High-frequency power spectra for systems subject to noise

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We examine the falloff of power spectra at high frequencies as a possible means of distinguishing systems exhibiting deterministic chaos from systems subject to noise. To this end, we derive the asymptotic series describing the high-frequency falloff of the power spectrum for systems subject to noise. Our analysis applies to systems with an arbitrary finite number of degrees of freedom and includes the cases of additive and multiplicative white noise and the case of continuous nonwhite noise appearing with a general parametric dependence. In the case of colored noise, we show that the frequency at which crossover to the asymptotic behavior occurs is the effective bandwidth of the noise, i.e., the inverse of the correlation time. This result should be particularly useful in cases where the noise is not directly observable.

I. INTRODUCTION

Until the 1970's, erratic behavior in physical systems was generally explained as an effect of noise, Brownian motion being the paradigmatic example. However, the work of Lorenz[1] in 1963 made it clear that erratic behavior can occur intrinsically in deterministic systems even if the number of degrees of freedom is small. Given these two sources of erratic behavior, it is important to be able to determine which is the cause in a given case. The important physical issue is whether the observed random behavior can be described by a small number of deterministic equations or is more adequately modeled by a stochastic process. In this paper we provide a partial answer to this problem by deriving criteria for the observed behavior to be essentially stochastic.

The identification of a known "route to chaos," the calculation of fractal Hausdorff dimension of attractors, and the determination of positive Lyapunov exponents are commonly taken as signatures of deterministic chaos. The measurement of finite, nonzero metric entropy of time series has also been suggested as an indicator of deterministic chaos.[2] All these methods, however, face various well-known technical difficulties when applied to time-series data of real physical systems.[3] What is needed is a tool that allows us to answer the basic question before trying to calculate Lyapunov numbers, dimensions, and entropies, namely, is the observed erratic behavior essentially deterministic or stochastic? In this paper we will show that the falloff of the power spectrum at high frequencies can furnish such a tool. Intuitive arguments have been advanced to indicate that systems that can be described by deterministic equations with a few variables should have power spectra that fall off faster than any inverse power of the frequency, e.g., exponentially, while on the other hand spectra for systems whose behavior is essentially stochastic should decay via a power law.[4,5] In spite of these intuitive arguments, a general theoretical solution to the problem of high-frequency behavior of power spectra of deterministic or stochastic systems has not been achieved up to now. Investigations are under way on this problem for deterministic systems.[6] In this paper we solve the problem for stochastic systems and derive the asymptotic form of the power spectrum for a general system subject to noise. We prove that this form always corresponds to power-law decay. This provides half of the answer to the issue raised in the first paragraph; we have a condition for the observed random behavior to be essentially stochastic.

Experimentally, both exponential and power-law decay of power spectra are seen. (Obviously at very high frequency the signal will be drowned in the instrumental noise which is white. Thus, as far as experimental applications are concerned, we consider the falloff of the power spectrum before it flattens out into the instrumental noise level.) Exponential decay is typical of the Taylor-Couette system in the turbulent regime (see Fig. 1).[7] It is seen also in model deterministic systems such as the Lorenz model.[4] On the other hand, power-law decays occur in

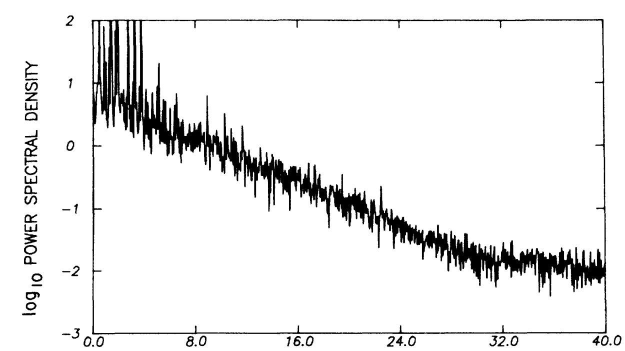

FIG. 1. A semilogarithmic plot of a power spectrum from the Taylor-Couette system showing exponential decay at high frequencies. The flat region at very high frequencies represents instrumental noise.

the Rayleigh-Bénard system. Both ω^{-2} and ω^{-4} decays have been reported (see Fig. 2).[4,8]

These power-law decays, together with other features of the transition to turbulence in the Rayleigh-Bénard system, have led Greenside *et al.*, to propose a stochastic model for the behavior of the system in the regime involved.[4] Clearly, it would be desirable to have a proof that stochastic systems do, indeed, show power-law decays together with criteria that show when ω^{-2}, ω^{-4}, or higher-order decay is to be expected.

