'-i(\alpha-\alpha')]}{4(1+\gamma^2)}\ ,$$ (5.23a)

$$\xi_2^\infty=\tfrac{1}{2}(\beta+\beta')-\frac{\beta+\beta'+i(\alpha-\alpha')+\gamma[-(\alpha+\alpha')-i(\beta-\beta')]}{4(1+\gamma^2)}\ ,$$ (5.23b)

and C is still given by (5.21c). The coherence kernel thus exists at least at $\tau=0$ and $\tau=\infty$.

2. Convergence problem of the representation

To study the convergence of the integral (5.19) we first note that for each fixed κ and ξ, $W^\infty_{\alpha\beta;\alpha'\beta'}(\xi,\kappa)$ in (5.22) and (5.23) is an exponential of a quadratic polynomial of the components of the eight-dimensional vector

$$X=(\alpha_r,\alpha_i,\beta_r,\beta_i,\alpha'_r,\alpha'_i,\beta'_r,\beta'_i)\ .$$

Let us denote with $\mathcal{M}$ the matrix of the quadratic form comprised of the second-order terms of this polynomial. $\mathcal{M}$ depends on γ but not on κ,ξ. We have studied numerically the eigenvalues of the real part of $\mathcal{M}$, and found that for $0\leq\gamma<\infty$ the largest eigenvalue is always smaller than $\tfrac{1}{4}$. When γ is fixed, a sufficient condition for the integral

(5.19) to converge at infinity, therefore, is that for every $\epsilon>0$ positive numbers N and M can be found such that

$$\mid\rho^0(\alpha\beta;\alpha'\beta')\mid e^{(1/4-\epsilon)(\mid\alpha\mid^2+\mid\beta\mid^2+\mid\alpha'\mid^2+\mid\beta'\mid^2)}<M$$

(5.24)

whenever $\mid\alpha\mid,\mid\beta\mid,\mid\alpha'\mid,\mid\beta'\mid>N$.

We provide an example of the use of (5.18) and (5.19) with the harmonic oscillator states (5.13). Suppose the initial density operator corresponds to a normalizable pure state $\mid\Psi\rangle$ written as a superposition of $\mid n_1 n_2\rangle$,

$$\mid\Psi\rangle=\sum_{n_1,n_2}a_{n_1 n_2}\mid n_1 n_2\rangle\ ,$$ (5.25)

where $\{a_{n_1 n_2}\}$ is a square-summable sequence of complex numbers. It follows from (5.14a) and (5.18b) that

$$\rho^0(\alpha\beta;\alpha'\beta')=\frac{1}{4\pi^2}e^{-(1/4)(\mid\alpha\mid^2+\mid\beta\mid^2+\mid\alpha'\mid^2+\mid\beta'\mid^2)}\sum_{\substack{n_1,n_2,\\n_1',n_2'}}a_{n_1 n_2}a^*_{n_1' n_2'}\frac{\left[\dfrac{\alpha^*}{\sqrt{2}}\right]^{n_1}\left[\dfrac{\beta^*}{\sqrt{2}}\right]^{n_2}\left[\dfrac{\alpha'}{\sqrt{2}}\right]^{n_1'}\left[\dfrac{\beta'}{\sqrt{2}}\right]^{n_2'}}{\sqrt{n_1!n_2!n_1'!n_2'!}}\ .$$ (5.26)

Every term in this fourfold sum separately satisfies (5.24). Hence, if the sum is *finite*, the integral (5.19) converges. Generalizing slightly, we have shown that every initial GIWF corresponding to a mixture of a finite number of pure states, each of which is a finite superposition of har-

monic oscillator states, leads to a *mathematically* well-defined stationary GIWF. Dropping the restriction "finite" in this statement, i.e., a convergence proof, is the part of the problem we have not been able to do. In fact, (5.24) is a quite crude sufficient condition for the integral

(5.19) to converge at infinity, and the convergence proof for an arbitrary square-summable sequence $\{a_{n_1 n_2}\}$ probably requires more delicate methods. It is also conceivable that the final state really does not exist for some physically realizable initial states, indicating a serious failure of our Wigner-function method. But we have not found such an example either.

C. Evolution of coherent states

Although we have shown that at least finite superpositions of harmonic oscillator states evolve into well-defined final Wigner functions, we do not know whether these in

general represent attainable physical density operators. However, we are now about to demonstrate that a stronger statement holds for the coherent states. We stress that, according to Appendix B, we again associate with the coherent state of the particle the GIWF that would apply *without* quantum fields.

