PHYSICAL REVIEW A VOLUME 35, NUMBER 6 MARCH 15, 1987

Brief Reports

Brief Reports are short papers which report on completed research which, while meeting the usual **Physical Review** *standards of scientific quality, does not warrant a regular article. (Addenda to papers previously published in the* **Physical Review** *by the same authors are included in Brief Reports.) A Brief Report may be no longer than $3\frac{1}{2}$ printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Exact analytical model of the classical Weibel instability in a relativistic anisotropic plasma

Peter H. Yoon and Ronald C. Davidson
Plasma Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 9 June 1986)

Detailed properties of the Weibel instability in a relativistic unmagnetized plasma are investigated for a particular choice of anisotropic distribution function $F(p_\perp^2,p_z)$ that permits an *exact* analytical solution to the dispersion relation for arbitrary energy anisotropy. The particular equilibrium-distribution function considered in the present analysis assumes that all particles move on a surface with perpendicular momentum $p_\perp=\hat{p}_\perp=$const and are uniformly distributed in parallel momentum from $p_z=-\hat{p}_z=$const to $p_z=+\hat{p}_z=$const. (Here, the propagation direction is the z direction.) The resulting dispersion relation is solved analytically, and detailed stability properties are determined for a wide range of energy anisotropy.

I. INTRODUCTION

A sufficiently large anisotropy in the average kinetic energy of the plasma electrons and/or ions can provide the free energy to drive various types of electromagnetic instabilities in a uniform plasma. Examples range from the classical Weibel instability[1,2] in an unmagnetized plasma, to the electron whistler[2–4] and cyclotron maser[5–8] instabilities for wave propagation in the presence of an applied magnetic field $B_0\hat{\mathbf{e}}_z$. The electron whistler instability[2–4] is a natural extension of the Weibel instability to the case of a magnetized plasma. On the other hand, the cyclotron maser instability[5–8] is inherently relativistic in nature and vanishes in the limit of zero magnetic field and/or sufficiently dense plasma (as measured by ω_p^2/ω_c^2). Generally speaking, anisotropy-driven electromagnetic instabilities have a wide range of applicability to astrophysical plasmas,[3–5] and to laboratory plasmas[4] with intense microwave heating. For *nonrelativistic* anisotropic plasma, detailed properties of the Weibel and whistler instabilities are readily calculated[1–3] for a wide range of equilibrium distribution functions $F_j(p_\perp^2,p_z)$. For *relativistic* anisotropic plasma, however, because of the coupling of the perpendicular and parallel particle motions through the relativistic mass factor $\gamma=(1+p_\perp^2/m_j^2c^2+p_z^2/m_j^2c^2)^{1/2}$, stability properties are usually calculated in the limit of extreme energy anisotropy, or long perturbation wavelength, which allows approximate analytical solutions to the electromagnetic dispersion relation. Here, "perpendicular" and "parallel" refer to directions relative to the propagation direction (the z direction).

The purpose of the present brief report is to investigate detailed properties of the Weibel instability in a relativistic unmagnetized plasma for a particular choice of anisotropic distribution function that permits an *exact* analytical solution to the dispersion relation for arbitrary energy anisotropy. This calculation is intended to provide qualitative insights regarding stability behavior for more general choices of equilibrium distribution function. The particular distribution function [Eq. (3)] considered in the present analysis assumes that all particles move on a surface with perpendicular momentum $p_\perp=\hat{p}_\perp=$const and are uniformly distributed in parallel momentum between $p_z=-\hat{p}_z=$const and $p_z=+\hat{p}_z=$const. The resulting dispersion relation [Eq. (8)] can be solved analytically, and detailed stability properties are determined for a wide range of energy anisotropy. Extension of the present analysis to include a uniform applied magnetic field $B_0\hat{\mathbf{e}}_z$ will be the subject of a future investigation.

II. THEORETICAL MODEL

We investigate the electromagnetic stability properties of relativistic anisotropic plasma for wave perturbations propagating in the z direction with wave vector $\mathbf{k}=k_z\hat{\mathbf{e}}_z$. Perturbations are about the class of uniform, field-free equilibria with distribution function

$$f_j^0(\mathbf{p})=\hat{n}_j F_j(p_\perp^2,p_z)\ , \tag{1}$$

where $\hat{n}_j=$const is the ambient density of the jth plasma component, $p_\perp=(p_x^2+p_y^2)^{1/2}$ is the particle momentum perpendicular to the propagation direction, and p_z is the parallel momentum. In the absence of applied magnetic field, the linear dispersion relation for transverse electromagnetic wave perturbations propagating in the z direction is given by

$$0=D_T(k_z,\omega)=1-\frac{c^2k_z^2}{\omega^2}+\sum_j\frac{\omega_{pj}^2}{\omega^2}\int\frac{d^3p}{\gamma}\frac{(p_\perp/2)}{(\omega-k_zp_z/\gamma m_j)}\left[\left(\omega-\frac{k_zp_z}{\gamma m_j}\right)\frac{\partial}{\partial p_\perp}+\frac{k_zp_\perp}{\gamma m_j}\frac{\partial}{\partial p_z}\right]F_j(p_\perp^2,p_z)\ . \tag{2}$$

