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Effective eigenvalue for the intensity correlations of single-mode and two-mode lasers

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Exact formulas for the effective eigenvalue characterizing the initial decay of intensity correlation functions are given in terms of stationary moments of the intensity. Spontaneous emission noise and nonwhite pump noise are considered. Our results are discussed in connection with earlier calculations, simulations, and experimental results for single-mode dye lasers, two-mode inhomogeneously broadened lasers, and two-mode dye ring lasers. The effective eigenvalue is seen to depend sensitively on noise characteristics and symmetry properties of the system. In particular, the effective eigenvalue associated with cross correlations of two-mode lasers is seen to vanish in the absence of pump noise as a consequence of detailed balance. In the presence of pump noise, the vanishing of this eigenvalue requires equal pump parameters for the two modes and statistical independence of spontaneous emission noise acting on each mode.

I. INTRODUCTION

Dynamical information on the statistical properties of laser light is obtained from the steady-state time-dependent intensity correlation function. Generally speaking, the intensity correlation function can be expressed as a superposition of decaying exponentials whose damping constants are calculated from a certain eigenvalue problem associated with a Fokker-Planck equation.[1] A global characterization of the correlation function is given by the relaxation time T, defined as the area under the normalized correlation function.[2] The correlation time includes, in principle, contributions from all the eigenvalues, but it gives no information on possible different time regimes of evolution. An alternative partial characterization of the correlation function is given by an effective eigenvalue λ_{eff} or weighted decay rate,[3(c)] which is the reciprocal time constant associated with the initial slope of the correlation function. This quantity also contains contributions from all the eigenvalues. The behavior of T and $\lambda_{\text{eff}}^{-1}$ is sometimes similar. In fact, if a single eigenvalue dominates the decay of the correlation function, $T=\lambda_{\text{eff}}^{-1}$. This is, for example, the case for a single-mode gas (He-Ne) laser. However, if different time scales are involved T and $\lambda_{\text{eff}}^{-1}$ can be quite different and in this case λ_{eff} gives precise information on the initial decay of the correlation function.

T and λ_{eff} depend sensitively on the characteristics of the noise present in the system. This is because the eigenvalue spectrum depends on the noise characteristics. The spectrum and correlation functions of a single-mode gas laser on resonance are by now well understood.[1] However, anomalous statistical properties have been observed in a single-mode dye laser.[3,4] These have been attributed to pumping fluctuations.[3,4] A phenomenological modeling of pumping fluctuations leads to equations which include multiplicative (state-dependent) noise and in which spontaneous emission noise is neglected.[5–12] Multiplicative noise gives rise to a continuous eigenvalue spectrum[5] and to a slowing-down phenomena in the sense that T grows with increasing noise intensity.[13–15] Experimental results for the intensity correlation functions of the single-mode dye laser indicate[4,6,16] that pumping fluctuations cannot be modeled by a white noise, but they rather have a finite correlation time. It has been recently shown[11(a)] that when taking into account nonwhite pumping fluctuations the intensity correlation function exhibits for short times a very slow decay with a characteristic plateau. This initial behavior should be characterized by λ_{eff}. In fact one can show that for the model under consideration the initial slope of the correlation function vanishes[11(b)] ($\lambda_{\text{eff}}=0$). This result seems to be in agreement with experimental measurements which discriminate times of 1 μsec.[16] However, these results were based on a model on which spontaneous emission was neglected. This approximation is known to be unreliable far below threshold.[17] The importance of spontaneous emission noise in the transient statistics of dye lasers have also been recently emphasized.[18–21] The theoretical characterization of the initial decay through the calculation of λ_{eff} taking into account spontaneous noise remains then an open question.

The statistical properties of two-mode ring lasers have been reviewed by Singh.[22] For an inhomogeneously broadened ring laser (gas), experimental and theoretical results do not show important discrepancies. The eigenvalue spectrum is exactly known for equal pump parameters of the two modes ($a_1=a_2$).[23] Several approximate calculations exist for $a_1\neq a_2$.[22,24,25] The autocorrelation seems to be fairly well represented by a sum of two exponentials.[22] The situation is not so clear for the homogeneously broadened two-mode dye laser. Similarly to what happens for the single-mode dye laser, it seems that a model for a proper description of this system requires taking into account pumping fluctuations.[26] An extensive experimental study of the autocorrelations and cross correlations of this system has been recently reported.[27] The main qualitative features which have been observed are reproduced by a simulation of a model which includes nonwhite pump fluctuations and spontaneous emission noise.[27] Both the autocorrelation and cross correlation exhibit in a short time scale a small initial slope. By analogy with the single-mode case[11] it has been suggested[27] that this is due to the nonwhite character of the pumping fluctuations, but no theoretical characterization of this regime through the calculation of the corresponding λ_{eff} exists so far.

In this paper we calculate the effective eigenvalue which characterizes the initial decay of the correlation function for single-mode and two-mode lasers taking into account spontaneous emission white noise and pumping fluctuations modeled by a nonwhite noise. For the two-mode case our results are valid for homogeneously broadened and inhomogeneously broadened lasers as well as for different values of the two pump parameters. An important difference between this and other studies is that we do not attempt a direct calculation of λ_{eff} by explicitly calculating the eigenvalue spectrum. We give exact results for λ_{eff} in terms of moments of the stationary distribution. We obtain these formulas by an analysis of the different role of white and nonwhite noise in the continuity properties of the time derivative of the correlation function at the initial time and by exploiting symmetry properties of the models under consideration. Our results indicate that the analysis of the effective eigenvalue gives a reliable method to determine the existence of different sources of noise in the system, as well as the main characteristics of the noise. It also gives a good test of the symmetry properties of the system.

The outline of the paper is as follows. In Sec. II we consider single-mode lasers. We first analyze a general mathematical model for a single variable including both white and nonwhite multiplicative fluctuations. Our general analysis is then applied to a single-mode dye laser. We obtain that λ_{eff} becomes different from zero due to spontaneous emission noise. Our results are discussed in connection with experimental and simulation data. The dependence of λ_{eff} on pumping fluctuations is elucidated. Section III is devoted to two-mode lasers. Again, we first analyze a mathematical model (Sec. III A). This analysis permits us to clarify the role of symmetry properties and of the statistical independence of different sources of noise. General results are then given for a two-mode laser valid both when pumping fluctuations are present or absent (Sec. III B). We then study the relevance of these results for previous theoretical and experimental studies of inhomogeneously broadened lasers (Sec. III C). The effect of mode competition in the effective eigenvalue of the autocorrelation is analyzed. We also show that the effective eigenvalue associated with the cross correlation vanishes for any value of the pump parameters due to the property of detailed balance. In Sec. III D we finally consider two-mode dye lasers. We analyze our results in the context of experimental and simulation data.[27] We discuss the dependence of λ_{eff} for the autocorrelation on pumping fluctuations. We also show that due to pumping fluctuations the effective eigenvalue associated with the cross correlation in general does not vanish except if the two pump parameters are equal. The Appendix contains an explicit calculation of correlation functions for a linear two variable model in which the different effects associated with symmetry properties and nonwhite noise are clearly displayed.