Various partial results have been reported. Brey *et al.*

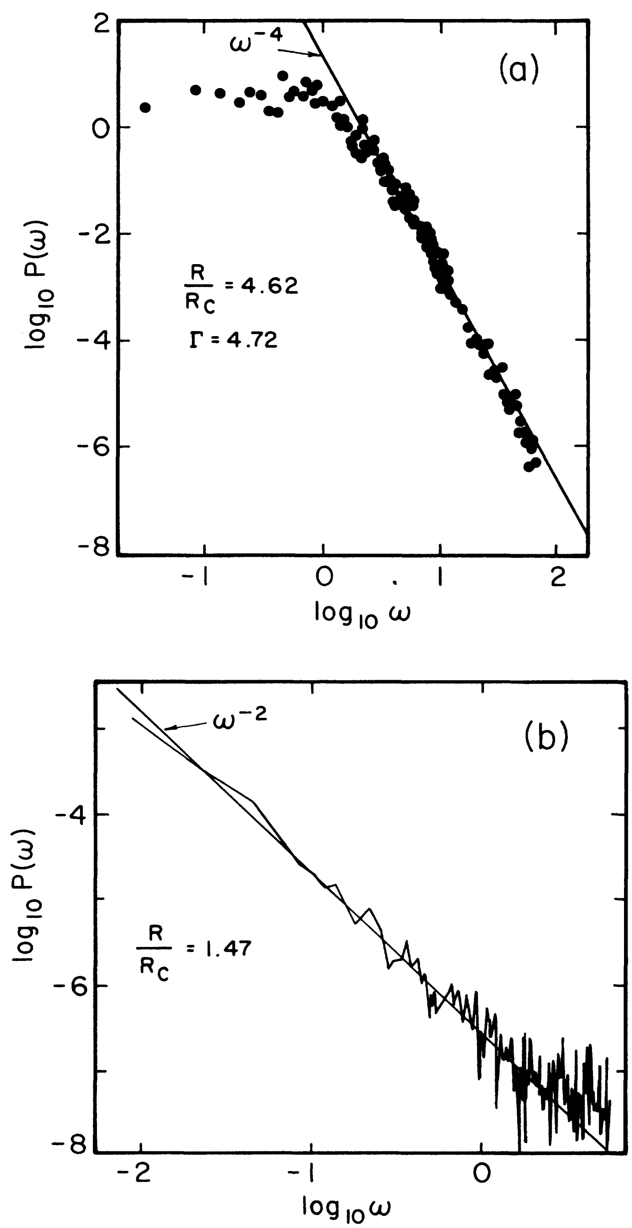

FIG. 2. Log-log plots of power spectra from the Rayleigh-Bénard system showing power-law decay at high frequencies (see Ref. 4).

have shown that, for a one-variable system subject to white noise via a first-order Langevin equation, the spectrum decays as ω^{-2}.[9] They state that the method they use may be extended to show that a one-variable system subject to white noise via a second-order Langevin equation exhibits an ω^{-4} decay. Caroli *et al.*[10] have shown that an N-variable system subject to additive white noise in a first-order equation will exhibit an ω^{-2} decay, while a system governed by a second-order equation exhibits ω^{-4} decay. Neither study examines the effect of colored noise nor considers situations that lead to higher-power-law decays.

We have examined an N-variable system subject to multiplicative white noise and have derived the general conditions under which the power spectrum decays as ω^{-2n}, n an integer. Because the number of variables is arbitrary, the analysis includes the case of second- and higher-order Langevin equations and the case of a system subject to general parametric colored noise. This is explained in detail in Sec. III A.

In Sec. IV we show that in the case of a colored noise we can estimate the correlation time of the noise from an examination of the power spectrum of a variable or a function of variables of the system itself without observing the noise directly.

In Sec. II we will outline the intuitive arguments that lead us to expect faster than power-law decay for deterministic systems and power-law decays for stochastic systems. In Sec. III we present the general system and prove the results mentioned above. In Sec. IV we examine the case of colored noise and show how the correlation time may be estimated.

II. INTUITIVE ARGUMENT

The following argument will show why high-frequency power spectra are expected to distinguish between deterministic and stochastic systems. Consider $x(t)$, a function of time which may be one variable in a deterministic dynamical system or a realization of a stochastic process. We consider $x(t)$ to have, in some sense, a Fourier transform $\hat{x}(\omega)$. Then, if $x(t)$ is differentiable with respect to time, the Fourier transform of dx/dt is $i\omega\hat{x}(\omega)$. Thus, if dx/dt exists, $\hat{x}(\omega)$ must fall off faster than ω^{-1} as $\omega \to \infty$ so that the inverse Fourier transform of $i\omega\hat{x}(\omega)$ will exist. Now, the power spectrum of $x(t)$, $S_x(\omega)$, is, roughly, the square of $\hat{x}(\omega)$. So, if $x(t)$ is once differentiable, $S_x(\omega)$ must fall off faster than ω^{-2}.

Similarly, one may argue that if the nth derivative of $x(t)$ exists then $S_x(\omega)$ falls off faster than ω^{-2n}. If one assumes that variables of a dynamical system are C^∞ functions of time, which is clearly true for model systems like the Lorenz model, then one will expect the power spectra of variables of a dynamical system to fall off faster than any power of ω^{-1}.

