Infrared radiative corrections: Extended treatment applicable to resonant scattering

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The calculation of infrared radiative corrections to scattering cross sections, to which a great deal of attention has been devoted in the past, is reconsidered here and carried to a higher order of accuracy within the context of a specific relativistic model. Earlier work, initiated by Low [Phys. Rev. **110**, 974 (1958)] and developed further by others, which led to soft-photon approximations for bremsstrahlung amplitudes, is extended to apply to *virtual*- as well as real-photon emission and absorption processes. As in the earlier work the requirement of gauge invariance plays a central role in the analysis. Evaluation of the new correction term obtained here requires, as input, the physical (on-mass-shell) amplitude for scattering in the absence of radiative interactions. The new term is expected to lead, in most cases, to only small modifications of the results obtained by standard methods; verification of this expectation would provide a useful check on the range of validity of these standard procedures. It is pointed out that for scattering in the neighborhood of a very narrow resonance the new correction term, which properly accounts for rapid variations of the scattering amplitude with energy, could become numerically significant. In an attempt to examine this possibility in a very preliminary yet quantitative manner the radiative correction terms derived here have been evaluated with the aid of a simple Breit-Wigner representation of the radiation-free scattering amplitude, and some typical results are reported.

I. INTRODUCTION

It is a well-known feature of scattering theory that, as a consequence of the imperfect energy resolution of particle detectors, analyses of collision processes involving charged particles must include the effect of energy loss caused by the emission of soft photons. After removal of infrared divergences one is left, in such analyses, with finite radiative corrections.[1] In energy regions where the cross section is rapidly varying, near narrow resonances and reaction thresholds in particular, the standard treatment of these radiative corrections can be inadequate. Here we indicate, in the context of a particular model, how to extend the standard treatment through the inclusion of corrections of higher order in the photon frequency. In doing so the effect of the energy variation of the scattering amplitude is accounted for to greater accuracy. If the variation is rapid enough—that is, if the cross section changes appreciably as the scattering energy is increased from E to $E+\Lambda$, where Λ is a suitably chosen soft-photon cutoff energy—the new correction term can be significant. This is illustrated in the numerical estimates presented below. If, as will be true in most cases, the cross section is slowly varying on this scale the correction should be small; the validity of the standard soft-photon approximation can be investigated quantitatively with an explicit form for the higher-order correction term in hand.

Our treatment of radiative corrections to resonant scattering may be thought of as an extension of the Feshbach-Yennie theory[2] of low-frequency bremsstrahlung—the emission of a single real photon during a resonant collision. In that work Low's soft-photon theorem[3] was generalized to account for a rapid energy variation of the cross section. Here we are concerned with the radiation of an undetected photon, with energy below some value ΔE corresponding to the detector resolution. This process may be treated in the manner of Feshbach and Yennie. In addition we must include the effect of virtual photons. With terms of higher order than e^2 ignored (e is the electric charge) this amounts to the inclusion of processes in which a single virtual photon is emitted and reabsorbed. Our treatment of this effect is based on the general analysis of soft-photon approximations given some time ago by Brown and Goble.[4] These authors showed how improved accuracy in the evaluation of multiphoton bremsstrahlung amplitudes can be obtained through application of gauge invariance requirements. They were primarily concerned with emission and absorption of real photons but as they remarked,[5] and as is shown explicitly below, it is a simple matter to adapt their methods to the problem of infrared corrections involving virtual photons.

It will be convenient to begin, in Sec. II, with a brief summary of one of the standard treatments of the infrared radiative correction problem. This provides the necessary preparation for the subsequent material, in Sec. III, in which a more accurate calculation is presented. The result of that calculation has the interesting feature that the radiative corrections are expressed in terms of the physical, on-shell cross section for scattering in the absence of any radiative interaction. We point out how the calculation could be extended to still higher order in the photon frequency; this would require, as additional input, the physical single-photon bremsstrahlung amplitude. For the sake of numerical orientation we have evaluated the relative size of the new correction term obtained here to the standard one over a range of scattering energies in the neighborhood of a resonance; a simple Breit-Wigner representation of the resonant radiation-free scattering amplitude was adopted. A typical set of results is shown graphically in Fig. 5 below.

The effect of infrared radiation on scattering in the neighborhood of a reaction threshold is discussed in a separate report.[6]

II. RADIATIVE CORRECTION IN LOWEST ORDER

While greater generality is possible it will be convenient to formulate our discussion of radiative corrections in the context of the model adopted by Brown and Goble.[4] Thus, we consider the scattering (due to some unspecified nonelectromagnetic interaction) of two spinless bosons, one of charge e and mass m and the other neutral, of mass M. The initial and final momenta of the charged particle are p and p', respectively, with P and P' representing the corresponding momenta of the neutral particle. This process is depicted in Fig. 1. The elastic scattering amplitude, in the absence of radiation, is denoted as $T(s,t)$, with scalar invariants chosen as $s=-pP$ and $t=-(P'-P)^2$, and with $p^2=p'^2=-m^2$, $P^2=P'^2=-M^2$. (Our notation is such that $pP=p^\mu P_\mu=\mathbf{p}\cdot\mathbf{P}-p^0P^0$, and we set $\hbar=c=1$.) The cross section for scattering with the emission of i photons, correct to order j in the electric charge will be denoted as $\sigma_{i,j}$. The elastic scattering cross section in the absence of any radiative corrections, $\sigma_{0,0}$, may be expressed in the form

$$\sigma_{0,0}=\int \tfrac{1}{4}(s^2-m^2M^2)^{-1/2}\,|T(s,t)|^2\frac{d^3p'}{(2\pi)^3 2E_{p'}}\times\frac{d^3P'}{(2\pi)^3 2E_{P'}}(2\pi)^4\delta(p+P-p'-P')\,, \tag{2.1}$$

with $E_p=(|\mathbf{p}|^2+m^2)^{1/2}$ and $E_P=(|\mathbf{P}|^2+M^2)^{1/2}$.