II. SINGLE-MODE DYE LASER

Before studying the problem of the single-mode dye laser, we consider the following general one-variable problem for a relevant variable x:

$$\partial_t x(t) = v(x) + g(x)p(t) + h(x)q(t) . \tag{2.1}$$

Here $p(t)$ stands for colored noise, which we do not need to specify at this point, and $q(t)$ is a Gaussian white noise with zero mean and correlation

$$\langle q(t)q(t')\rangle = 2D\delta(t-t') . \tag{2.2}$$

We wish to obtain the value of the initial slope of the correlation function $C(s)$ in the steady state. $C(s)$ is defined by $C(s) = \langle x(t+s)x(t)\rangle_{\text{st}} - \langle x\rangle_{\text{st}}^2$. The initial slope is given by

$$\dot{C}(0^+) = \lim_{\substack{s\to 0\\(s>0)}} \frac{d}{ds} C(s) . \tag{2.3}$$

Taking Eq. (2.1) at time $t+s$, and multiplying it by $x(t)$, we get

$$\dot{C}(s) = \langle x(t)v(x(t+s))\rangle_{\text{st}} + \langle x(t)g(x(t+s))p(t+s)\rangle_{\text{st}} + \langle x(t)h(x(t+s))q(t+s)\rangle_{\text{st}} . \tag{2.4}$$

In (2.4) the two first terms are continuous functions of s at $s=0$. On the other hand, because $q(t)$ is a white noise, the last term has a discontinuity at $s=0$, so that[28]

$$\lim_{\substack{s\to 0\\(s>0)}} \langle x(t)h(x(t+s))q(t+s)\rangle_{\text{st}} = D\langle xhh'\rangle_{\text{st}} , \tag{2.5a}$$

$$\lim_{\substack{s\to 0\\(s<0)}} \langle x(t)h(x(t+s))q(t+s)\rangle_{\text{st}} = D\langle xhh'\rangle_{\text{st}} + D\langle h^2\rangle_{\text{st}} . \tag{2.5b}$$

In the steady state we have $\partial_t\langle x^2(t)\rangle_{\text{st}} = 0$. This implies the relation

$$\langle xv\rangle_{\text{st}} + \langle xgp\rangle_{\text{st}} + \langle xhq\rangle_{\text{st}} = 0 . \tag{2.6}$$

Using[28]

$$\langle xhq\rangle_{\text{st}} = D\langle xhh'\rangle_{\text{st}} + D\langle h^2\rangle_{\text{st}} \tag{2.7}$$

in (2.6), and taking also into account (2.4) and (2.5a), we obtain in the limit $s\to 0$ $(s>0)$

$$\dot{C}(0^+) = -D\langle h^2\rangle_{\text{st}} . \tag{2.8}$$

We conclude from this result that when there is no white noise, $D=0$, the initial slope vanishes.[29] This is not surprising, since $C(s)$ is an even function of s, and when there is no white noise present, it has a continuous derivative at $s=0$. On the contrary, if white noise acts on the system, $D\neq 0$, we have

$$\dot{C}(0^-)=\lim_{\substack{s\to 0\\(s<0)}}\frac{d}{ds}\langle x(t+s)x(t)\rangle_{st}$$

$$=-\dot{C}(0^+)=D\langle h^2\rangle_{st}\ . \tag{2.9}$$

Note that the relation $\dot{C}(0^+)+\dot{C}(0^-)=0$ always holds. This relation is equivalent to $\partial_t\langle x^2(t)\rangle_{st}=0$.

We now consider the problem of the single-mode dye laser. This system can be described by[17,19]

$$\partial_t\bar{E}=(\bar{a}-A\,|\bar{E}\,|^2)\bar{E}+p(\bar{t})\bar{E}+q(\bar{t})\ , \tag{2.10}$$

where $\bar{E}$ is the laser complex amplitude. The random term $p(\bar{t})$ models the fluctuations of the pump parameter $\bar{a}$. It is assumed to be Gaussian with zero mean and correlation function of the form

$$\langle p^*(\bar{t})p(\bar{t}\,')\rangle=\bar{Q}\,\bar{\Gamma}e^{-\bar{\Gamma}|\bar{t}-\bar{t}'|}\ . \tag{2.11a}$$

$q(\bar{t})$ stands for the spontaneous emission fluctuations, whose real q^R and imaginary part q^I are independent Gaussian white noise of zero mean and correlation

$$\langle q^I(\bar{t})q^I(\bar{t}\,')\rangle=\langle q^R(\bar{t})q^R(\bar{t}\,')\rangle$$

$$=2D\delta(\bar{t}-t')\ . \tag{2.11b}$$

For later reference we consider the dimensionless form of (2.10),

$$\partial_t E=(a-|E\,|^2)E+p(t)E+q(t)\ , \tag{2.12}$$

where we have introduced the following dimensionless variables:

$$E=\left[\frac{D}{A}\right]^{-1/4}\bar{E},\ \ t=(DA)^{1/2}\bar{t},\ \ a=(DA)^{-1/2}\bar{a}\ . \tag{2.13}$$

In these variables we have $D=1$ and

$$Q=(DA)^{-1/2}\bar{Q},\ \ \Gamma=(DA)^{-1/2}\bar{\Gamma}\ . \tag{2.14}$$

From (2.10) we obtain an equation for the laser intensity $|\bar{I}\,|=|\bar{E}\,|^2$

$$\partial_t\bar{I}=2(\bar{a}-A\bar{I})\bar{I}+2p^R(\bar{t})\bar{I}+2q^R(\bar{t})\bar{E}^{\,R}+2q^I(\bar{t})\bar{E}^{\,I}\ . \tag{2.15}$$

This equation is similar to (2.1), but we have now two variables, due to the complex character of $\bar{E}$. However, taking into account that q^R, q^I are independent, we can obtain the initial slope for the intensity correlation following the same method than for (2.1),

$$\dot{C}(0^+)=\lim_{\substack{\bar{s}\to 0\\(\bar{s}>0)}}\frac{d}{d\bar{s}}\langle\bar{I}(\bar{t}+\bar{s})\bar{I}(\bar{t})\rangle_{st}=-4D\langle\bar{I}\,\rangle_{st}\ . \tag{2.16}$$

It was shown in Ref. 11(b) that when the spontaneous emission fluctuations are not considered, $\dot{C}(0^+)=0$. However, these fluctuations cannot be neglected, specially for small intensities,[17] and therefore $\dot{C}(0^+)$ is not strictly zero.

One quantity of interest to characterize the correlation function is the effective eigenvalue $\lambda_{\rm eff}$ given by[3(c),30]

$$\lambda_{\rm eff}=-\frac{\dot{C}(0^+)}{C(0)}=\frac{4D}{\lambda(0)\langle\bar{I}\,\rangle_{st}}\ , \tag{2.17}$$

where

$$\lambda(\bar{s})=\frac{C(\bar{s})}{\langle\bar{I}\,\rangle_{st}^2}\ . \tag{2.18}$$

The first thing we would like to point out is that (2.17) is an exact result, given in terms of moments of the stationary distribution. Earlier approaches to the general problem of calculating $\lambda_{\rm eff}$ were based on different approximation methods.[22,31]

Concerning experimental data,[16] it is not easy to obtain the initial slope in order to compare it with the one given by (2.17). Such comparison requires accurate data in short time scales. However, it seems that the initial slope is small, which corresponds to a small value of the additive noise intensity D. It is also observed that the decay of $\lambda(\bar{s})$ is slower when $\langle\bar{I}\,\rangle_{st}$ grows. This agrees, at least qualitatively, with (2.17). At this point we also note that the values for $\lambda_{\rm eff}$ obtained by approximating $C(\bar{s})$ by a sum of a few exponentials do not give a good test of (2.17). In fact, to obtain a good estimation of $\lambda_{\rm eff}$, we must consider a time scale short enough for the curvature to be negligible. The effective eigenvalue $\lambda_{\rm eff}$ predicted by Eq. (2.17) is shown in Fig. 1 using experimental and simulation values of $\lambda(0)$ versus $\langle I\,\rangle_{st}$ (Ref. 17). The validity of the model (2.10) for the dynamical properties of the system could be tested by a comparison of the results in Fig. 1 with a direct measurement of $\lambda_{\rm eff}$.