On the other hand, it is well known that realizations of stochastic processes are not C^∞ functions of time. Indeed, they are often not even once differentiable. Therefore, we expect power spectra of stochastic processes to fall off as some power of ω^{-1}.

Clearly, this argument does not constitute a proof. We have not defined the sense in which the Fourier transform of a variable of a general dynamical system or of a general stochastic process exists nor have we proved that the standard manipulations may be performed. Sec. III will provide a rigorous version of the argument for the case of a stochastic system.

III. ASYMPTOTIC FORM OF THE POWER SPECTRUM FOR A GENERAL SYSTEM

A. Definition of the general system

The general system with which we will work is an N-variable stationary stochastic process $\mathbf{X}=(X_1,X_2,\ldots,X_N)$ satisfying the stochastic differential equation

$$d\mathbf{X}=\mathbf{f}(\mathbf{X})dt+\underline{\sigma}(\mathbf{X})d\mathbf{W}\ , \tag{1}$$

where $\mathbf{W}=(W_1,W_2,\ldots,W_N)$ is a vector of N independent Wiener processes, $\mathbf{f}(\mathbf{X})$ is the drift vector for $\mathbf{X}$, and $\underline{\sigma}^T(\mathbf{X})\underline{\sigma}(\mathbf{X})$ is the diffusion matrix for $\mathbf{X}$. We assume that $\mathbf{f}$ and $\underline{\sigma}$ are such as to admit a single stationary solution to (1). For mathematical convenience, we will interpret (1) as an Itô equation. This is no restriction since any Stratonovic equation can be transformed into an equivalent Itô equation.

Equation (1) corresponds to the Langevin equation

$$\dot{\mathbf{X}}=\mathbf{f}(\mathbf{X})+\underline{\sigma}(\mathbf{X})\xi\ , \tag{2}$$

where $\xi=(\xi_1,\xi_2,\ldots,\xi_N)$ is a vector of independent Gaussian white noises. Form (1) is preferred mathematically because neither the white noise ξ nor, as a consequence, the derivative $\dot{\mathbf{X}}$ can be defined as an ordinary stochastic process. Nevertheless, form (2) will prove useful in interpreting our results intuitively. The fact that the vector $\mathbf{X}$ has an arbitrary number of components allows us to include a wide variety of cases in (1).

Clearly, we may include second order Langevin equations of the form

$$\ddot{\mathbf{X}}=\mathbf{g}(\mathbf{X},\dot{\mathbf{X}})+\underline{\pi}(\mathbf{X},\dot{\mathbf{X}})\xi \tag{3}$$

by the common trick of substituting $\mathbf{V}$ for $\dot{\mathbf{X}}$ and appending the equation

$$\dot{\mathbf{X}}=\mathbf{V}$$

to (3) to obtain

$$\begin{aligned}&\dot{\mathbf{X}}=\mathbf{V}\ ,\\&\dot{\mathbf{V}}=\mathbf{g}(\mathbf{X},\mathbf{V})+\underline{\pi}(\mathbf{X},\mathbf{V})\xi\ .\end{aligned} \tag{4}$$

If $\mathbf{X}$ is an N-dimensional vector, this is, of course, a $2N$-dimensional version of (1). Note that in this case (1) has a degenerate diffusion matrix since the X_i are not directly driven by white noise. Thus we may expect (4) to exhibit behavior which is nongeneric for the general $2N$-dimensional version of (1). This technique can, of course, be extended to higher-order Langevin equations.

We may include nonwhite noise in the system (1) if the equations divide into two parts, a "system" which we will again call $\mathbf{X}$, and a "noise" $\mathbf{Z}$ which evolves independently of $\mathbf{X}$,

$$\dot{\mathbf{X}}=\mathbf{g}(\mathbf{X},\mathbf{Z})\ , \tag{5a}$$

$$\dot{\mathbf{Z}}=\mathbf{h}(\mathbf{Z})+\underline{\pi}(\mathbf{Z})\xi\ . \tag{5b}$$

Note again that this system, when regarded as a special case of (1), has a degenerate diffusion matrix and, thus, may exhibit nongeneric behavior.

Equation (5a) allows a general parametric dependence on the noise $\mathbf{Z}$ so we are not restricted to additive or multiplicative (colored) noise. Moreover, (5b) is the equation for a general (stationary) diffusion process. Thus, we may consider any nonwhite noise which (1) has almost surely continuous sample paths and (2) is part of an M-variable Markov process for some $M>0$.

B. Statement of the theorem

Returning to (1) in its general form, we are interested in a general function $Y(\mathbf{X})$ and its power spectrum $S_Y(\omega)$. Let us discuss how we expect $S_Y(\omega)$ to behave as $\omega\rightarrow\infty$ based on the intuitive discussion in Sec. II.