We shall include only the order-e^2 correction to the cross section arising from interactions between the charged particle and the low-frequency modes of the radiation field. The dominant corrections associated with the emission and reabsorption of a low-frequency virtual photon correspond, diagramatically, to graphs in which each end of the virtual-photon line is attached to an external charged-particle line. These diagrams are shown in Fig. 2. An evaluation of the associated amplitudes leads to an approximate determination of the cross section $\sigma_{0,2}$. The observed cross section, correct to order e^2, is $\sigma_{\rm obs}=\sigma_{0,2}+\sigma_{1,2}$, where $\sigma_{1,2}$ represents the cross section for emission of a single undetected soft photon. The dominant contributions to the bremsstrahlung amplitude, corresponding to emission in initial or final states, are represented graphically in Fig. 3. In the remainder of this section we shall review the standard procedure for evaluating the dominant contributions to $\sigma_{0,2}$ and $\sigma_{1,2}$.

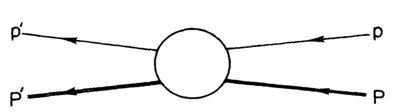

FIG. 1. Diagram representing the elastic process in the absence of radiation.

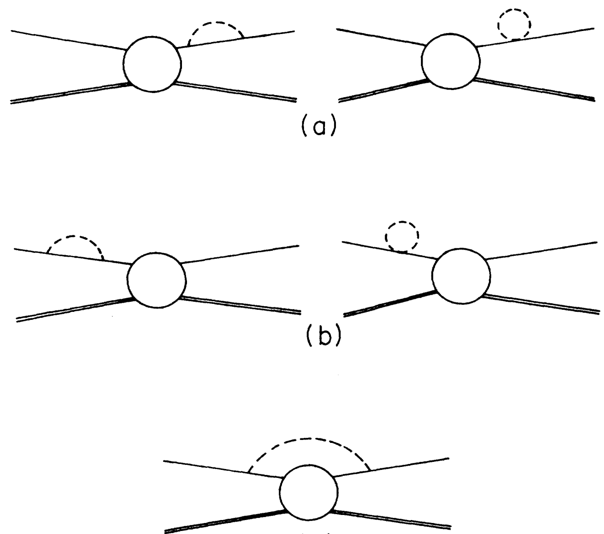

FIG. 2. Diagrams representing the dominant corrections due to virtual photon emission and reabsorption in initial and final states.

We begin by considering the virtual-photon corrections depicted in Fig. 2. These amplitudes have the structure

$$\int{}' d^4k\, D_{\mu\nu}(k)T^{\mu\nu}\,. \tag{2.2}$$

The prime indicates that the integration is restricted to the range $\lambda\leq|\mathbf{k}|\leq\Lambda$. Here Λ is an energy low enough to justify the introduction of soft-photon approximations and λ is a cutoff which serves to prevent infrared divergences at intermediate stages of the calculation.[7] The photon propagator is

$$D_{\mu\nu}(k)=\frac{-i}{(2\pi)^4}\frac{g_{\mu\nu}}{k^2-i\varepsilon}\,. \tag{2.3}$$

The first diagram in Fig. 2(a) corresponds to the self-energy insertion

$$T(s,t)\Sigma(p^2)(p^2+m^2-i\varepsilon)^{-1}\,,$$

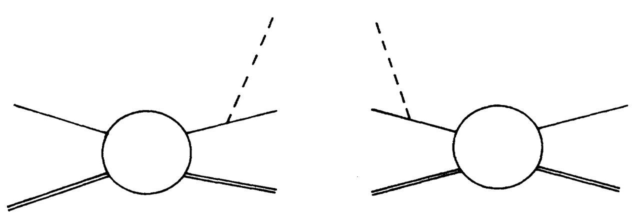

FIG. 3. Diagrams representing the dominant contributions to the amplitude for scattering with one soft photon emitted.

with

$$\Sigma(p^2)=\frac{-ie^2}{(2\pi)^4}g_{\mu\nu}\int'\frac{d^4k}{k^2-i\varepsilon}\frac{(2p+k)^\mu(2p+k)^\nu}{(p+k)^2+m^2-i\varepsilon}\ . \tag{2.4}$$

The condition $p^2+m^2=0$ has temporarily been lifted at this point in order that we may first carry out the mass-renormalization procedure in second order. As a first step[8] we make the subtraction

$$[(p+k)^2+m^2]^{-1}\to[(p+k)^2+m^2]^{-1}-(2pk+k^2)^{-1}$$
$$=-[(p+k)^2+m^2]^{-1}(p^2+m^2)(2pk+k^2)^{-1}\ .$$