An interesting piece of information that can be extracted from (2.17) concerns the influence of the multiplicative noise $p(t)$. Although the expression for $\lambda_{\rm eff}$ is the same, the dependence on $\langle\bar{I}\,\rangle_{st}$ is different with and without multiplicative noise. This is due to the different variation of $\lambda(0)$ with $\langle\bar{I}\,\rangle_{st}$ in both cases. The effective eigenvalue for the system (2.12) without multiplicative noise is shown in Fig. 2 in the dimensionless units of (2.13). The stationary moments needed are taken from Ref. 1. Comparing Figs. 1 and 2 we observe that pump fluctuations cause a large reduction of $\lambda_{\rm eff}$ due to the peak of $\lambda(0)$ and suppress the minimum of $\lambda_{\rm eff}$. Measurements of $\lambda_{\rm eff}$ give a way to check the existence and characteristics of pump fluctuations and also to determine their intensity. We recall that if $p(t)$ were a white noise there would be an additional contribution to (2.17) proportional to Q. An experimental determination of the value of Q from measurements of $\lambda_{\rm eff}$ require, in principle, an evaluation of (2.17) for different values of Q, but no analytical results are known for $\lambda(0)$ and $\langle\bar{I}\,\rangle_{st}$, and values obtained from simulations are only known for a single value of Q.[17] However an estimate of Q can be obtained recalling that for large values of $\langle\bar{I}\,\rangle_{st}$, spontaneous fluctuations can be neglected and pump noise can be approximated by a white

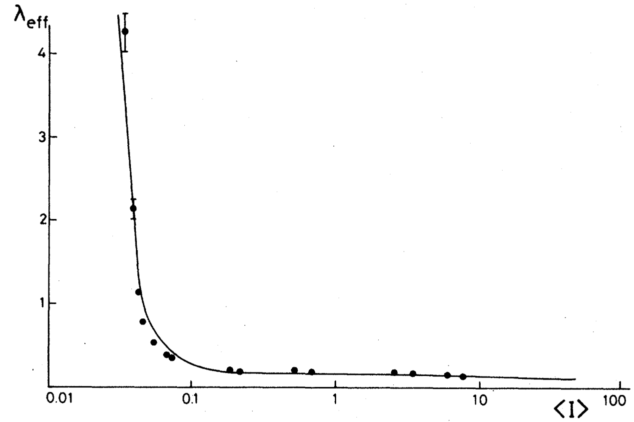

FIG. 1. $\lambda_{\rm eff}$ vs $\langle I \rangle$ for a single-mode dye laser with pumping fluctuations. Intensity scale in arbitrary units. Continuous line follows from the simulation of (2.12) with $Q=300$, $\Gamma=5$ (Ref. 17). Dots follow from experimental results of Ref. 17.

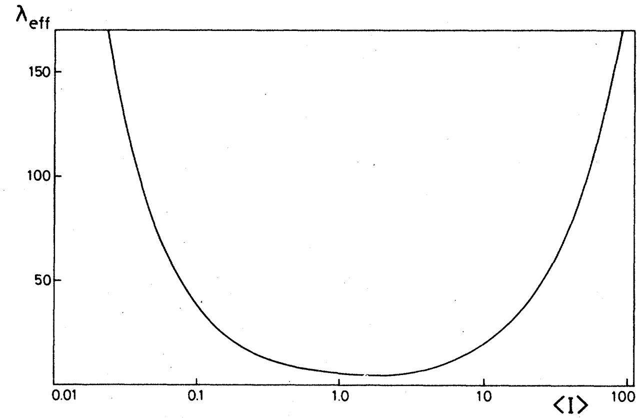

FIG. 2. $\lambda_{\rm eff}$ vs $\langle I \rangle$ for a single-mode laser (no pumping fluctuations).

noise in the calculation of $\lambda(0)$.[17] In this approximation $\lambda(0)=Q/\langle \bar I \rangle_{\rm st}$ so that for large $\langle \bar I \rangle_{\rm st}$, $\lambda_{\rm eff}(Q)Q=$const.

III. TWO-MODE LASERS

In the case of two-mode lasers new aspects are faced: Different sources of noise must be considered for each mode, and we can also study the initial slope of the cross correlation. In order to discuss these new aspects in a general framework we first consider a general two-variable model.

A. A general two-variable model

We consider a general system described by the following equations for the two variables x_1,x_2:

$$\partial_t x_1 = v_1(x_1,x_2)+g_1^k(x_1,x_2)p_k(t)+h_1^r(x_1,x_2)q_r^1 \, ,$$
$$\partial_t x_2 = v_2(x_1,x_2)+g_2^k(x_1,x_2)p_k(t)+h_2^r(x_1,x_2)q_r^2 \, , \tag{3.1}$$

where $p_k(t)$ are colored noise which we do not need to specify here and q_r^i are Gaussian white noise with zero mean and correlation

$$\langle q_r^i(t) q_l^j(t')\rangle = 2D_{rl}^{ij}\delta(t-t') \, . \tag{3.2}$$

In Eq. (3.1) summation over repeated indices is assumed. Following the approach used in Sec. II it is straightforward to get the initial slope of the autocorrelations $C_{ii}(s)\equiv\langle x_i(t+s)x_i(s)\rangle_{\rm st}-\langle x_i\rangle_{\rm st}^2$

$$\dot C_{ii}(0^+) = -D_{rl}^{ii}\langle h_i^r(x_1,x_2)h_i^l(x_1,x_2)\rangle_{\rm st} \quad (i=1,2) \, . \tag{3.3}$$

It is clear that the initial slope of the autocorrelation vanishes in the absence of white noise. On the contrary, the initial slope of the cross correlation, $C_{12}(s) = \langle x_1(t+s)x_2(t)\rangle_{\rm st} - \langle x_1\rangle_{\rm st}\langle x_2\rangle_{\rm st}$, can be different than zero, even in the absence of white noise. A vanishing $\dot C_{12}(0^+)$ is rather related to a symmetry of the system (3.1) under the interchange of x_1 and x_2. In this connection, the linear problem studied in the Appendix shows by an explicit solution that this symmetry is a necessary and sufficient condition for $\dot C_{12}(0^+)$ to vanish when q^1,q^2 are independent white noise. We will now show that for a two-variable system (3.1) symmetric under the interchange of x_1 and x_2, and with independent white noise for each variable ($D_{rl}^{12}=0$), $\dot C_{12}(0^+)$ vanishes. The reason for this result is the following. When the white noise acting on each variable is independent, the cross correlation has a continuous derivative at the initial time. Then, due to the even parity of $C_{12}(s)+C_{21}(s)$, we get $\dot C_{12}(0)+\dot C_{21}(0)=0$. Now, using the symmetry $x_1\leftrightarrow x_2$, it can be shown that $\dot C_{12}(0)=\dot C_{21}(0)$. Therefore, we obtain $\dot C_{12}(0)=0$. However, when the white noise acting on each variable are not independent ($D_{rl}^{12}\neq 0$), the continuity argument for $\dot C_{12}(0)$ does not hold, and we can have $\dot C_{12}(0^+)\neq 0$.