We may write the Itô stochastic differential of $Y(\mathbf{X})$ as

$$\begin{aligned}dY&=\mathscr{A}Y\,dt+\sum_{i,j}\frac{\partial Y}{\partial X_i}\sigma_{ij}(\mathbf{X})dW_j\\&=\mathscr{A}Y\,dt+(\underline{\sigma}^T\nabla_x Y)\cdot d\mathbf{W}\end{aligned}$$

or, in Langevin form,

$$\dot{Y}=\mathscr{A}Y+(\underline{\sigma}^T\nabla_x Y)\cdot\xi\ . \tag{6}$$

Here $||\sigma_{ij}||=\underline{\sigma}$ and $\mathscr{A}$ is the Kolmogorov backward operator for the process $\mathbf{X}$,

$$\mathscr{A}=\sum_i f_i\frac{\partial}{\partial X_i}+\tfrac{1}{2}\sum_{i,j,k}\sigma_{ik}\sigma_{jk}\frac{\partial^2}{\partial X_i\partial X_j}\ . \tag{7}$$

$\mathscr{A}$ is the formal adjoint of the Fokker-Planck operator for $\mathbf{X}$.

If the quantity $\underline{\sigma}^T\nabla_x Y$ occurring in Eq. (6) is not identically zero than $\dot{Y}$ will have a white-noise part. In that case, we will expect the power spectrum for $\dot{Y}$, $S_{\dot{Y}}$, to behave like the spectrum of white noise as $\omega\rightarrow\infty$. In other words, we expect

$$S_{\dot{Y}}(\omega)\doteq C\neq 0\ ,$$

where $\doteq$ means asymptotic equivalence as $\omega\rightarrow\infty$.

As argued in Sec. II, integrating $\dot{Y}$ with respect to time to get Y will bring down a power of ω^{-2} in the power spectrum, giving

$$S_Y(\omega)\doteq C\omega^{-2}\ .$$

If the quantity $\underline{\sigma}^T\nabla_x Y$ *is* identically zero, then $\mathscr{A}Y(\mathbf{X})$ is a function of $\mathbf{X}$ which may be regarded as the derivative of Y with respect to time. We may then look at the stochastic differential of $\mathscr{A}Y$. If this has a contribution from white noise ($\underline{\sigma}^T\nabla_x\mathscr{A}Y$ is not identically zero) then

$$S_{\dot{Y}} \doteq C\omega^{-2} ,$$

so

$$S_Y \doteq C\omega^{-4} .$$

Clearly, this process may be continued indefinitely. We are thus led to the following definition and theorem

Definition. We say that $Y(\mathbf{X})$ is once removed from white noise if $\underline{\sigma}^T \nabla_x Y$ is not identically zero. We say Y is n times removed from white noise ($n > 1$) if

(i) $\underline{\sigma}^T \nabla_x \mathscr{A}^p Y \equiv 0$ for all $p < (n-1)$ and

(ii) $\underline{\sigma}^T \nabla_x \mathscr{A}^{n-1} Y$ is not identically zero.

Note that Y being n times removed from white noise is rigorously equivalent to the sample paths of Y being almost surely $n-1$ times differentiable.

Theorem. If Y is n times removed from white noise then $S_Y(\omega) \doteq C\omega^{-2n}$.

C. Proof of the theorem

We now prove the theorem that concluded Sec. IIIB. The power spectrum of Y is the Fourier transform of the correlation function of Y,

$$S_Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} C_Y(\tau) ,$$

where

$$C_Y(\tau) = \langle Y(t=0) Y(t=\tau) \rangle .$$

The symmetry of $C(\tau)$ allows us to write

$$S_Y(\omega) = \mathrm{Re} \left[\frac{1}{\pi} \int_0^{\infty} d\tau\, e^{-i\omega\tau} C_Y(\tau) \right] .$$

Following Caroli *et al.* we apply the standard asymptotic expansion for a one-sided Fourier transform,[10,11]

$$S(\omega) \doteq \mathrm{Re} \left[\frac{1}{\pi} \sum_{s=0}^{\infty} \left[\frac{-i}{\omega} \right]^{s+1} \left. \frac{d^s C_Y}{d\tau^s} \right|_{\tau \downarrow 0} \right]$$
$$= \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \left[\frac{1}{\omega} \right]^{2n+2} \left. \frac{d^{2n+1} C_Y}{d\tau^{2n+1}} \right|_{\tau \downarrow 0} , \quad (8)$$

where

$$\left. \frac{d^{2n+1} C_Y}{d\tau^{2n+1}} \right|_{\tau \downarrow 0}$$

is the limit of the $(2n+1)$th derivative of $C_Y(\tau)$ as τ approaches zero from above.

Clearly, the leading term in the asymptotic series will be determined by the first nonzero odd derivative of $C_Y(\tau)$ as $t \downarrow 0$. The rest of the proof consists of finding the conditions under which the $(2n+1)$th derivative is the first nonzero odd derivative of C_Y as $\tau \downarrow 0$.