(The $-i\varepsilon$ contribution to the energy denominators is implied when not explicitly written.) A similar subtraction applied to the term depicted in the second diagram of Fig. 2(a) removes that term completely. Before taking the limit $p^2+m^2\to 0$, as required, we must evaluate the factor $(p^2+m^2)(p^2+m^2)^{-1}$ in an unambiguous way. It may be shown[9] that the proper choice for this factor is $\frac{1}{2}$. The contribution from the diagrams of Fig. 2(a) then becomes

$$T(s,t)\frac{i}{(2\pi)^4}\tfrac{1}{2}e^2\int'\frac{d^4k}{k^2}\frac{(2p+k)^2}{(2pk+k^2)^2}\ ,$$

to which must be added a similar term with $p\to p'$ corresponding to the diagrams of Fig. 2(b). Finally, from the diagram of Fig. 2(c) we obtain the contribution

$$\frac{-i}{(2\pi)^4}e^2\int'\frac{d^4k}{k^2}T(s-kP,t)\frac{(2p'+k)(2p+k)}{(2p'k+k^2)(2pk+k^2)}\ .$$

Note that in this expression we have kept the elastic scattering amplitude at its physical, mass-shell value. That is (following Brown and Goble[4]) we have anticipated that off-mass-shell contributions cancel. We will be more explicit about this point below. (Cancellations of this type are familiar from the work of Low[3] and of Feshbach and Yennie.[2]) Collecting results we have, from the diagrams of Fig. 2, an approximation to the tensor $T^{\mu\nu}$ of Eq. (2.2) which may be written as

$$T^{\mu\nu}\cong-\tfrac{1}{2}e^2\left[T(s,t)\left[\frac{2p'+k}{2p'k+k^2}-\frac{2p+k}{2pk+k^2}\right]^\mu\left[\frac{2p'+k}{2p'k+k^2}-\frac{2p+k}{2pk+k^2}\right]^\nu -2[T(s-kP,t)-T(s,t)]\frac{(2p'+k)^\mu}{2p'k+k^2}\frac{(2p+k)^\nu}{2pk+k^2}\right]\ . \tag{2.5}$$

Taking into account the energy dependence of the elastic scattering amplitude we see that the approximation (2.5) fails to satisfy the gauge invariance requirement

$$k_\mu T^{\mu\nu}=k_\nu T^{\mu\nu}=0\ . \tag{2.6}$$

We postpone consideration of this point until Sec. III, where an improved approximation which does satisfy (2.6) is obtained, and proceed, in the remainder of this section, with a sketch of the standard derivation of the cross section in which only the most singular terms are retained.

The cross section $\sigma_{0,2}$ is obtained from the right-hand side of Eq. (2.1) by replacing $T(s,t)$ with the amplitude, denoted as $M_{0,2}$, which includes the correction terms shown graphically in Fig. 2. If we retain only the terms in Eq. (2.5) which are most singular in the infrared limit we obtain the approximation

$$M_{0,2}\cong T(s,t)\left[1+\frac{i}{(2\pi)^4}\frac{1}{2}e^2\int'\frac{d^4k}{k^2}\left[\frac{p'}{p'k}-\frac{p}{pk}\right]^2\right]\ . \tag{2.7}$$

The k^0 integration can be carried out by the method of residues, as described, for example, in Ref. 7. The expression (2.7) then becomes

$$M_{0,2}\cong T(s,t)\left[1-\tfrac{1}{4}\frac{e^2}{(2\pi)^3}\int_\lambda^\Lambda\frac{d\omega}{\omega}\int d^2\Omega\left[\frac{p'}{p'l}-\frac{p}{pl}\right]^2\right]. \tag{2.8}$$

Here l is the four vector $(\hat{1},1)$, with $\hat{1}\cdot\hat{1}=1$ and the angular integration is over the directions of the unit vector $\hat{1}$. All of the angular integrals which will be required here are obtained from the formula[10]

$$I(p_1,p_2)\equiv(p_1p_2)\frac{1}{4\pi}\int d^2\Omega\frac{1}{(p_1l)(p_2l)}$$
$$=-\frac{1}{2\beta_{p_1p_2}}\ln\left[\frac{1+\beta_{p_1p_2}}{1-\beta_{p_1p_2}}\right]\ , \tag{2.9}$$

with

$$\beta_{p_1p_2}=\left[1-\frac{p_1^2p_2^2}{(p_1p_2)^2}\right]^{1/2}\ . \tag{2.10}$$

This formula holds for $p_1\neq p_2$; for $p_1=p_2=p$ we have $I(p,p)=-1$. These results, applied to Eq. (2.8), then give

$$M_{0,2}\cong T(s,t)\left[1-\tfrac{1}{2}A\ln\frac{\Lambda}{\lambda}\right]\ , \tag{2.11}$$

with

$$A=\frac{e^2}{4\pi^2}\left[-2+\frac{1}{\beta_{p'p}}\ln\left[\frac{1+\beta_{p'p}}{1-\beta_{p'p}}\right]\right]\ ; \tag{2.12}$$

this leads, in turn, to the approximation

$$\sigma_{0,2}\cong\sigma_{0,0}\left[1-A\ln\frac{\Lambda}{\lambda}\right]\ . \tag{2.13}$$

As remarked earlier one should add to $\sigma_{0,2}$ the bremsstrahlung cross section $\sigma_{1,2}$ appropriate to the emission of a photon with energy anywhere in the interval between λ and ΔE, where ΔE is the energy resolution of the detector.