Thus, select an arbitrary coherent state $|\alpha\beta\rangle$. The corresponding initial density operator is $\hat{\rho}_P^0 = |\alpha\beta\rangle\langle\alpha\beta|$, and the GIWF is $W^0_{\alpha\beta;\alpha\beta}(\xi,\kappa)$, a special case of coherence kernels studied in Sec. V B. Instead of the complex numbers α,β we for convenience denote the coherent states by their center point (η,ν) from (5.15). Equation (5.20) then gives

$$W_{\eta^0,\nu^0}(\xi,\kappa) = W^0_{\alpha\beta;\alpha\beta}(\xi,\kappa)$$

$$= \frac{1}{\pi^2}\exp\{-[(\xi-\eta^0)^2 + 2(\kappa-\nu^0)^2 + 2(\kappa_2-\nu_2^0)(\xi_1-\eta_1^0) - 2(\kappa_1-\nu_1^0)(\xi_2-\eta_2^0)]\} . \tag{5.27}$$

We now invoke (5.22) and (5.23). The initial GIWF (5.27) evolves smoothly into a stationary GIWF at $\tau=\infty$ which is precisely of the form (5.27) representing a coherent state. Only the average position and kinetic momentum have changed,

$$\eta_1^\infty = \eta_1^0 + \frac{\gamma\nu_1^0 + \nu_2^0}{1+\gamma^2} , \quad \eta_2^\infty = \eta_2^0 + \frac{\gamma\nu_2^0 - \nu_1^0}{1+\gamma^2} , \tag{5.28a}$$

$$\nu^\infty = 0 . \tag{5.28b}$$

Most importantly, the average velocity is damped to zero.

Using (5.7) it is, in fact, easy to show that if the initial GIWF is that of a coherent state, the solution of (5.2) [and (4.11)] is a GIWF of a coherent state at all times. The center point just moves according to the equations

$$\dot{\eta} = \nu, \quad \dot{\nu} = \nu \times e^3 - \gamma\nu . \tag{5.29}$$

These are recognized as the dimensionless form of the classical equations of motion for the position r and kinetic momentum k of a charged particle in the presence of a constant magnetic field be^3 and damping force $-\Gamma k$. A coherent state retains its form even when radiation damping is taken into account, and its center point spirals along the classical trajectory.

D. Final remarks about the Fokker-Planck equation

The manifestly physical behavior of the coherent states and the proof that at least finite linear combinations of harmonic oscillator states stay mathematically well behaved after time evolution suggest that the FPE (5.2) may be useful and essentially correct when restricted to the space of physically allowable Wigner functions. Nevertheless, serious problems remain.

We have not been able to show that an arbitrary physical GIWF evolves into a mathematically well-defined object under (5.2). As the nonexistence of the propagator shows, this difficulty is in some way associated with nar-

rowness of the initial distribution. Mathematically, the derivation of (4.11) or (5.2) is based on an expansion in $1/\Delta r$ and $1/\Delta k$, and (5.2) is not valid for too narrow distributions. On the other hand, the uncertainty relations imply that physical distributions must not be arbitrarily narrow simultaneously in all directions in (r,k) space, so the δ-function initial distribution is physically meaningless. The roots of the mathematical troubles may lie in either one of these aspects.

However, we have found no counterexample either. For instance, the highly singular but physical GIWF $\delta(\xi_1)\delta(\kappa_2)$ leads to a stationary state.

Even if the stationary GIWF existed, there still is no guarantee that it corresponds to a physical density operator. At present we have obtained no concrete results in our attempts to address this issue in its full generality.

Finally, the even deeper question "what is the GIWF corresponding to a given particle state" (and, a fortiori, the question "which Wigner functions are physical") remains open. We forward the opinion that the connection between the instantaneous state of the particle $\hat{\rho}_P(t)$ and the GIWF $W(r,k,t)$ should be laid down by assuming that no quantized fields are present.

VI. SUMMARY AND CONCLUSION

In this paper we have exploited the property of our GIWO and GIWF that they can handle quantized electromagnetic fields. We have outlined a general method to eliminate the radiation reaction fields from the equation of motion of the GIWO so as to arrive at a closed equation for the GIWF, and demonstrated the required procedures with a charged particle in a constant magnetic field. We have obtained a Fokker-Planck equation for the GIWF which explicitly displays the classical radiation damping and the lowest diffusive quantum correction proportional to $\hbar$. Although the FPE has defied a comprehensive analysis, the example of the coherent states neatly demonstrates how the quantum diffusion maintains

the Heisenberg uncertainty relations.