To proceed we follow Caroli *et al.* and rewrite $C_Y(\tau)$ as

$$C_Y(\tau) = \langle Y e^{\mathscr{A}\tau} Y \rangle .$$

So, we find

$$\left. \frac{d^m C_Y}{d\tau^m} \right|_{\tau \downarrow 0} = \langle Y \mathscr{A}^m Y \rangle .$$

Thus, our asymptotic series may be rewritten as[10]

$$S_Y(\omega) \doteq \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\omega^{2n+2}} \langle Y \mathscr{A}^{2n+1} Y \rangle . \quad (9)$$

We now derive some properties of the operator $\mathscr{A}$. If $\eta(\mathbf{X})$ and $\zeta(\mathbf{X})$ are arbitrary functions of $\mathbf{X}$, then we have

$$\mathscr{A}(\eta\zeta) = \eta \mathscr{A} \zeta + \zeta \mathscr{A} \eta + \frac{1}{2} \sum_{i,j,k} \sigma_{ik} \sigma_{jk} \left[\frac{\partial \eta}{\partial X_i} \frac{\partial \zeta}{\partial X_j} + \frac{\partial \eta}{\partial X_j} \frac{\partial \zeta}{\partial X_i} \right]$$
$$= \eta \mathscr{A} \zeta + \zeta \mathscr{A} \eta + (\underline{\sigma}^T \nabla \eta) \cdot (\underline{\sigma}^T \nabla \zeta) .$$

Hence, it follows that

$$\eta \mathscr{A} \zeta = \mathscr{A}(\eta\zeta) - \zeta \mathscr{A} \eta - (\underline{\sigma}^T \nabla \eta) \cdot (\underline{\sigma}^T \nabla \zeta) . \quad (10)$$

If we set $\eta = \zeta$, we obtain

$$\eta \mathscr{A} \eta = \tfrac{1}{2} \mathscr{A}(\eta^2) - \tfrac{1}{2} \, | \underline{\sigma}^T \nabla \eta |^2 . \quad (11)$$

Finally, note that since $\mathscr{A}$ is the adjoint of the Fokker-Planck operator, in the stationary state, we have

$$\langle \mathscr{A} \eta \rangle = 0 \quad (12)$$

for arbitrary η.

We now apply (10) to the expression $Y \mathscr{A}^{2n+1} Y = Y \mathscr{A}(\mathscr{A}^{2n} Y)$ to obtain

$$Y \mathscr{A}^{2n+1} Y = \mathscr{A}(Y \mathscr{A}^{2n} Y) - (\mathscr{A} Y)(\mathscr{A}^{2n} Y) - (\underline{\sigma}^T \nabla Y) \cdot (\underline{\sigma}^T \nabla \mathscr{A}^{2n} Y) .$$

We may then apply (10) to the second term on the right-hand side to obtain

$$(\mathscr{A} Y)(\mathscr{A}^{2n} Y) = \mathscr{A}[(\mathscr{A} Y)(\mathscr{A}^{2n-1} Y)] - (\mathscr{A}^2 Y)(\mathscr{A}^{2n-1} Y) - (\underline{\sigma}^T \nabla \mathscr{A} Y) \cdot (\underline{\sigma}^T \nabla \mathscr{A}^{2n-1} Y) .$$

Repeating the procedure, we obtain

$$(\mathscr{A}^q Y)(\mathscr{A}^{2n+1-q} Y) = \mathscr{A}[(\mathscr{A}^q Y)(\mathscr{A}^{2n-q} Y)] - (\mathscr{A}^{q+1} Y)(\mathscr{A}^{2n-q} Y) - (\underline{\sigma}^T \nabla \mathscr{A}^q Y) \cdot (\underline{\sigma}^T \nabla \mathscr{A}^{2n-q} Y) .$$

When we get to $q = n$, the term on the left-hand side is $(\mathscr{A}^n Y) \mathscr{A} (\mathscr{A}^n Y)$ so we may apply (11) to obtain

$$(\mathscr{A}^n Y) \mathscr{A} (\mathscr{A}^n Y) = \tfrac{1}{2} \mathscr{A}[(\mathscr{A}^n Y)^2] - \tfrac{1}{2} \, | \underline{\sigma}^T \nabla \mathscr{A}^n Y |^2 .$$

Taking this all into account, we have

$$Y \mathscr{A}^{2n+1} Y = \sum_{q=0}^{n-1} (-1)^q \mathscr{A}[(\mathscr{A}^q Y)(\mathscr{A}^{2n-q} Y)] - \sum_{q=0}^{n-1} (-1)^q (\underline{\sigma}^T \nabla \mathscr{A}^q Y) \cdot (\underline{\sigma}^T \nabla \mathscr{A}^{2n-q} Y) + (-1)^n \tfrac{1}{2} \mathscr{A}[(\mathscr{A}^n Y)^2] - (-1)^n \tfrac{1}{2} \, | \underline{\sigma}^T \nabla \mathscr{A}^n Y |^2 .$$

The mean of the first and third terms on the right-hand side is zero because of (12). So we have

$$\langle Y\mathscr{A}^{2n+1}Y\rangle=\langle \tfrac{1}{2}(-1)^{n+1}\,|\,\underline{\sigma}^T\nabla\mathscr{A}^nY\,|^2+\sum_{q=0}^{n-1}(-1)^{q+1}(\underline{\sigma}^T\nabla\mathscr{A}^qY)\cdot(\sigma^T\nabla\mathscr{A}^{2n-q}Y)\rangle . \tag{13}$$

Induction then yields that $[(-1)^{n+1}/\pi]\langle Y\mathscr{A}^{2n+1}Y\rangle$ is the first nonzero coefficient in the series if and only if
(i) $\underline{\sigma}^T\nabla\mathscr{A}^qY\equiv 0$ for all $q<n$ and
(ii) $\underline{\sigma}^T\nabla\mathscr{A}^nY$ is not identically zero.
This is the expected result.