Let us assume that

$$v_2(x_1,x_2)=v_1(x_2,x_1)\equiv v(x_2,x_1) \, ,$$
$$g_2^k(x_1,x_2)=g_1^k(x_2,x_1)\equiv g^k(x_2,x_1) \, , \tag{3.4}$$
$$h_2^r(x_1,x_2)=h_1^r(x_2,x_1)\equiv h^r(x_2,x_1) \, ,$$

so that Eqs. (3.1) are symmetric under the interchange $x_1\leftrightarrow x_2$, $q_r^1\leftrightarrow q_r^2$. Multiplying Eq. (3.1) for $\dot x_1(t+s)$, by $x_2(t)$, we get

$$\dot C_{12}(0^+) = \lim_{\substack{s\to 0\\(s>0)}} \frac{d}{ds}\langle x_1(t+s)x_2(t)\rangle_{\rm st}$$
$$= \langle x_2 v(x_1,x_2)\rangle_{\rm st} + \langle x_2 g^k(x_1,x_2)p_k\rangle_{\rm st} + \left\langle x_2 \frac{\partial h^r(x_1,x_2)}{\partial x_1} h^l(x_1,x_2)\right\rangle_{\rm st} D_{rl}^{11} + \left\langle x_2 \frac{\partial h^r(x_1,x_2)}{\partial x_2} h^l(x_2,x_1)\right\rangle_{\rm st} D_{rl}^{12} \, , \tag{3.5}$$

where we have calculated $\langle x_2(t)h^r(x_1(t+s),x_2(t+s))q_r^1(t+s)\rangle_{st}$ in the limit $s\to 0$ $(s>0)$.[28] In the steady state $\partial_t\langle x_1(t)x_2(t)\rangle_{st}=0$, so that[28]

$$\langle x_2 v(x_1,x_2)\rangle_{st}+\langle x_2 g^k(x_1,x_2)p_k\rangle_{st}+\left\langle x_2\frac{\partial h^r(x_1,x_2)}{\partial x_1}h^l(x_1,x_2)\right\rangle_{st}D_{rl}^{11}$$
$$+\left\langle x_2\frac{\partial h^r(x_1,x_2)}{\partial x_2}h^l(x_2,x_1)\right\rangle_{st}D_{rl}^{12}+\langle h^r(x_1,x_2)h^l(x_2,x_1)\rangle_{st}D_{rl}^{12}=0\ . \quad (3.6)$$

To get (3.6) we have used the following symmetry relations for an arbitrary function $\varphi(x_1,x_2)$:

$$\langle\varphi(x_1,x_2)\rangle_{st}=\langle\varphi(x_2,x_1)\rangle_{st}\ ,$$
$$\langle\varphi(x_1,x_2)p_k\rangle_{st}=\langle\varphi(x_2,x_1)p_k\rangle_{st}\ . \quad (3.7)$$

Using (3.6) in (3.5) we get

$$\dot{C}_{12}(0^+)=-D_{rl}^{12}\langle h^r(x_1,x_2)h^l(x_2,x_1)\rangle_{st}\ . \quad (3.8)$$

This is the main result of this section. It agrees with the one obtained for the linear system in the Appendix and it shows that when the symmetry properties (3.4) are assumed $\dot{C}_{12}(0^+)=0$ if q_1 and q_2 are independent.

We finally note that in the particular case of a Markovian system (no colored noise), it is possible to study other symmetry properties associated with detailed balance.[32] Such properties can also lead to the vanishing of $\dot{C}_{12}(0^+)$, without assuming (3.4). When the variables x_i are even under time reversal, detailed balance implies that $C_{12}(s)=C_{21}(s)$. Then, if $C_{12}(s)$ has a continuous derivative at $s=0$, as it happens when the noise acting on each variable is independent, $\dot{C}_{12}(0)$ vanishes.

B. Effective eigenvalues for two-mode lasers

The approach used in Sec. III A to get the initial slope of the autocorrelation and cross-correlations for a two-variable model, can be applied to two-mode lasers, described by the following equations:[22–27]

$$\partial_t\bar{E}_1=(\bar{a}_1-A\,|\bar{E}_1|^2-\xi\,|\bar{E}_2|^2)\bar{E}_1+p(\bar{t})\bar{E}_1+q_1(\bar{t})\ ,$$
$$\partial_t\bar{E}_2=(\bar{a}_2-A\,|\bar{E}_2|^2-\xi\,|\bar{E}_1|^2)\bar{E}_2+p(\bar{t})\bar{E}_2+q_2(\bar{t})\ , \quad (3.9)$$

where $q_i(\bar{t})$ are independent Gaussian white noise, with zero mean and correlation

$$\langle q_i^*(\bar{t})q_j(\bar{t}')\rangle=4\delta_{ij}D\delta(\bar{t}-\bar{t}')\ . \quad (3.10)$$

q_1 and q_2 model spontaneous emission fluctuations. We have also included in (3.9) a colored noise $p(t)$ [with the properties (2.11a)] associated with pumping fluctuations which are known to be important for dye lasers.[26,27] The mode-coupling parameter in the dimensionless units of (2.13) is smaller than one, $\xi\le 1$, for inhomogeneously broadened lasers. It is clear that (3.9) is symmetric under the interchange $\bar{E}_1\leftrightarrow\bar{E}_2$, $q_1\leftrightarrow q_2$, when $\bar{a}_1=\bar{a}_2=\bar{a}$. We apply now the method used in Sec. III A to get the initial slope of the autocorrelation and cross correlation of the field intensity $\bar{I}_i=|\bar{E}_i|^2$. We take as a starting point the equations for $\bar{I}_i$,

$$\partial_t\bar{I}_1=2(\bar{a}_1-A\bar{I}_1-\xi\bar{I}_2)\bar{I}_1+2p^R(\bar{t})\bar{I}_1+2q_1^R(\bar{t})\bar{E}_1^R+2q_1^I(\bar{t})\bar{E}_1^I\ ,$$
$$\partial_t\bar{I}_2=2(\bar{a}_2-A\bar{I}_2-\xi\bar{I}_1)\bar{I}_2+2p^R(\bar{t})\bar{I}_2+2q_2^R(\bar{t})\bar{E}_2^R+2q_2^I(\bar{t})\bar{E}_2^I\ , \quad (3.11)$$

where R (I) means real (imaginary) part. Defining normalized correlations $\lambda_{ij}(\bar{s})\equiv C_{ij}(\bar{s})/(\langle\bar{I}_i\rangle_{st}\langle\bar{I}_j\rangle_{st})$ we obtain

$$\dot{\lambda}_{ii}(0^+)=\frac{\dot{C}_{ii}(0^+)}{\langle\bar{I}_i\rangle_{st}^2}=-\frac{4D}{\langle\bar{I}_i\rangle_{st}}\quad(i=1,2) \quad (3.12)$$

$$\dot{\lambda}_{12}(0^+)=\frac{\dot{C}_{12}(0^+)}{\langle\bar{I}_1\rangle_{st}\langle\bar{I}_2\rangle_{st}}=0\quad(\bar{a}_1=\bar{a}_2)\ . \quad (3.13)$$