Here, $\omega_{pj}^2=4\pi\hat{n}_je_j^2/m_j$ is the nonrelativistic plasma frequency squared; e_j and m_j are the charge and rest mass, respectively, of a jth component particle; c is the speed of light *in vacuo*; $\gamma=(1+p_\perp^2/m_j^2c^2+p_z^2/m_j^2c^2)^{1/2}$ is the relativistic mass factor; the range of integration is $\int d^3p\cdots=2\pi\int_0^\infty dp_\perp p_\perp\int_{-\infty}^\infty dp_z\cdots$; and the normalization of F_j is $\int d^3p\,F_j(p_\perp^2,p_z)=1$. In obtaining Eq. (2), the perturbations are assumed to have z and t dependence proportional to $\exp[i(k_zz-\omega t)]$, where k_z is the wave number and ω is the complex oscillation frequency with $\mathrm{Im}\omega>0$, which corresponds to instability (temporal growth). For relativistic anisotropic plasma, we note that the perpendicular and parallel particle motions in Eq. (2) are inexorably coupled through the relativistic mass factor $\gamma=(1+p_\perp^2/m_j^2c^2+p_z^2/m_j^2c^2)^{1/2}$.

In the analysis that follows, we specialize to the case of stationary ions ($m_i\to\infty$) and consider a single active component of relativistic anisotropic electrons. Moreover, for simplicity of notation, the electron species labels are omitted from ω_{pe}^2, m_e, $F_e(p_\perp^2,p_z)$, etc. The resulting dispersion relation [Eq. (8)] is readily generalized to the multicomponent case.

III. WATERBAG DISTRIBUTION IN PARALLEL MOMENTUM

The dispersion relation (2) can be used to investigate detailed electromagnetic stability properties for a wide range of anisotropic distribution functions $F(p_\perp^2,p_z)$. For purposes of elucidating the essential features of the Weibel instability in relativistic anisotropic plasma, we make a particular choice of $F(p_\perp^2,p_z)$ for which the momentum integrals in Eq. (2) can be carried out in closed analytical form. In particular, it is assumed that the electrons move on a surface with perpendicular momentum $p_\perp=\hat{p}_\perp=$const and are uniformly distributed in parallel momentum between $p_z=-\hat{p}_z=$const and $p_z=+\hat{p}_z=$const. That is, $F(p_\perp^2,p_z)$ is specified by

$$F(p_\perp^2,p_z)=\frac{1}{2\pi p_\perp}\delta(p_\perp-\hat{p}_\perp)\frac{1}{2\hat{p}_z}H(\hat{p}_z^2-p_z^2)\ , \tag{3}$$

where $H(x)$ is the Heaviside step function defined by $H(x)=+1$ for $x>0$, and $H(x)=0$ for $x<0$. Note from Eq. (3) that $\int d^3p\,F(p_\perp^2,p_z)=1$. Because the electrons are uniformly distributed in parallel momentum for $|p_z|<\hat{p}_z$, we refer to the p_z dependence of the distribution function in Eq. (3) as a waterbag distribution in p_z. For future reference, it is useful to introduce the maximum energy $\hat{\gamma}mc^2$, parallel speed $c\hat{\beta}_z$, and perpendicular speed $c\hat{\beta}_\perp$ defined by

$$\hat{\beta}_z=\frac{\hat{p}_z}{\hat{\gamma}mc},\quad \hat{\beta}_\perp=\frac{\hat{p}_\perp}{\hat{\gamma}mc}\ ,$$
$$\hat{\gamma}=\left[1+\frac{\hat{p}_\perp^2}{m^2c^2}+\frac{\hat{p}_z^2}{m^2c^2}\right]^{1/2}$$
$$=(1-\hat{\beta}_\perp^2-\hat{\beta}_z^2)^{-1/2}\ . \tag{4}$$