IV. COLORED NOISE

We now apply our formula (9) to the example of colored noise, i.e., (5). In this case the operator $\underline{\sigma}^T\nabla$ becomes $\underline{\pi}^T\nabla_z$. We are interested in the power spectrum of $Y(\mathbf{X})$. The coefficient of ω^{-2} in the asymptotic series for $S_Y(\omega)$ is

$$\frac{-1}{\pi}\langle Y\mathscr{A}Y\rangle=\frac{1}{2\pi}\langle\,|\,\underline{\pi}^T\nabla_zY\,|^2\rangle=0 .$$

So, the spectrum must fall off as a higher power.

The coefficient of ω^{-4} is then

$$\frac{1}{\pi}\langle Y\mathscr{A}^3Y\rangle=\frac{1}{2\pi}\langle\,|\,\underline{\pi}^T\nabla_z\mathscr{A}Y\,|^2\rangle . \tag{14}$$

Thus, if the special condition

$$|\,\underline{\pi}^T\nabla_z\mathscr{A}Y\,|^2\equiv 0$$

is not satisfied, a colored noise will produce an ω^{-4} falloff.

However, if the time scale of the system τ_X, and the correlation time of the noise, τ_Z, are well separated ($\tau_X\gg\tau_Z$) and we look at frequencies in an intermediate range ($\tau_X^{-1}\ll\omega\ll\tau_Z^{-1}$) then the noise "looks" white. In such an intermediate frequency range we therefore expect an ω^{-2} behavior of the power spectrum of a function of $\mathbf{X}$.

Clearly, then, if we plot $\log S_X(\omega)$ versus $\log\omega$, we will observe the behavior shown in Fig. 3. For intermediate frequencies, the plot will be a straight line with slope -2. For high frequencies, the plot will be a straight line with slope -4. The change will occur in a region around the inverse correlation time of the noise.

This result will be particularly useful in experimental situations when the noise is not directly observable. The correlation time of the noise perturbing the system can be estimated from the frequency at which crossover to the asymptotic regime occurs in the power spectrum of the system.

We may illustrate this by looking at the linear system

$$\dot{X}=-\gamma X+\sigma Z , \tag{15a}$$

$$\dot{Z}=-\beta Z+\beta\xi . \tag{15b}$$

The stationary solution to (15b) is an Ornstein-Uhlenbeck process with correlation time β^{-1}. The characteristic time for X is γ^{-1} so, for $\gamma^{-1}\ll\omega\ll\beta^{-1}$ we expect ω^{-2} behavior for $S_X(\omega)$.

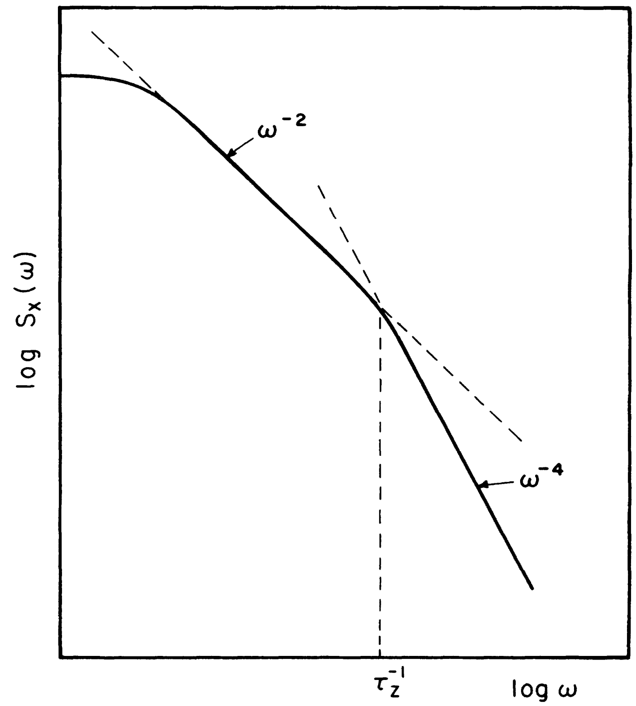

FIG. 3. Power spectrum of system described by Eqs. (15) showing ω^{-4} behavior at high frequencies and ω^{-2} behavior at intermediate frequencies.