This last result is obtained using the symmetry under the interchange of the two modes and the independence of q_1 and q_2. When $\bar{a}_1\neq\bar{a}_2$, a straightforward calculation leads to

$$\dot{\lambda}_{12}(0^+)=(\bar{a}_1-\bar{a}_2)\frac{\langle\bar{I}_1\bar{I}_2\rangle_{st}}{\langle\bar{I}_1\rangle\langle\bar{I}_2\rangle_{st}}+\frac{(1-\xi)}{\langle\bar{I}_1\rangle_{st}\langle\bar{I}_2\rangle_{st}}(\langle\bar{I}_2^2\bar{I}_1\rangle_{st}-\langle\bar{I}_1^2\bar{I}_2\rangle_{st})+2D\left[\frac{1}{\langle\bar{I}_1\rangle_{st}}-\frac{1}{\langle\bar{I}_2\rangle_{st}}\right]\ . \quad (3.14)$$

When $\bar{a}_1=\bar{a}_2$, $\langle\bar{I}_1\rangle_{st}=\langle\bar{I}_2\rangle_{st}$, $\langle\bar{I}_2^2\bar{I}_1\rangle_{st}=\langle\bar{I}_1^2\bar{I}_2\rangle_{st}$ and we recover (3.13).

The effective eigenvalue $\lambda_{\rm eff}^{ii}$ can be obtained from (3.12),

$$\lambda_{\rm eff}^{ii}\equiv-\frac{\dot{\lambda}_{ii}(0^+)}{\lambda_{ii}(0)}=\frac{4D}{\langle\bar{I}_i\rangle_{st}\lambda_{ii}(0)}\ . \quad (3.15)$$

We also define $\lambda_{\rm eff}^{12}$ by analogy with $\lambda_{\rm eff}^{ii}$ as

$$\lambda_{\rm eff}^{12}=-\frac{\dot{\lambda}_{12}(0^+)}{\lambda_{12}(0)}\ . \quad (3.16)$$

Our results in this section clarify a previous suggestion[27] of a vanishing slope for the cross correlation of a two-mode dye laser due to nonwhite pumping fluctuations. This is a necessary condition, but we have seen that the result can be derived only when pump parameters are equal, and the spontaneous emission noise for each mode are independent. However, for inhomogeneously broadened gas lasers pumping fluctuations are not important, $p(t)=0$, and the resulting Markovian system satisfies the property of detailed balance.[23] In this case, according to the discussion at the end of Sec. III A, we also have $\lambda_{\rm eff}^{12}=0$, for any $\Delta\bar{a}=\bar{a}_1-\bar{a}_2$. When pumping fluctuations are present, the effective eigenvalue $\lambda_{\rm eff}^{12}$ vanishes in general only when $\bar{a}_1=\bar{a}_2$. A more detailed account of these results is given in Secs. III C and III D.

C. Inhomogeneously broadened two-mode lasers

We now discuss the consequences of the general results (3.12) and (3.14) for inhomogeneously broadened lasers. These formulas are valid for any value of the coupling parameter ξ and in particular for the weak-coupling case $\xi\leq 1$ corresponding to inhomogeneously broadened gas lasers. In this case pumping fluctuations are not believed to be important, but the formulas (3.12) and (3.14) are still valid. A first obvious consequence of (3.12) is that a difference in pump parameters $\Delta a=a_1-a_2$ can be identified from the initial slopes of λ_{11} and λ_{22}: For $\xi\lesssim 1$ and $\Delta a>0$, $\langle I_1\rangle_{\rm st}$ becomes distinctively larger[23] than $\langle I_2\rangle_{\rm st}$ so that the initial slope of λ_{11} should be much smaller than that of λ_{22}. Our results can be compared at least qualitatively, with the explicit calculations of $\lambda_{11}(s)$ and $\lambda_{22}(s)$ by Tehrani and Mandel[23] for $\xi=1$ and $\Delta a=0$. The first thing to be noticed is the different initial behavior of $\lambda_{11}(s)$ and $\lambda_{12}(s)$ (Figs. 6 and 7 of Ref. 23). Such behavior is compatible with a vanishing initial slope for $\lambda_{12}(s)$ and a nonvanishing slope for $\lambda_{11}(s)$, as predicted here in general. Also, their eigenvalue expansion formula for $\langle I_{s'}(t+s)I_s(t)\rangle_{\rm st}$, $s,s'=1,2$, implies that $\dot{\lambda}_{11}(0^+)$ is a negative definite quantity, while for $\dot{\lambda}_{12}(0^+)$ a cancellation of terms is possible compatible with $\dot{\lambda}_{12}(0^+)=0$. In addition, the initial relaxation of $\lambda_{11}(s)$ becomes slower with increasing pump parameter. This is also compatible with (3.12) since $\langle I_1\rangle_{\rm st}$ is a monotonous growing function of the pump parameter.[23]

The effective eigenvalues predicted by (3.15) and (3.16) for the autocorrelation are shown in Figs. 3—5 for different values of Δa and ξ in dimensionless units [see (2.13)] for model (3.9) with $p(t)=0$. We include results for strong coupling $\xi=2$ as appropriate for homogeneously broadened lasers in the absence of pumping fluctuations. The stationary moments needed are taken from the results of Ref. 23. We obtain here exact results for these effective eigenvalues without need of addressing the dynamical problem through the calculation of eigenvalues. This was the method followed in other approaches, as, for example, using a one-dimensional approximation to calculate $\lambda_{\rm eff}^{11}$ (Ref. 22). In Fig. 3 we observe that this approximation is reliable well above threshold (large a_1). However, it leads to different values of $\lambda_{\rm eff}^{11}$ for $\Delta a=\pm 1$ below threshold, whereas our exact results show that in this region $\lambda_{\rm eff}^{11}$ does not change appreciably with Δa. In fact, in the weak-coupling case ($\xi\leq 1$) the effective eigenvalue $\lambda_{\rm eff}^{11}$ tends below threshold to the one of the single-mode case (see Fig. 5). In Fig. 4 we observe that when $\xi=0.5$ the behavior of $\lambda_{\rm eff}^{11}$ with a_1 is similar to

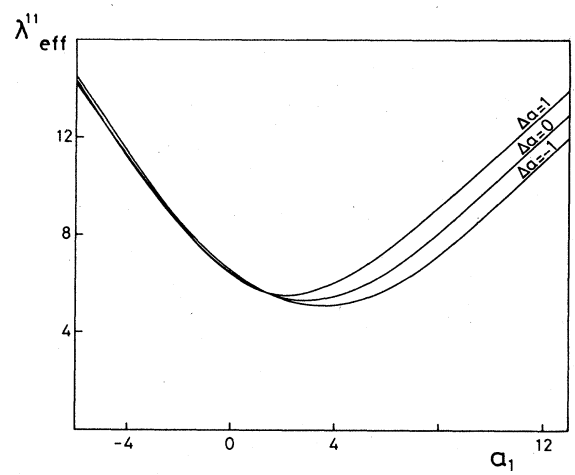

FIG. 4. $\lambda_{\rm eff}^{11}$ vs a_1 for a two-mode laser, $\xi=0.5$ (no pumping fluctuations).