We further introduce the effective perpendicular and parallel temperatures defined by

$$T_\perp=\int d^3p\frac{p_\perp^2}{2\gamma m}F(p_\perp^2,p_z)\ ,$$
$$\tfrac{1}{2}T_\parallel=\int d^3p\frac{p_z^2}{2\gamma m}F(p_\perp^2,p_z)\ . \tag{5}$$

Substituting Eq. (3) into Eq. (5) and carrying out the required integrations over $p_\perp$ and p_z give

$$T_\perp=\tfrac{1}{2}\hat{\gamma}mc^2\hat{\beta}_\perp^2G(\hat{\beta}_z)\ ,$$
$$T_\parallel=\tfrac{1}{2}\hat{\gamma}mc^2[1-G(\hat{\beta}_z)+\hat{\beta}_z^2G(\hat{\beta}_z)]\ , \tag{6}$$

where $G(\hat{\beta}_z)$ is defined by

$$G(\hat{\beta}_z)=\frac{1}{2\hat{\beta}_z}\ln\left[\frac{1+\hat{\beta}_z}{1-\hat{\beta}_z}\right]\ . \tag{7}$$

From Eq. (7) we note that $G(\hat{\beta}_z)$ is a slowly increasing function of $\hat{\beta}_z$ with $G(\hat{\beta}_z)=1+\hat{\beta}_z^2/3+\cdots$ for $\hat{\beta}_z^2\ll1$. Moreover, in the limit of a nonrelativistic plasma with $\hat{\beta}_z^2\ll1$ and $\hat{\beta}_\perp^2\ll1$, Eq. (6) reduces to the expected results, $T_\perp\to\frac{1}{2}mc^2\hat{\beta}_\perp^2$ and $T_\parallel\to\frac{1}{3}mc^2\hat{\beta}_z^2$. Depending on the relative values of $\hat{\beta}_\perp$ and $\hat{\beta}_z$, it is clear that the choice of distribution function in Eq. (3) can cover a wide range of energy anisotropy.

For the choice of distribution function $F(p_\perp^2,p_z)$ in Eq. (3), the $p_\perp$- and p_z-integrations required in Eq. (2) can be carried out in closed analytical form. Some straightforward algebra that makes use of Eqs. (2), (3), (4), and (7) gives the dispersion relation

$$
\begin{aligned}
0 &= D_T(k_z,\omega) \\
&= 1-\frac{c^2k_z^2}{\omega^2} \\
&\quad -\frac{\omega_p^2/\hat{\gamma}}{\omega^2}\left[G(\hat{\beta}_z)+\frac{1}{2}\frac{\hat{\beta}_\perp^2}{(1-\hat{\beta}_z^2)}\left[\frac{c^2k_z^2-\omega^2}{\omega^2-c^2k_z^2\hat{\beta}_z^2}\right]\right] . \quad (8)
\end{aligned}
$$

Equation (8) is readily extended to the case of a multicomponent plasma by making the replacements $(\omega_p^2/\hat{\gamma})\cdots \rightarrow \sum_j(\omega_{pj}^2/\hat{\gamma}_j)\cdots$, $\hat{\beta}_z\rightarrow\hat{\beta}_{zj}$, $\hat{\beta}_\perp\rightarrow\hat{\beta}_{\perp j}$, etc. For a single active (electron) component, Eq. (8) can be expressed in the equivalent form

$$
\begin{aligned}
0 &= \omega^4-\omega^2\left[c^2k_z^2(1+\hat{\beta}_z^2)+\frac{\omega_p^2}{\hat{\gamma}}\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2}\right. \\
&\qquad \left. -\frac{\omega_p^2}{\hat{\gamma}}\left[\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2(1-\hat{\beta}_z^2)}-G(\hat{\beta}_z)\right]\right] \\
&\quad +c^2k_z^2\hat{\beta}_z^2\left[c^2k_z^2-\frac{\omega_p^2}{\hat{\gamma}}\left[\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2(1-\hat{\beta}_z^2)}-G(\hat{\beta}_z)\right]\right] , \quad (9)
\end{aligned}
$$

which is a quadratic equation for ω^2. In Eq. (9), $\omega_p^2=4\pi\hat{n}_e e^2/m$ is the nonrelativistic electron plasma frequency squared, and $G(\hat{\beta}_z)$ is defined in Eq. (7).