Let us write this cross section as

$$\sigma_{1,2}=\int_{\lambda\leq|\mathbf{k}|\leq\Delta E}\tfrac{1}{4}(s-m^2M^2)^{1/2}|M_{1,1}|^2\frac{d^3p'}{(2\pi)^3 2E_{p'}}\frac{d^3P'}{(2\pi)^3 2E_{P'}}\frac{d^3k}{(2\pi)^3 2|\mathbf{k}|}(2\pi)^4\delta(p+P-p'-P'-k)\ . \tag{2.14}$$

Here $M_{1,1}=M_{1,1}{}^{\mu}\epsilon_{\mu}$ is the single-photon bremsstrahlung amplitude correct to first order in the charge and ϵ_{μ} is the photon polarization vector. As a first approximation we have (following Brown and Goble)

$$M_{1,1}{}^{\mu}\cong e\left[\frac{(2p'+k)^{\mu}}{2p'k+k^2}T(s,t)+T(s+kP,t)\frac{(2p-k)^{\mu}}{-2pk+k^2}\right]. \tag{2.15}$$

This form (which, it should be noted, involves only the on-shell value of the elastic amplitude) satisfies the gauge-invariance condition

$$k_{\mu}M_{1,1}{}^{\mu}=0 \tag{2.16}$$

only in the approximation that the energy dependence of the elastic amplitude may be ignored. When this is not the case additional terms must be added to Eq. (2.15) to enforce the condition (2.16), as discussed in Sec. III. With only the most singular part retained we have

$$M_{1,1}{}^{\mu}\cong eT(s,t)\left[\frac{p'^{\mu}}{p'k}-\frac{p^{\mu}}{pk}\right]. \tag{2.17}$$

Since the condition (2.16) is satisfied by the approximation (2.17) the sum over transverse photon polarization states can be performed as

$$\sum_{\text{pol}}|M_{1,1}{}^{\mu}\epsilon_{\mu}|^2=M_{1,1}{}^{\mu}M^{*}_{1,1\mu}\ . \tag{2.18}$$

A brief calculation then gives

$$\sigma_{1,2}\cong\sigma_{0,0}A\ln\frac{\Delta E}{\lambda}\ . \tag{2.19}$$

The observed cross section, correct to second order, is

$$\sigma_{0,2}+\sigma_{1,2}\cong\sigma_{0,0}\left[1+A\ln\frac{\Delta E}{\Lambda}\right], \tag{2.20}$$

the logarithmic singularity associated with the limit $\lambda\to 0$ having cancelled in forming the sum.[11]

III. IMPROVED TREATMENT OF INFRARED CORRECTIONS

A. Amplitude corrections

As pointed out in Ref. 4 a deficiency in the approximation for the bremsstrahlung amplitude shown in Eq. (2.15), namely, that it fails to satisfy the gauge invariance condition (2.16), may be corrected through the addition of the term

$$e[T(s+kP,t)-T(s,t)]\frac{P^{\mu}}{Pk}\ , \tag{3.1}$$

which is nonsingular in the limit $k\to 0$. The *exact* amplitude may then be expressed as

$$M_{1,1}{}^{\mu}=e\left[T(s,t)\left[\frac{(2p'+k)^{\mu}}{2p'k+k^2}+\frac{(2p-k)^{\mu}}{-2pk+k^2}\right]+[T(s+kP,t)-T(s,t)]\left[\frac{P^{\mu}}{Pk}+\frac{(2p-k)^{\mu}}{-2pk+k^2}\right]\right]+R^{\mu}\ , \tag{3.2}$$

where the remainder R^{μ} is nonsingular. Nominally, R^{μ} is of order unity in the infrared limit, approaching a constant, independent of k, for $k\to 0$. However, the fact that R^{μ} satisfies the gauge invariance condition

$$k_{\mu}R^{\mu}=0 \tag{3.3}$$

allows one to conclude that the limiting value is zero. This information is obtained by differentiating Eq. (3.3) with respect to k and setting $k=0$. Since R^{μ} vanishes for $k\to 0$ the terms in the expansion of $M_{1,1}{}^{\mu}$ of order k^{-1} and k^0 are both given correctly in the approximation obtained by neglecting R^{μ} in Eq. (3.2), and this is the content of the low-frequency theorem for the bremsstrahlung amplitude.[2–4]