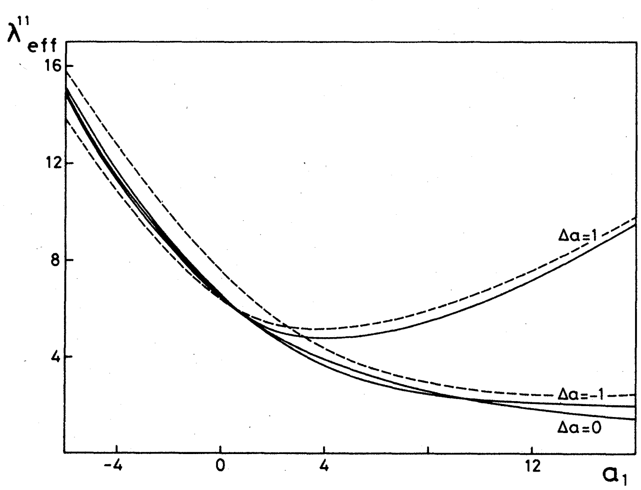

FIG. 3. $\lambda_{\rm eff}^{11}$ vs a_1 for a two-mode laser, $\xi=1$ (no pumping fluctuations). Dotted lines correspond to a one-dimensional approximation (Ref. 22).

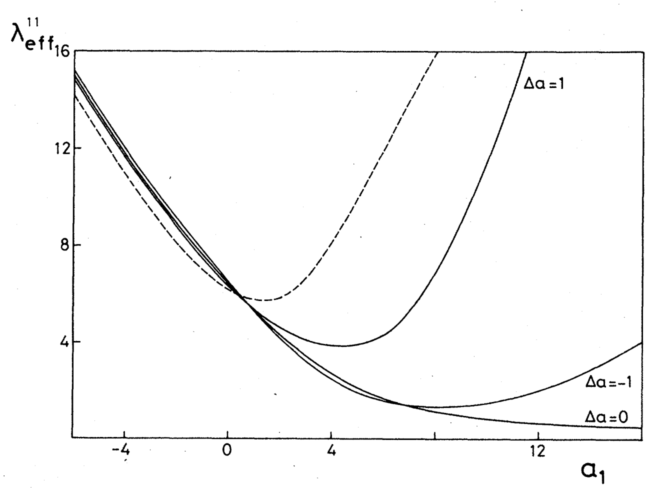

FIG. 5. $\lambda_{\rm eff}^{11}$ vs a_1 for a two-mode laser, $\xi=2$ (no pumping fluctuations). Dotted line corresponds to the single-mode case $\xi=0$.

that of the single-mode case. This corresponds to a weak coupling. When the coupling is stronger, $\xi \gtrsim 1$, and there is no dominant mode, $\Delta a = 0$, $\lambda_{\rm eff}^{11}$ has no minimum (Figs. 3 and 5). The behavior of $\lambda_{\rm eff}^{11}$ is different from that of the single-mode case due to the competition between the two modes. However, when one mode dominates ($\Delta a = 1$, and $\xi \geq 1$), $\lambda_{\rm eff}^{11}$ behaves again as in the single-mode case. Figures 3—5 show that useful information on the values of ξ and Δa can be obtained from measurements of $\lambda_{\rm eff}^{11}$.

Concerning experimental data for $\lambda_{\rm eff}^{11}$, they fit well only for $a < 4$ with theoretical results for $\xi = 1$, $\Delta a = 0$ (see Fig. 5.5 of Ref. 22). Note, however, that the experimental results were obtained by using a sum of two exponentials for the correlation function, which might not be reliable for estimating the initial slope.

It can be shown that the effective eigenvalue for the cross correlation vanishes. As we noted in Sec. III A, for a Markovian system it is easy to verify if detailed balance holds. This is the case for model (3.9) without pumping fluctuations. Consequently, the cross correlation is an even function. Due to the independence of the white noise for each mode the cross correlation has a continuous derivative at the initial time. Therefore, the initial slope of the cross correlation vanishes. This result also follows from Eq. (3.14). A lengthy but straightforward calculation of the time-independent correlations in the right-hand side of (3.14) using the results in Ref. 23 shows that $\dot{\lambda}_{12}(0^+)$ vanishes identically. A similar result is obtained in the Appendix. Note, however, that now $\lambda_{\rm eff}^{12}$ vanishes even when $a_1 \neq a_2$. In this case there is no symmetry under the interchange of E_1 and E_2. The vanishing of $\lambda_{\rm eff}^{12}$ is due exclusively to the detailed balance property. We remark that this result can be obtained only when the colored pumping fluctuations are neglected (Markovian case).

D. Two-mode dye laser

We now turn our attention to ring dye lasers for which pumping fluctuations are known to be important.[26,27] We first consider the autocorrelations. An asymmetry in the pumping parameters will be again reflected in different initial slopes of $\lambda_{11}(s)$ and $\lambda_{22}(s)$. Other than that, the main qualitative features of the autocorrelations are the same as for the single-mode case. In the following we restrict ourselves to the symmetric case $a = 0$. The initial slope of $\lambda_{ii}(s)$ is predicted to be very small well above threshold, but for small $\langle I_i \rangle_{\rm st}$ spontaneous emission noise becomes important and the slope is not necessarily small.

The experimental and simulation results of Lett and Mandel[27] indicate a slower relaxation of $\lambda_{11}(s)$ with increasing $\langle I_1 \rangle_{\rm st}$ which is in agreement with the implications of Eq. (3.12). The effective eigenvalue $\lambda_{\rm eff}^{11}$ predicted by Eq. (3.12) is shown in Figs. 6—8 in the dimensionless units of (2.13). Figure 6 gives $\lambda_{\rm eff}^{11}$ using for the time-independent correlation $\lambda_{11}(0)$ the values obtained in the simulation of Ref. 27. In Fig. 7 the time-independent correlations are the experimental ones.[27] Figure 8 gives $\lambda_{\rm eff}^{11}$ in the absence of pump noise using the time-independent moments needed as calculated in Ref. 23. We observe that the simulation gives much smaller values for $\lambda_{\rm eff}^{11}$ than those in Fig. 7. This is due to the fact that

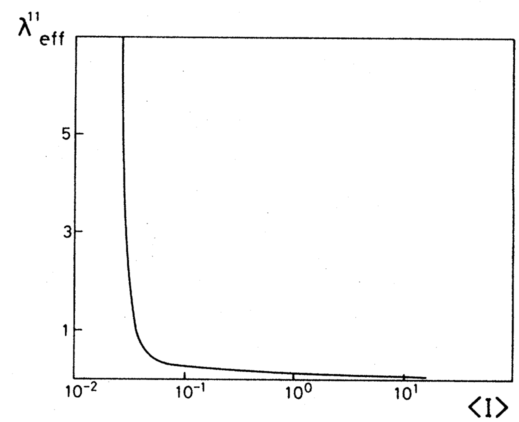

FIG. 6. $\lambda_{\rm eff}^{11}$ vs $\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle$ for a two-mode laser, $\xi = 2$, $\Delta a = 0$ including pumping fluctuations. $\lambda_{11}(0)$ obtained from simulation with $Q = 500$, $\Gamma = 5$ (Ref. 27).