The dispersion relation (9) can be solved exactly for the complex oscillation frequency ω. In this regard, a careful examination of Eq. (9) shows that there are two classes of solutions for ω^2, namely, a fast-wave branch corresponding to stable oscillations with $\mathrm{Im}\,\omega=0$ and $(\mathrm{Re}\,\omega)^2>c^2k_z^2$, and a slow-wave branch which may or may not exhibit instability, depending on the degree of energy anisotropy. It is readily shown that the necessary and sufficient condition for the slow-wave branch to exhibit instability ($\mathrm{Im}\,\omega>0$) is given by

$$
\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2}>(1-\hat{\beta}_z^2)G(\hat{\beta}_z) . \quad (10)
$$

Moreover, when Eq. (10) is satisfied, the corresponding range of k_z^2 corresponding to instability is given by

$$
0<k_z^2<k_0^2\equiv\frac{\omega_p^2}{\hat{\gamma}c^2}\left[\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2(1-\hat{\beta}_z^2)}-G(\hat{\beta}_z)\right] . \quad (11)
$$

When Eq. (10) is satisfied, and k_z^2 is in the range specified by Eq. (11), the real oscillation frequency of the slow-wave branch satisfies $\mathrm{Re}\,\omega=0$ and the growth rate of the unstable mode is given by

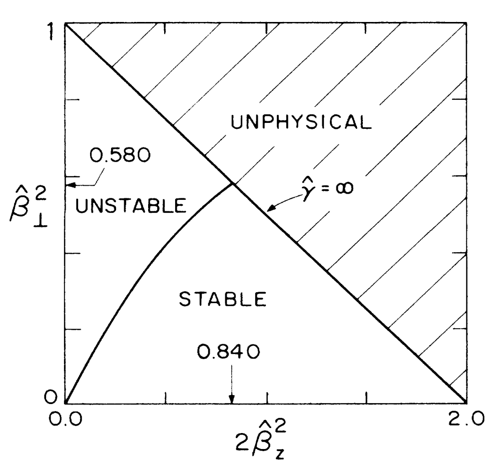

FIG. 1. Region of $(\hat{\beta}_\perp^2, 2\hat{\beta}_z^2)$ parameter space corresponding to instability [Eq. (10)].

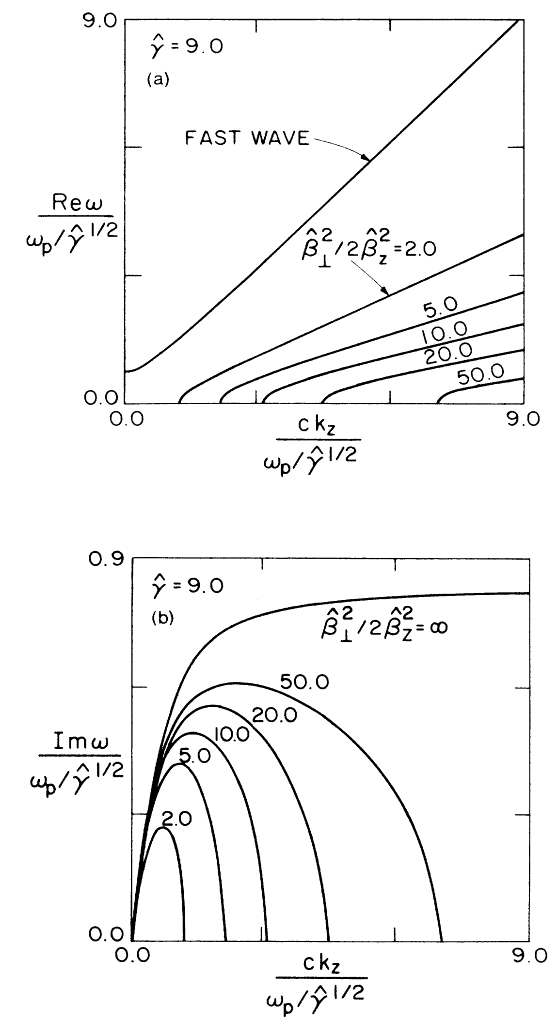

FIG. 2. Plots of (a) normalized real frequency $\mathrm{Re}\,\omega/[\omega_p/\hat{\gamma}^{1/2}]$ and (b) normalized growth rate $\mathrm{Im}\,\omega/[\omega_p/\hat{\gamma}^{1/2}]$ vs $ck_z/[\omega_p/\hat{\gamma}^{1/2}]$, for $\hat{\gamma}=9$ and several values of $\hat{\beta}_\perp^2/2\hat{\beta}_z^2$ [Eq. (9)].

$$\mathrm{Im}\omega=\frac{1}{\sqrt{2}}\left\{\left[\left[c^2k_z^2\hat{\beta}_z^2+\frac{\omega_p^2}{\hat{\gamma}}\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2}-c^2(k_0^2-k_z^2)\right]^2+4c^4k_z^2\hat{\beta}_z^2(k_0^2-k_z^2)\right]^{1/2}-\left[c^2k_z^2\hat{\beta}_z^2+\frac{\omega_p^2}{\hat{\gamma}}\frac{\hat{\beta}_\perp^2}{2\hat{\beta}_z^2}-c^2(k_0^2-k_z^2)\right]\right\}^{1/2}. \tag{12}$$

Note from Eq. (12) that $\mathrm{Im}\omega=0$ for $k_z=0$ and $k_z^2=k_0^2$, and that $\mathrm{Im}\omega$ passes through a maximum for some value of k_z^2 intermediate between 0 and k_0^2.