The derivation given above has been simplified by the neglect of the dependence of the elastic amplitude on the off-shell variables $\xi=p^2+m^2$ and $\xi'=p'^2+m^2$. That is, we should have introduced the extrapolated amplitude $T(s,t,\xi',\xi)$, with $T(s,t,0,0)\equiv T(s,t)$ representing the on-shell limit. The result of the derivation would have been the same, however. To see this consider, for example, the first term in the approximation (2.15). This should, more properly, have been written as

$$e\frac{(2p'+k)^{\mu}}{2p'k+k^2}T(s,t,2p'k+k^2,0)\ .$$

Expanding about $\xi'=0$ we have

$$T(s,t,2p'k+k^2,0)\cong T(s,t,0,0)+(2p'k+k^2)\frac{\partial T}{\partial\xi'}\ ; \tag{3.4}$$

the off-shell extrapolation has led to the introduction of an addition term $e(2p'+k)^{\mu}\partial T/\partial\xi'$. In a similar way, extrapolation in the variable ξ brings in the additional term $e(2p-k)^{\mu}\partial T/\partial\xi$. Both terms should be included in the representation (3.2) of the exact bremsstrahlung amplitude. However, since these additional terms are nonsingular we may consider them to have been absorbed in the remainder R^{μ}, thereby leaving the form of Eq. (3.2) unchanged. We may then argue as before that R^{μ} is gauge invariant as well as nonsingular, and therefore vanishes as k goes to zero; inclusion of the effects of off-mass-shell extrapolation has not altered our previous conclusion. It should be noted that the correction term (3.1) may play a significant role for scattering near a resonance whose width is comparable to ΔE. If one does retain the correction then, for consistency, the treatment of virtual photon corrections must be improved in a similar way. The method for doing so will now be described.

We wish to improve on the accuracy of the approximation (2.5) for the tensor $T^{\mu\nu}$; recall that the second-order virtual photon correction to the elastic scattering amplitude is obtained by inserting $T^{\mu\nu}$ in the integral (2.2). In obtaining Eq. (2.5) we accounted for diagrams in which the virtual photon line is attached at both ends to external charged-particle lines. We now wish to include diagrams, of the type shown in Fig. 4, in which one end of the photon line is attached to an external line and the other to an internal line; the corresponding amplitudes contain single rather than double poles and are therefore of higher order in the photon frequency than the terms already included. We omit those nonsingular contributions corresponding to purely internal radiative corrections. One might expect that this will introduce an error of order unity for $k \to 0$. It will be shown, however, that these nonsingular terms are in fact of order k; their neglect constitutes the low-frequency approximation for the tensor $T^{\mu\nu}$. The argument is based on the gauge-invariance condition and is analogous to the one presented above in connection with the bremsstrahlung amplitude.

The amplitudes depicted in Fig. 4 may be evaluated from a knowledge of the bremsstrahlung matrix element. It is not the full amplitude which is to be inserted, however, since to do so would be to include the terms already accounted for in Eq. (2.5). The proper part to insert is the difference between the full amplitude $M_{1,1}{}^{\mu}$ and the pole terms shown in Eq. (2.15). With the aid of Eq. (3.2) one sees that this difference, denoted as $\widetilde{M}_{1,1}{}^{\mu}$, can be expressed as

$$\widetilde{M}_{1,1}{}^{\mu}(p',p)=e\,[T(s+kP,t)-T(s,t)]\frac{P^{\mu}}{Pk}+R^{\mu}(p',p)\ . \tag{3.5}$$

The dependence of the functions $\widetilde{M}_{1,1}{}^{\mu}$ and R^{μ} on the charged-particle momenta is indicated explicitly here, with all other momentum dependence left implicit. It is understood that it is the on-mass-shell elastic amplitude which appears in Eq. (3.5); as discussed earlier, the off-shell contributions have been separated off and defined to be part of the nonsingular remainder R^{μ}. The graphs of Fig. 4 are now evaluated with $\widetilde{M}_{1,1}{}^{\mu}$ used to define the bremsstrahlung subgraph—the vertex to which the four-particle lines and single-photon line are attached. The result may be interpreted as a correction of the form

$$e^2\left\{[T(s,t)-T(s-kP,t)]\frac{P^{\mu}}{Pk}\frac{(2p+k)^{\nu}}{2pk+k^2}+[T(s+kP,t)-T(s,t)]\frac{P^{\mu}}{Pk}\frac{(2p'-k)^{\nu}}{-2p'k+k^2}\right\}$$
$$+e\left\{R^{\mu}(p',p+k)\frac{(2p+k)^{\nu}}{2pk+k^2}+R^{\mu}(p'-k,p)\frac{(2p'-k)^{\nu}}{-2p'k+k^2}\right\}, \tag{3.6}$$

which is to be added to the approximation (2.5) for $T^{\mu\nu}$. The function R^{μ} which appears in the expression (3.6) may be evaluated with each particle momentum on the appropriate mass shell. To see this, consider first the photon momentum. The photon is virtual, with $k^2 \neq 0$. However, we may expand about $k^2=0$ and note that all terms except the first may be neglected. These higher-order terms introduce at least two additional powers of k. Thus they contribute to corrections to $T^{\mu\nu}$ which vanish in the limit $k \to 0$ and terms of this order are omitted in the low-frequency approximation adopted here. Turning now to the dependence on the off-shell variables $\xi=p^2+m^2$ and $\xi'=p'^2+m^2$ we observe that the expression (3.6) is to be contracted with $g_{\mu\nu}$ and that $R^{\mu}k_{\mu}=0$. Let us now write the function $R^{\mu}(p',p+k)p_{\mu}$, which appears in the contraction of the expression (3.6), as the scalar function $Y(\xi',\xi)$ with $\xi=(p+k)^2+m^2=2pk+k^2$ and $\xi'=0$. (The dependence of Y on the remaining scalar variables which enter into a complete definition of the function is suppressed here since it plays no role in the analysis.) Expanding about $\xi=0$ we have