the intensity Q of pump fluctuations in the simulation was not adjusted to fit the experimental results,[27] Q being too large. This leads to a larger peak of $\lambda_{11}(0)$ than the experimental one. The range of values of $\langle I \rangle_{\rm st}$ is also larger than the experimental one. As a consequence, the experimental $\lambda_{\rm eff}^{11}$ is larger than the effective eigenvalue predicted from simulation results. As we already discussed for the single-mode case, Eq. (3.12) is the same when pumping fluctuations do not exist. The actual value of $\lambda_{\rm eff}^{11}$ depends on pumping fluctuations through the different values of $\lambda_{11}(0)\langle I \rangle_{\rm st}$. A comparison of $\lambda_{\rm eff}^{11}$ including or not including pump fluctuations is seen from Figs. 6 and 8. The observed differences can be used to check the existence and determine the intensity of pump noise. In particular we observe that pumping fluctuations are predicted to cause a large reduction of $\lambda_{\rm eff}^{11}$ due to the peak of $\lambda_{11}(0)$ (compare scales of Figs. 6 and 8). In Fig. 6 we

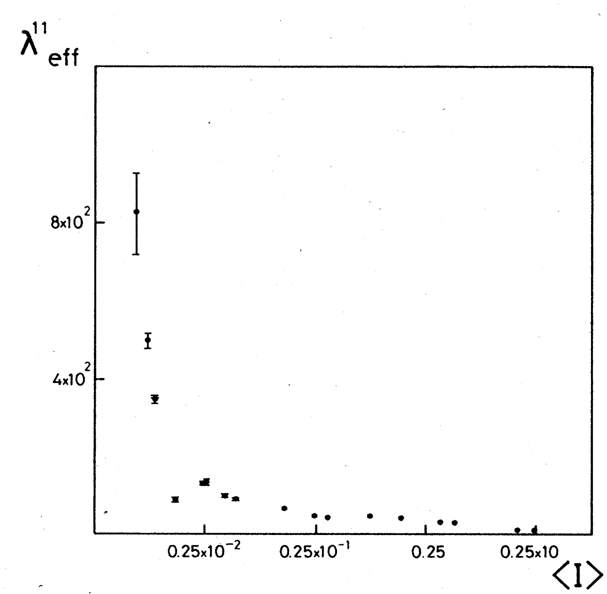

FIG. 7. $\lambda_{\rm eff}^{11}$ vs $\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle$ for a two-mode dye laser. $\lambda_{11}(0)$ obtained from experimental data (Ref. 27).

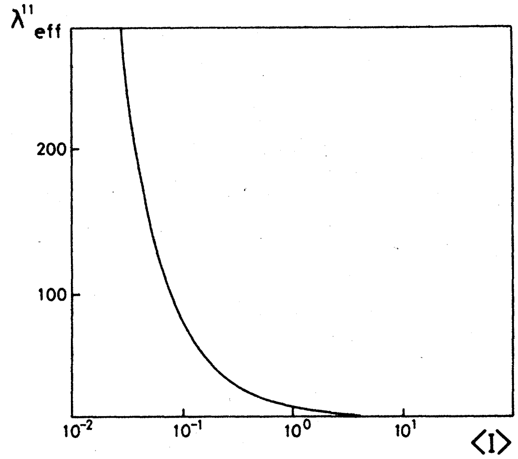

FIG. 8. $\lambda_{\rm eff}^{11}$ vs $\langle I\rangle=\langle I_1\rangle+\langle I_2\rangle$ for a two-mode laser, $\xi=2$, $\Delta a=0$ (no pumping fluctuations).

also see that the initial decreasing of $\lambda_{\rm eff}^{11}$ with the mean intensity is much faster than in Fig. 8. This is originated in the increasing part of $\lambda_{11}(0)$ with $\langle I\rangle_{\rm st}$. A quantitative determination of Q and Γ from measurements of $\lambda_{\rm eff}^{11}$ requires curve fitting with previous knowledge of $\lambda(0)$ for different noise parameters.[33] Measurements of $\lambda(0)$ already give a way to determine noise parameters but $\lambda_{\rm eff}$ seems to be more sensitive to the actual values of the noise parameters. In particular we recall that (3.15) is only valid for Γ^{-1} strictly different from zero. In any case the determination of noise parameters through (3.15) gives a stringent consistency test of the accuracy of parameters determined through measurements of $\lambda(0)$.

Concerning cross correlations, the simulations of Ref. 27 for short times indicate a clearly different behavior of $\lambda_{11}(s)$ and $\lambda_{12}(s)$. The cross correlation $\lambda_{12}(s)$ exhibits a very flat initial decay and certainly much slower than λ_{11}. This is compatible with our result of zero initial slope for λ_{12} when $a_1=a_2$ and nonzero slope for λ_{11}. This difference is also seen in larger time scales of the same simulation.[27] We have already mentioned that our result $\dot{\lambda}_{12}(0^+)=0$ for $a_1=a_2$ is based on symmetry properties and on the assumption of independence of the spontaneous noise driving each mode. Therefore, the nonwhite character of the pumping fluctuations is a necessary but not sufficient condition to have $\dot{\lambda}_{12}(0^+)=0$. In fact, our prediction based on model (3.9) is that generally $\dot{\lambda}_{12}(0^+)\neq0$ for $a_1\neq a_2$ even if pumping fluctuations are not white. In summary, a vanishing $\lambda_{\rm eff}^{12}$ indicates either the absence of pumping fluctuations or $\Delta a=0$. The first possibility can be ruled out considering the behavior of $\lambda_{\rm eff}^{11}$ versus $\langle I\rangle_{\rm st}$. In addition, $\Delta a=0$ implies that $\lambda_{\rm eff}^{11}=\lambda_{\rm eff}^{22}$. The explicit values of $\lambda_{\rm eff}^{12}$ for $a_1\neq a_2$ cannot be given since no analytical or simulation results exist at present for the stationary moments of the intensity distribution $P(I_1,I_2)$ when $a_1\neq a_2$.

IV. SUMMARY AND CONCLUSIONS

In this paper we have obtained exact expressions for effective eigenvalues characterizing the initial decay of the intensity correlations in single-mode and two-mode lasers. These expressions are given in terms of stationary moments. The effect of pump noise and symmetry properties on the effective eigenvalues have been analyzed in detail. In particular we have discussed the conditions under which the effective eigenvalue vanishes. Our main results are the following. For the single-mode dye laser we obtain that the existence of spontaneous emission noise implies a nonvanishing initial slope. The slope becomes larger where spontaneous emission is more important relatively to pumping fluctuations, that is for small mean value of the intensity. The actual value of $\lambda_{\rm eff}$ depends implicitly on pumping fluctuations through the mean value of the intensity. Measurements of $\lambda_{\rm eff}$ give then a way to probe the existence of pumping fluctuations. Additional information on the dependence of static properties as $\lambda(0)$ on noise parameters can be used as an alternative consistent way to experimentally determine noise parameters from measurements of $\lambda_{\rm eff}$.

The results for two-mode lasers given an interesting approach to compute the effective eigenvalue for the inhomogeneously broadened case in which pumping fluctuations are not crucial. Our explicit results show that when the coupling is weak the behavior of the effective eigenvalue $\lambda_{\rm eff}^{11}$ is similar to the single-mode case for different values of Δa. But if the coupling becomes stronger and there is no dominant mode ($\Delta a=0$), $\lambda_{\rm eff}^{11}$ has no minimum in contrast with the single-mode case. However, if one mode dominates the other, $\lambda_{\rm eff}^{11}$ behaves again as in the single-mode case. Measurements of the effective eigenvalue give then a way to analyze the competition between the two modes.

Concerning cross correlations, we have shown that, in the absence of pump noise, they have a vanishing initial slope. This result is due to the existence of detailed balance and to the assumption of statistical independence of the spontaneous emission noise acting on each mode. This result is valid for arbitrary values of Δa.