In the nonrelativistic limit with $\hat{\beta}_\perp^2,\hat{\beta}_z^2 \ll 1$, the necessary and sufficient condition for instability in Eq. (10) becomes $\hat{\beta}_\perp^2/2\hat{\beta}_z^2>1$, and the range of instability is given by $0<k_z^2<k_0^2\equiv(\omega_p^2/c^2)(\hat{\beta}_\perp^2/2\hat{\beta}_z^2-1)$. In the relativistic regime, however, the instability criterion in Eq. (10) is more complicated, which is illustrated in Fig. 1. In Fig. 1, the region of $(\hat{\beta}_\perp^2,2\hat{\beta}_z^2)$ parameter space corresponding to instability is above the contour connecting the origin to $(\hat{\beta}_\perp^2,2\hat{\beta}_z^2)=(0.580,\ 0.840)$, which corresponds to $\hat{\gamma}=\infty$. While the detailed form of the instability criterion in Eq. (10) differs from the nonrelativistic case, it is evident from Fig. 1 that the condition for instability in the relativistic regime is qualitatively the same, i.e., the Weibel instability exists for sufficiently large values of $\hat{\beta}_\perp^2/2\hat{\beta}_z^2$. Put another way, for the choice of distribution function in Eq. (3), the Weibel instability in a relativistic anisotropic plasma can be completely stabilized by increasing the thermal anisotropy $2\hat{\beta}_z^2/\hat{\beta}_\perp^2$ to sufficiently large values.

As a numerical example, shown in Fig. 2 are plots of normalized real frequency $\mathrm{Re}\omega/(\omega_p/\hat{\gamma}^{1/2})$ [Fig. 2(a)] and growth rate $\mathrm{Im}\omega/(\omega_p/\hat{\gamma}^{1/2})$ [Fig. 2(b)] versus normalized wave number $ck_z/[\omega_p/\hat{\gamma}^{1/2}]$ for $\hat{\gamma}=9$ and several values of energy anisotropy $\hat{\beta}_\perp^2/2\hat{\beta}_z^2$. The real oscillation frequencies for both the fast-wave and slow-wave branches are presented in Fig. 2(a). Moreover, for $\hat{\gamma}=9$, the slow-wave branch becomes completely stable ($\mathrm{Im}\omega=0$ and $k_0^2=0$) for $\hat{\beta}_\perp^2/2\hat{\beta}_z^2=0.697$. It is evident from Fig. 2(b) that the strongest instability occurs for the largest energy anisotropy, i.e., $\hat{\beta}_z^2=0$ and $\hat{\beta}_\perp^2=80/81$ (for $\hat{\gamma}=9$). Moreover, depending on the value of $\hat{\beta}_\perp^2/2\hat{\beta}_z^2$, the maximum growth rate in Fig. 2(b) can be a substantial fraction of $\omega_p/\hat{\gamma}^{1/2}$.

ACKNOWLEDGMENTS

This research was supported in part by the Office of Naval Research and in part by the Department of Energy.

[1]E. S. Weibel, Phys. Rev. Lett. **2**, 83 (1959).
[2]R. C. Davidson, *Handbook of Plasma Physics*, edited by A. A. Galeev and R. N. Sudan (North-Holland, New York, 1983), Vol. 1, pp. 521–585, cf. pp. 552–573, and references therein.
[3]C. F. Kennel and H. E. Petschek, J. Geophys. Res. **72**, 3303 (1967).
[4]N. T. Gladd, Phys. Fluids **26**, 974 (1983), and references therein.
[5]C. S. Wu and L. C. Lee, Astrophys. J. **230**, 621 (1979).
[6]K. R. Chu and J. L. Hirschfield, Phys. Fluids **21**, 461 (1978).
[7]K. T. Tsang, Phys. Fluids **27**, 1659 (1984).
[8]P. L. Pritchett, Phys. Fluids **27**, 2393 (1984).