The power spectrum of X may be obtained exactly,[12]

$$S_X(\omega)=\frac{\beta^2\sigma^2}{2\pi}\frac{1}{(\gamma^2+\omega^2)(\beta^2+\omega^2)} .$$

Clearly, when ω dominates γ and β dominates ω

$$S_X(\omega)\simeq\frac{\sigma^2}{2\pi}\frac{1}{\omega^2} ,$$

while, when ω dominates both γ and β,

$$S_X(\omega)\simeq\frac{\beta^2\sigma^2}{2\pi}\frac{1}{\omega^4} .$$

A simple calculation shows that the lines defined by

$$\log S_X=\log\left[\frac{\sigma^2}{2\pi}\frac{1}{\omega^2}\right]$$

and

$$\log S_X=\log\left[\frac{\sigma^2\beta^2}{2\pi}\frac{1}{\omega^4}\right]$$

intersect at $\omega=\beta$, the correlation time of the noise.

It must be stressed that the ω^{-2} behavior does not arise from the series (9). Equation (9) is an asymptotic expansion that breaks down for ω sufficiently small. A standard test for the validity of an asymptotic expansion is to find where the magnitude of the first neglected term is equal to the magnitude of the truncated series. In this case, the first neglected term is the ω^{-6} term. For the case of an additive Ornstein-Uhlenbeck noise in N dimensions, we show in the Appendix that this term is equal in magnitude to the ω^{-4} term when ω equals β, the inverse correlation time of the noise, provided the time scales of the system and the noise are well separated. This confirms our observation that the ω^{-4} decay must break down for frequencies less than the inverse correlation time of the noise.

APPENDIX

In this appendix we demonstrate that, for the case of an additive Ornstein-Uhlenbeck noise, the asymptotic approximation to $S_X(\omega)$ breaks down when ω equals the inverse correlation time of the noise, provided the time scales of the system and the noise are well separated. We do this by showing that the first neglected term, the ω^{-6} term, is equal in magnitude to the ω^{-4} term when ω equals τ_Z^{-1}.

The system we consider is

$$\dot{\mathbf{X}}=\mathbf{f}(\mathbf{X})+\gamma \underline{g}\mathbf{Z} \, , \qquad \dot{\mathbf{Z}}=-\gamma^2\mathbf{Z}+\gamma\xi \, . \tag{16}$$

Here, $\mathbf{Z}$ is a vector of independent Ornstein-Uhlenbeck noises with correlation time γ^{-2} and stationary probability density

$$p_S(\mathbf{Z})=(\pi)^{-1/2}\exp(-|\mathbf{Z}|^2) \, , \tag{17}$$

independent of γ. This independence is a consequence of the γ scaling of (16) which is the standard scaling for investigating nearly white noise.[13]

The coefficient of ω^{-6} in the asymptotic expansion for $S_{X_m}(\omega)$ is, according to (13),

$$\frac{-1}{\pi}\langle X_m \mathscr{A}^5 X_m\rangle=\frac{-1}{\pi}\langle \tfrac{1}{2}(-1)|\underline{\sigma}^T\nabla(\mathscr{A}^2X_m)|^2 +(\underline{\sigma}^T\nabla\mathscr{A}X_m)\cdot(\underline{\sigma}^T\nabla\mathscr{A}^3X_m)\rangle \, . \tag{18}$$

In this case we have

$$\mathscr{A}=(\mathbf{f}+\gamma\underline{g}\mathbf{Z})\cdot\nabla_x-\gamma^2\mathbf{Z}\cdot\nabla_z+\tfrac{1}{2}\gamma^2\nabla_z^2 \, ,$$

$$\underline{\sigma}^T\nabla=\gamma\nabla_z \, .$$

Plugging these formulas into the two terms on the right-hand side of (18), we have

$$\langle \tfrac{1}{2}(-1)|\underline{\sigma}^T\nabla(\mathscr{A}^2X_m)|^2\rangle =\left\langle \frac{-\gamma^4}{2}\sum_k\left[\sum_p g_{pk}\frac{\partial}{\partial X_p}f_m-\gamma^2 g_{mk}\right]^2\right\rangle \, ,$$

and

$$\langle(\underline{\sigma}^T\nabla\mathscr{A}X_m)\cdot(\underline{\sigma}^T\nabla\mathscr{A}^3X_m)\rangle=\gamma^4\Big\langle\gamma^4\sum_k g_{mk}^2-\gamma^2\sum_{p,k}g_{mk}g_{pk}\frac{\partial}{\partial X_p}f_m+\sum_{p,k}g_{mk}g_{pk}(\mathbf{f}\cdot\nabla_x)\frac{\partial}{\partial X_p}f_m+\sum_{p,k}g_{mk}g_{pk}\frac{\partial}{\partial X_p}(\mathbf{f}\cdot\nabla_x f_m)$$
$$+\gamma\sum_{p,j,k}g_{mk}g_{pj}Z_j\sum_q g_{qk}\frac{\partial}{\partial X_q}\frac{\partial}{\partial X_p}f_m+\gamma\sum_{p,k}g_{mk}g_{pk}\sum_{q,i}g_{qi}Z_i\frac{\partial}{\partial X_q}\frac{\partial}{\partial X_p}f_m\Big\rangle \, .$$