A proper description of a two-mode dye laser requires considering colored pump noise. In this case the cross correlations are found to have a vanishing initial slope only when $\Delta a=0$. This result clarifies a previous suggestion[27] of a vanishing slope due to colored pumping fluctuations. This is a necessary condition if pumping fluctuations exist, but the result only follows if $\Delta a=0$. An additional requirement of statistical independence of the spontaneous emission noise for each mode is also needed. We have also given a formula (3.14) for the initial slope of the cross correlation in a general case. The effective eigenvalue $\lambda_{\rm eff}^{11}$ of the autocorrelation is never exactly zero due to the spontaneous emission noise. We have computed $\lambda_{\rm eff}^{11}$ using simulation and experimental results for time-independent moments, concluding that $\lambda_{\rm eff}^{11}$ can be used to determine the parameters characterizing the pumping fluctuations.

As a final general conclusion we point out that the effective eigenvalue is a quantity very sensitive to the characteristics of the noise acting on the system and, as a consequence, it provides a good way of determining which are the sources of noise acting on the system and their characteristic parameters.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we study the following two-variable linear model:

$$\partial_t x_1 = -x_1 - a x_2 + \eta_1(t) ,$$
$$\partial_t x_2 = -x_2 - b x_1 + \eta_2(t) , \tag{A1}$$

where we take $a,b>0$ and $ab<1$ in order to have a stable system when $t\to\infty$.

The explicit solution of (A1) is given by

$$x_1(t)=\tfrac{1}{2}\int_0^t du[(e^{-\lambda_+ u}+e^{-\lambda_- u})\eta_1(t-u) + \mu(e^{-\lambda_+ u}-e^{-\lambda_- u})\eta_2(t-u)]+x_1(0) ,$$
$$x_2(t)=\tfrac{1}{2}\int_0^t du[(e^{-\lambda_- u}+e^{-\lambda_+ u})\eta_2(t-u) + \mu^{-1}(e^{-\lambda_+ u}-e^{-\lambda_- u})\eta_1(t-u)]+x_2(0) , \tag{A2}$$

where

$$\lambda_\pm = 1 \pm (ab)^{1/2} > 0, \quad \mu = \left[\frac{a}{b}\right]^{1/2} . \tag{A3}$$

(i) We consider first the case of independent noise sources with zero mean and correlations

$$\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij} A_{ij}(t-t') . \tag{A4}$$

In the steady state we get the following correlation function:

$$C_{12}(s)=\tfrac{1}{8}\int_0^\infty du \left[\{\mu^{-1}[A_{11}(s-u)+A_{11}(s+u)]+\mu[A_{22}(s-u)+A_{22}(s+u)]\} \left[\frac{e^{-\lambda_+ u}}{\lambda_+}-\frac{e^{-\lambda_- u}}{\lambda_-}\right] + \{\mu^{-1}[A_{11}(s-u)-A_{11}(s+u)]-\mu[A_{22}(s-u)-A_{22}(s+u)]\}(e^{-\lambda_- u}-e^{-\lambda_+ u}) \right] . \tag{A5}$$

When $\eta_i(t)$ is a colored noise, the initial slope of the cross correlation is

$$\dot C_{12}(0)=\tfrac{1}{4}\int_0^\infty du[\mu A_{22}(u)-\mu^{-1}A_{11}(u)](\lambda_- e^{-\lambda_- u}-\lambda_+ e^{-\lambda_+ u}) . \tag{A6}$$

The vanishing of $\dot C_{12}(0)$ is therefore equivalent to the condition

$$a\langle \eta_2(t)\eta_2(t')\rangle = b\langle \eta_1(t)\eta_1(t')\rangle . \tag{A7}$$

When η_1 and η_2 are independent and with the same correlation function this implies $a=b$. This corresponds to a system (A1) symmetric under the interchange of x_1 and x_2. We also note that the vanishing of $\dot C_{12}(0)$ is equivalent to $C_{21}(s)=C_{12}(s)$ [see (A5)]. This result is due to the linearity of the system (A1).

When η_1 and η_2 are white noise,

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{ij}D_i\delta(t-t') , \tag{A8}$$

we obtain

$$\dot C_{12}(0)=\tfrac{1}{2}(bD_1-aD_2) . \tag{A9}$$

The condition for $\dot C_{12}(0)=0$ is then the same as when η_1 and η_2 are independent colored noise.

(ii) As a second case we consider the same source of noise for each variable, $\eta_1=\eta_2$. We then have from (A2)

$$C_{12}(s)=\tfrac{1}{8}\int_0^\infty du \left[[A(s-u)+A(s+u)] \left[\frac{(1+\mu)(1+\mu^{-1})}{\lambda_+}e^{-\lambda_+ u}+\frac{(1-\mu)(1-\mu^{-1})}{\lambda_-}e^{-\lambda_- u}\right] + [A(s-u)-A(s+u)][(\mu-\mu^{-1})(e^{-\lambda_+ u}e^{-\lambda_- u})] \right] . \tag{A10}$$

For colored noise the initial slope of $C_{12}(s)$ is

$$\dot C_{12}(0)=\frac{a-b}{4(ab)^{1/2}}\int_0^\infty du\, A(u)(\lambda_- e^{-\lambda_- u}-\lambda_+ e^{-\lambda_+ u}) . \tag{A11}$$

The vanishing of $\dot C_{12}(0)$ is again equivalent to the condition $a=b$. Note that this is also equivalent to $C_{12}(s)=C_{21}(s)$. However, when we consider that $\eta_1=\eta_2$ is a white noise the situation changes. In this case we obtain

$$\dot C_{12}(0^+)=-D\left[1+\frac{(a-b)}{2}\right] , \tag{A12}$$

which is not zero when $a=b$. This is due to the discontinuity of $\dot{C}_{12}$ at the initial time. But we see from (A10) that the symmetry of $C_{12}(s)$ under time reversal is again equivalent to $a=b$.

We can summarize our results for the linear system (A1) as follows. The even parity of $C_{12}(s)$, i.e., $C_{12}(s)=C_{21}(s)$, is equivalent to condition (A7). Then, when, η_1 and η_2 have the same correlation function, $A_{11}(u)=A_{22}(u)$, this is equivalent to $a=b$, corresponding to equations symmetric under the interchange of x_1 and x_2. Now, for a cross correlation with continuous derivative at the initial time the parity of C_{12} leads to the vanishing of the initial slope of C_{12}. This situation happens in the cases of colored noise and independent white noise for each variable. We remark that in these cases the condition $a=b$ is equivalent to the vanishing of $\dot{C}_{12}(0)$ but not for $\eta_1=\eta_2$ white noise. We finally note that in the particular case of a Markovian system (η_1,η_2 white noise), condition (A7) is equivalent to the existence of the detailed balance property.[32] This means that the variables x_1,x_2 are even under time reversal. Then the invariance of $C_{12}(s)$ under time reversal can be derived as a consequence of this property. Note, however, that in the Markovian case the equivalence of detailed balance and the symmetry under the interchange of x_1 and x_2 ($a=b$) is due to the linearity of the system (A1) (see Sec. III C).

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[31]Direct calculations of correlation functions through continued fraction expansion methods also give dynamical information in terms of moments of the stationary distribution [see Refs. 11(a) and 14]. At least for white noise, it is known that the lowest-order truncation in the calculation of $C(s)$ gives the exact value of $\dot{C}(0^+)$ and therefore reproduces the exact value of λ_{eff}.

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