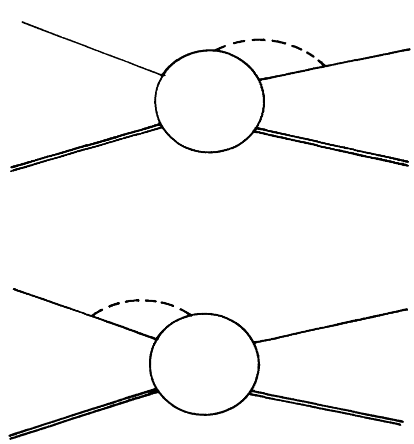

FIG. 4. Radiative corrections in which one end of the virtual-photon line is attached to an internal charged-particle line with the other end attached to an external line.

$$Y(0,2pk+k^2)\frac{1}{2pk+k^2}\cong Y(0,0)\frac{1}{2pk+k^2}+\left.\frac{\partial Y}{\partial \xi}\right|_{\substack{\xi=0,\\ \xi'=0}}\ . \tag{3.7}$$

The leading term is on the mass shell. The next term is nonsingular in the infrared limit and is neglected in the low-frequency approximation for $T^{\mu\nu}$. The last term in the expression (3.6) is analyzed in a similar way, and the neglect of off-mass-shell extrapolations of the functions appearing in (3.6) is thereby justified.

Proceeding with our analysis we note that the photon propagator in the integral (2.2) is even in k so that the sign of k may be changed freely in any one of the terms contributing to $T^{\mu\nu}$. In particular, the second term in the expression (3.6) will be rewritten as

$$e^2[T(s,t)-T(s-kP,t)]\frac{P^\mu}{Pk}\frac{(2p'+k)^\nu}{2p'k+k^2} .$$

We now add a term to $T^{\mu\nu}$ which is nonsingular in the limit $k\to 0$ and which leads to a gauge-invariant approximation; the procedure being followed here is analogous to that which led us to add the term (3.1) to the approximate bremsstrahlung amplitude. The term added to $T^{\mu\nu}$ is

$$e^2\left[T(s-kP,t)-T(s,t)+kP\frac{\partial T}{\partial s}\bigg|_{k=0}\right]\frac{P^\mu}{Pk}\frac{P^\nu}{Pk} . \tag{3.8}$$

The derivative term was included to provide us with a nonsingular function but this term may in fact be neglected since it is odd in k while the remaining factors in the integrand are even.

The foregoing discussion has led us to a representation of the amplitude $M_{0,2}$ in the form

$$M_{0,2}=T(s,t)-\frac{i}{(2\pi)^4}\int'\frac{d^4k}{k^2}g_{\mu\nu}(\widetilde{T}^{\mu\nu}+R^{\mu\nu}) \tag{3.9}$$

with

$$\widetilde{T}^{\mu\nu}=-\tfrac{1}{2}e^2\left[T(s,t)\left[\frac{2p'+k}{2p'k+k^2}-\frac{2p+k}{2pk+k^2}\right]^\mu\left[\frac{2p'+k}{2p'k+k^2}-\frac{2p+k}{2pk+k^2}\right]^\nu\right.$$
$$\left.-2[T(s-kP,t)-T(s,t)]\left[\frac{2p'+k}{2p'k+k^2}-\frac{P}{Pk}\right]^\mu\left[\frac{2p+k}{2pk+k^2}-\frac{P}{Pk}\right]^\nu\right] . \tag{3.10}$$

Since the approximation $\widetilde{T}^{\mu\nu}$ satisfies the same gauge invariance condition (2.6) as the exact tensor the same must be true for the remainder, that is

$$k_\mu R^{\mu\nu}=k_\nu R^{\mu\nu}=0 . \tag{3.11}$$

An approximation to $R^{\mu\nu}$ is provided by the last two terms, involving R^μ, in the expression (3.6). In the absence of resonance effects it would be reasonable to ignore the small momentum shifts in the arguments of R^μ and write $R^{\mu\nu}\cong\widetilde{R}_1{}^{\mu\nu}$, with

$$\widetilde{R}_1{}^{\mu\nu}=eR^\mu(p',p)\left[\frac{(2p+k)^\nu}{2pk+k^2}+\frac{(2p'-k)^\nu}{-2p'k+k^2}\right] . \tag{3.12}$$

Since this approximation is gauge invariant the same can be said for the error $R^{\mu\nu}-\widetilde{R}_1{}^{\mu\nu}$. Furthermore, this error must be nonsingular since all the singular contributions to $T^{\mu\nu}$ have been explicitly included. It follows that the error is of order k in the limit $k\to 0$, that is, we have the low-frequency approximation

$$T^{\mu\nu}=\widetilde{T}^{\mu\nu}+\widetilde{R}_1{}^{\mu\nu}+O(k) . \tag{3.13}$$

Consider now an improved treatment in which the possibility of a strong momentum dependence of the function R^μ is accounted for. Following the now familiar procedure we add to the original approximation to $R^{\mu\nu}$ [the last two terms in the expression (3.6)] a term, shown below to be nonsingular, of the form

$$-e[R^\mu(p',p+k)-R^\mu(p'-k,p)]\frac{P^\nu}{Pk} \tag{3.14}$$

resulting in the gauge-invariant approximation $R^{\mu\nu}\cong\widetilde{R}_2{}^{\mu\nu}$, with

$$\widetilde{R}_2{}^{\mu\nu}\cong e\left[R^\mu(p'-k,p)\left[\frac{(2p+k)^\nu}{2pk+k^2}+\frac{(2p'-k)^\nu}{-2p'k+k^2}\right]+[R^\mu(p',p+k)-R^\mu(p'-k,p)]\left[\frac{(2p+k)^\nu}{2pk+k^2}-\frac{P^\nu}{Pk}\right]\right] . \tag{3.15}$$

The error involved in this approximation is of order k since it is both gauge invariant and nonsingular. The improved low-frequency approximation to $T^{\mu\nu}$ is obtained by adding the expressions (3.10) and (3.15).