Because our formulation is intrinsically gauge independent, so, automatically, are all our approximations. In order to justify the commutator (3.25) simplifying the calculations we have worked out the final results in the dipole approximation and neglecting recoil effects. However, we see no fundamental reason why these approximations could not be relaxed if necessary. Then the inherent gauge invariance of the theory would likely be a valuable asset. In the general fashion of Wigner functions the results are also easily rendered $\hbar$ expansions around the classical Liouville equation, which offers a lot of fresh insight into quantum mechanics.

Among the disadvantages of our theory we must count the general property of Wigner-function formulations that the algebra tends to be lengthy and cumbersome. Moreover, our approach has got its fair share of both mathematical and conceptual difficulties, whose resolution may call for new innovations and methods.

One immediate application of the Wigner-function machinery we have in mind is to investigate the influence of the micromotion due to the trapping fields on the performance of laser cooling of an ion in a Paul trap: The cooling problem has been formulated using ordinary Wigner functions already,[15] and recently optical resonances derived from the micromotion may have been observed.[16] Quantum treatments of the scattered field are also possible using the GIWO, which might open a new approach to photon statistics of the free-electron laser.[17] As a formal extension of the theory we plan to develop a fully relativistic GIWF approach to the Dirac electron. In summary, the GIWF has potentials both as a formal device and as a tool in practical calculations.

APPENDIX A: ON A COMMUTATOR

We shall demonstrate that, under the Markov approximation and when the particle oscillates in a region smaller than the spatial scale of the variation of the external classical fields, the commutator (3.25),

$$[\hat{b}_q(t_0), \hat{k}_\alpha^{(+)}(t)] = 0 , \tag{3.25}$$

holds true.

We begin with the Heisenberg equations of motion for the kinetic momentum and position operators,

$$\dot{\hat{k}} = Q\left[\hat{E} + \frac{1}{2M}(\hat{k} \times \hat{B} - \hat{B} \times \hat{k})\right] , \tag{A1a}$$

$$\dot{\hat{r}} = \hat{k}/M . \tag{A1b}$$

Here $\hat{E}$ and $\hat{B}$ contain the classical external fields, the radiation reaction fields and the free quantized fields. However, to avoid inessential complications we neglect the quantized magnetic field, and assume that the classical component of the magnetic field is time independent. We also assume that the secular approximation is valid,

$$\hat{k}(t) = \sum_\alpha [\hat{k}_\alpha^{(+)}(t) + \text{H.c.}] , \tag{A2}$$

where the components $\hat{k}_\alpha^{(\pm)}(t)$ roughly oscillate at the fre-

quencies ω_α. As these frequencies are well separated and the positive frequency components of $\hat{k}$ obviously only couple to the positive frequency components of $\hat{E}$, we may write for $\hat{k}_\alpha^{(+)}$ the equation

$$\dot{\hat{k}}_\alpha^{(+)} = Q\left[\hat{E}_\alpha^{(+)} + \frac{1}{2M}(\hat{k}_\alpha^{(+)} \times \hat{B} - \hat{B} \times \hat{k}_\alpha^{(+)})\right] . \tag{A3}$$

Here $\hat{E}_\alpha^{(+)}$ stands for a linear projection of $\hat{E}^{(+)}$ that accounts for the statement that the components of various operators at each frequency ω_α only couple among themselves. Finally, we assume that the secular motion commences with an amplitude smaller than the wavelength of the external fields, so that in the arguments of the classical fields we may replace $\hat{r}$ with $\hat{r}(t_0)$.

Within this framework, and within the Markov approximation, we write the electric field from (3.11) and (3.12) as

$$\hat{E}_\alpha^{(+)}(\hat{r}, t) \approx E_{C,\alpha}^{(+)}(\hat{r}(t_0), t)$$
$$+ \sum_q E_\alpha^{(+)}(q, t) e^{iq \cdot \hat{r}(t_0)} \hat{b}_q(t_0) - \frac{K_\alpha}{Q} \hat{k}_\alpha(t) . \tag{A4}$$

$E_{C,\alpha}^{(+)}$ is the ω_α component of the classical electric field, $E_\alpha^{(+)}(q, t)$ contains the filter (if any) used to select the ω_α component from the free field, and K_α is a positive number. The magnetic field is just

$$\hat{B}(\hat{r}, t) = B_C(\hat{r}(t_0)) . \tag{A5}$$

We now define the commutators

$$\hat{C}_{\alpha,i}^{(+)}(t) = [\hat{b}_q(t_0), \hat{k}_\alpha^{(+)}(t)_i] . \tag{A6}$$