We now note that $(\partial/\partial X_p)f_m$ is a local characteristic frequency of the deterministic system

$$\dot{\mathbf{X}}=\mathbf{f}(\mathbf{X})$$

and, so, is much less than γ^2. Similarly, $(\mathbf{f}\cdot\nabla_x)(\partial/\partial X_p)f_m$ and $(\partial/\partial X_p)(\mathbf{f}\cdot\nabla_x f_m)$ are local characteristic quantities of dimension frequency squared which must, thus, be much less than γ^4. This allows us to write

$$\langle -\tfrac{1}{2}|\underline{\sigma}^T\nabla(\mathscr{A}^2X_m)|^2\rangle\simeq\frac{-\gamma^8}{2}\sum_k\langle g_{mk}^2\rangle \, ,$$

and

$$\langle(\underline{\sigma}^T\nabla\mathscr{A}X_m)\cdot(\underline{\sigma}^T\nabla\mathscr{A}^3X_m)\rangle \simeq\gamma^8\sum_k\langle g_{mk}^2\rangle +\gamma^5\left\langle\sum_{p,j,k}g_{mk}g_{pj}Z_j\sum_q g_{qk}\frac{\partial}{\partial X_q}\frac{\partial}{\partial X_p}f_m\right\rangle +\gamma^5\left\langle\sum_{p,k}g_{mk}g_{pk}\sum_{q,i}g_{qi}Z_i\frac{\partial}{\partial X_q}\frac{\partial}{\partial X_p}f_m\right\rangle \, . \tag{19}$$

The last result may be further simplified by realizing that the second and third terms involve the expectation of Z_j multiplied by a function of $\mathbf{X}$. In the limit of large γ, the stationary joint probability density $p(\mathbf{X},\mathbf{Z})$ is of the form[13]

$$p(\mathbf{X},\mathbf{Z})=p_0(\mathbf{X})\bar{p}(\mathbf{Z})+\frac{1}{\gamma}p_1(\mathbf{X},\mathbf{Z})+O(1/\gamma^2) \, .$$

Here $\bar{p}(\mathbf{Z})$ is the stationary density (17) of the Ornstein-Uhlenbeck process $\mathbf{Z}$. Since $\langle\mathbf{Z}\rangle=\mathbf{0}$, the expectations in the second and third terms are of order γ^{-1}. Therefore, these terms can be neglected with respect to the first term for γ sufficiently large and we may write

$$\langle(\underline{\sigma}^T\nabla\mathscr{A}X_m)\cdot(\underline{\sigma}^T\nabla\mathscr{A}^3X_m)\rangle\simeq\gamma^8\sum_k\langle g_{mk}^2\rangle \, .$$

Hence we have

$$\frac{-1}{\pi}\langle X_m\mathscr{A}^5X_m\rangle\simeq\frac{-\gamma^8}{2\pi}\sum_k\langle g_{mk}^2\rangle \, .$$

A quick calculation shows that the coefficient of ω^{-4} is

$$\frac{1}{\pi}\langle X_m \mathscr{A}^3 X_m \rangle = \frac{\gamma^4}{2\pi}\sum_k \langle g_{mk}^2 \rangle .$$

So, the condition that the ω^{-4} and ω^{-6} terms be equal in magnitude is

$$\gamma^4 \left[\sum_k \langle g_{mk}^2 \rangle \right] \omega^{-4} = \gamma^8 \left[\sum_k \langle g_{mk}^2 \rangle \right] \omega^{-6}$$

or

$$\omega = \gamma^2 .$$

QED .

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[1]E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[2]For a recent review, see J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).

[3]*Dimensions and Entropies in Chaotic Systems: Quantification of Complex Behavior,* edited by G. Mayer-Kress (Springer-Verlag, Heidelberg, 1986).

[4]H. S. Greenside, Guenter Ahlers, R. C. Hohenberg, and R. W. Walden, Physica **5D**, 322 (1982).

[5]Uriel Frisch and Rudolf Morf, Phys. Rev. A **23**, 2673 (1981).

[6]David Ruelle, Phys. Rev. Lett. **56**, 405 (1986).

[7]H. L. Swinney (private communication).

[8]G. Ahlers and R. W. Walden, Phys. Rev. Lett. **44**, 445 (1980).

[9]J. J. Brey, J. M. Casado, and M. Morillo, Phys. Rev. A **30**, 1535 (1984).

[10]B. Caroli, C. Caroli, and B. Roulet, Physica **112A**, 517 (1982).

[11]Frank W. Olver, Jr., *Asymptotics and Special Functions* (Academic, New York, 1974).

[12]C. W. Gardner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1983).

[13]G. Blankenship and G. C. Papanicolaou, SIAM J. Appl. Math. **34**, 437 (1978).