The claim that the expression (3.14) is nonsingular in the limit $k\to 0$ can be verified as follows. Consider the scalar $Z=R^\mu(p',p)P_\mu$. Since all the momenta are on the mass shell Z is a function of five invariants which may be chosen as $s=-pP$, $s'=-p'P$, $t=-(P'-P)^2$, $u=pk$, and $u'=p'k$. The added term (3.14), after contraction with $g_{\mu\nu}$, is then expressed as

$$-e(kP)^{-1}[Z(s-kP,s',t,u,u')-Z(s,s'+kP,t,u,u')] . \tag{3.16}$$

This approaches the nonsingular form $e(\partial Z/\partial s+\partial Z/\partial s')$ in the limit $k\to 0$.

B. Cross-section formulas

Let us summarize the results obtained up to this point. The cross section of interest is the sum $\sigma_{\rm obs}=\sigma_{1,2}+\sigma_{0,2}$ with $\sigma_{1,2}$ given by Eq. (2.14) and $\sigma_{0,2}$ by the version of

(2.1) in which T is replaced by $M_{0,2}$. With R^{μ} neglected in Eq. (3.2) we have an approximation to $M_{1,1}{}^{\mu}$ which correctly reproduces the terms of order k^{-1} and k^{0} in the bremsstrahlung amplitude. With $R^{\mu\nu}$ neglected in Eq. (3.9) we have an approximation to $M_{0,2}$ correct to order k^{-1}. Still greater accuracy can be achieved if R^{μ} is known to first order in k. Then, with R^{μ} assumed to be a slowly varying function of the particle momenta, Eq. (3.12) provides an approximation to the function $R^{\mu\nu}$ correct to order unity in the limit $k \to 0$. It is interesting to note that in this approximation the terms involving R^{μ} cancel, in forming the sum $\sigma_{0,2}+\sigma_{1,2}$, in the special case where ΔE and Λ are taken to be equal.[12] An improved approximation to $R^{\mu\nu}$, accounting for the possibility that R^{μ} is strongly momentum dependent, is given by Eq. (3.15).

In order to obtain a more explicit expression for the cross section $\sigma_{\rm obs}$ we now introduce a number of simplifications, the primary one being the neglect of terms involving R^{μ}. We also introduce the approximation

$$\frac{(2p+k)^{\mu}}{2pk+k^{2}} \cong \frac{p^{\mu}}{pk}, \quad \frac{(2p'+k)^{\mu}}{2p'k+k^{2}} \cong \frac{p'^{\mu}}{pk} . \qquad (3.17)$$

There is some inconsistency in this neglect of corrections of higher order in k when, at the same time, we retain higher-order corrections involving terms such as $T(s-kP,t)-T(s,t)$. However, we have in mind applications to resonant scattering where corrections of the latter type can be substantial, far outweighing in importance the error incurred in the approximation (3.17). We shall adopt the laboratory reference frame in which $P^{\mu}=(0,M)$. Then $kP=-k^{0}M$, independent of the orientation of $\mathbf{k}$ and this, in conjunction with the approximation (3.17), simplifies the evaluation of the angular integrations. To simplify notation a bit we suppress the dependence of $T(s,t)$ on the momentum transfer variable t. The energy variable is $s=-pP=p^{0}M$ in the laboratory frame; setting $p^{0}=E$ we write $T(E)$ in place of $T(s,t)$.

By extension of the methods which led to the infrared correction shown in Eq. (2.20) one finds that all of the required angular integrals appear in the form

$$A(p_1,p_2;p_3,p_4) \equiv \frac{1}{2}\frac{e^{2}}{(2\pi)^{3}}\int d^{2}\Omega \left[\frac{p_1^{\mu}}{p_1 l}-\frac{p_2^{\mu}}{p_2 l}\right] \times \left[\frac{p_{3\mu}}{p_3 l}-\frac{p_{4\mu}}{p_4 l}\right] , \qquad (3.18)$$

with $l=(\hat{1},1)$. This may be expanded and expressed in terms of the functions $I(p_i,p_j)$, which are defined and evaluated in Eq. (2.9). The integrals over the magnitude of the photon momentum are given in terms of the two functions

$$F(E,\Delta E) = |T(E)|^{-2}\int_0^{\Delta E}\frac{d\omega}{\omega}2\,{\rm Re}T^{*}(E)[T(E-\omega)-T(E)] , \qquad (3.19)$$

$$G(E,\Delta E)=|T(E)|^{-2}\int_0^{\Delta E}\frac{d\omega}{\omega}|T(E-\omega)-T(E)|^{2} . \qquad (3.20)$$