From the commutator of $\hat{b}_q(t_0)$ with Eq. (A3) we obtain the equation

$$\dot{\hat{C}}_{\alpha,i}^{(+)} = -K_\alpha \hat{C}_{\alpha,i}^{(+)}$$
$$+ \frac{Q}{2M} \sum_{j,k} \epsilon_{ijk}(\hat{C}_{\alpha,j}^{(+)} B_{C,k} - B_{C,j} \hat{C}_{\alpha,k}^{(+)}) , \tag{A7}$$

where we have used (A4), (A5), and the commutator

$$[\hat{b}_q(t_0), \hat{r}(t_0)] = 0 . \tag{A8}$$

Obviously at t_0,

$$\hat{C}_{\alpha,i}^{(+)}(t) = 0 . \tag{A9}$$

But since (A7) is a homogeneous equation, $\hat{C}_{\alpha,i}^{(+)}(t) = 0$ for all $t \geq t_0$. Hence (3.25) holds true. In fact, (A7) contains an explicit damping, and the zero commutator would result even from nonzero initial values.

The essential premises are the Markov approximation and the restriction on the amplitude of the particle's oscillations, i.e., the dipole approximation. The rest of the assumptions are basically technical, chosen with an eye on the problem of Secs. IV and V. In the cases where the cross term $\hat{T}_{\Delta v \times B}$ in the equations of motion is expected to be of decisive importance the commutator (3.25) should be reexamined without the dipole approximation.

APPENDIX B: WIGNER FUNCTION WITH AND WITHOUT QUANTUM FIELDS

In this appendix we show that the GIWF has a divergence even if the quantum field is in the vacuum state, and suggest that the correct way to associate the GIWF to the state of the particle is to calculate the GIWF without quantized fields.

Let us begin by assuming that at some instant of time the state of the system (particle + quantized field) is

$$\hat{\rho} = \hat{\rho}_F \hat{\rho}_P , \tag{B1}$$

where the field state $\hat{\rho}_F$ satisfies (3.15), and $\hat{\rho}_P$ refers to the particle only. The expectation value of the operator $\hat{P}$ in this state is very much analogous to (2.6);

$$P(r,u) = \mathrm{Tr}_F \left[\hat{\rho}_F \exp \left[\frac{i}{\hbar} Qu \cdot \int_{-1/2}^{1/2} d\tau \, \hat{A}(r+\tau u) \right] \right]$$

$$\times \langle r + \tfrac{1}{2}u \mid \hat{\rho}_P \mid r - \tfrac{1}{2}u \rangle . \tag{B2}$$

To exploit the property that $\hat{\rho}_F$ represents the vacuum state we write the exponential operator in (B2) in normal order,

$$\exp \left[\frac{i}{\hbar} Qu \cdot \int_{-1/2}^{1/2} d\tau \, \hat{A}(r+\tau u) \right]$$

$$= \exp \left[\frac{i}{\hbar} Qu \cdot \int_{-1/2}^{1/2} d\tau \, \hat{A}^{(-)}(r+\tau u) \right]$$

$$\times \exp \left[\frac{i}{\hbar} Qu \cdot \int_{-1/2}^{1/2} d\tau \, \hat{A}^{(+)}(r+\tau u) \right] e^{-f(u)} . \tag{B3}$$

Then the operators $\hat{A}^{(\pm)}$ do not contribute at all in (B2). Unfortunately, the commutator term

$$f(u) = 2 \sum_q \left[\frac{Qu \cdot g(q) \sin(q \cdot u/2)}{\hbar q \cdot u} \right]^2 \tag{B4}$$

diverges for $u \neq 0$, implying that $P(r,u)$ is ill defined.

We stress that this divergence is not an artifact of the GIWF. In fact, the expectation value of the square of the kinetic momentum $\hat{k}^2$, and hence of the kinetic energy of the particle, is infinite in any state of the form (B1). Divergent fluctuations of the position and velocity operators that we believe to be of related origin have also surfaced in earlier Heisenberg-picture treatments of the motion of an unbound electron.[7]

The reason for the divergencies probably is that the particle cannot be separated from its own electromagnetic field, and the state (B1) with an uncorrelated particle and field is therefore unphysical. It seems that the well-known renormalization procedure (see, e.g., Ref. 7) needed to remove the divergent principal-value part from (3.9) must be *supplemented* with a redefinition ("dressing") of the field and particle states and operators in such a way that they represent the true physical states and observables.

However, we adopt another approach. First, in view of the intended use of the GIWF we assume that it refers to the experimentally observable dynamical variables of the dressed particle. Second, we assume that to the lowest approximation the dressed quantized field can be ignored, and that precisely the ordinary quantum mechanics then emerges for the dressed particle. We end up demanding that the correspondence between the GIWF and the state of the particle should be set up as if no quantized fields were present, as is done in the main text.

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