We may then write

$$\sigma_{\rm obs}=\sigma_{0,0}(1+C^{(1)}+C^{(2)}) , \qquad (3.21)$$

with

$$C^{(1)}=A(p',p;p',p)\ln\frac{\Delta E}{\Lambda} \qquad (3.22)$$

representing the standard correction, given in Eq. (2.20); the new correction term is

$$C^{(2)}=A(p',p;P,p)F(E,\Delta E)+A(P,p;P,p)G(E,\Delta E) -2A(p',P;p,P)F(E,\Lambda) . \qquad (3.23)$$

The first two terms on the right-hand side of Eq. (3.23) arise from the improved estimate of $\sigma_{1,2}$ while the last term corresponds to the correction to $\sigma_{0,2}$. Note that in the special case where the target mass greatly exceeds that of the projectile, so that the energy of the projectile is conserved in the collision, the momenta p and p' are equal for scattering in the forward direction. In this case the function $A(p',p;p',p)$, and hence the standard correction $C^{(1)}$, vanishes. However, $C^{(2)}$ is nonvanishing, so that the higher-order correction is dominant for forward scattering.

The correction term $C^{(2)}$ should make its most significant contribution in the neighborhood of a scattering resonance. Without entering into elaborate calculations at this point we can obtain some quantitative indication of the effect of the new correction term by adopting the simple resonant form

$$T(E)=\frac{\gamma}{E-E_0+i\Gamma/2} \qquad (3.24)$$

for the elastic scattering amplitude. The integrals in Eqs. (3.19) and (3.20) can then be evaluated as

$$F(E,\Delta E)=\ln\left[\frac{(E-E_0)^{2}+(\Gamma/2)^{2}}{(E-E_0-\Delta E)^{2}+(\Gamma/2)^{2}}\right] , \qquad (3.25)$$

$$G(E,\Delta)=g(E)\{\tan^{-1}[g(E)]-\tan^{-1}[g(E-\Delta E)]\} -\tfrac{1}{2}F(E,\Delta E) , \qquad (3.26)$$

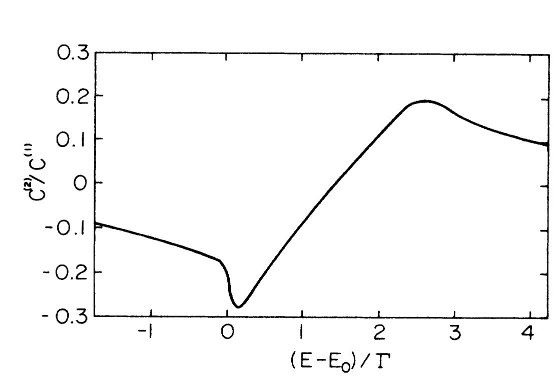

FIG. 5. Plot of the ratio $C^{(2)}/C^{(1)}$ of correction terms, defined in Eqs. (3.22) and (3.23), as a function of scattering energy. The graph corresponds to the choices $v=0.5$, $\Delta E=\Gamma/10$, and $\Lambda=5\Gamma/2$ for the input parameters, with the scattering angle taken to be 90°.

with

$$g(E)=\frac{(E-E_0)}{(\Gamma\Delta E/4)^{1/2}} . \tag{3.27}$$

We have evaluated the ratio $C^{(2)}/C^{(1)}$ over a range of energies in the resonance region, for a number of different choices of input parameters. For simplicity we have taken $M \gg m$ and have fixed the scattering angle at 90°, in which case we have, for the parameters β of Eq. (2.10), $\beta_{p'p}=v(2-v^2)^{1/2}$, $\beta_{pP}=\beta_{p'P}=v$, where v is the speed of the incoming projectile. (The variation of the β parameters as E ranges over the narrow resonance region has been ignored.) A typical set of results, corresponding to the parameters $v=0.5$, $\Delta E=\Gamma/10$, $\Lambda=5\Gamma/2$, is plotted in Fig. 5. The form of the fluctuation with energy of the correction term is common to a number of interference phenomena involving radiation[13] or ionization[14] during a resonant scattering process. The calculations indicate that the correction can be a significant one in the resonance region for ΔE an appreciable fraction of the resonance width. Undoubtedly, circumstances can be found under which this condition is satisfied, but it is a rather restrictive one, requiring very narrow resonances. In most cases of interest, in which ΔE is very much less than Γ, the effect of the resonance on the radiative correction is expected to be small. However, an evaluation of the new correction term will still be of interest in such cases since this provides the means for studying the accuracy and range of validity of the standard form of the infrared radiative correction.

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[10] The integrals are easily evaluated using the Feynman parametrization technique; see, for example, K. E. Ericksson, Nuovo Cimento **19**, 1010 (1961).

[11] The approximation (2.20) breaks down for $\Delta E \to 0$. The correct limiting behavior is obtained by summing an infinite subseries of perturbation terms, as shown, for example, in Ref. 1. Here we assume that ΔE is large enough to justify the use of second-order perturbation theory. It is quite possible that results obtained in Sec. III, which provide corrections of higher order in the photon frequency but only to order e^2, could be extended to higher orders in the charge but we shall not attempt to do so here.

[12] The calculation required to verify this statement is straightforward and we omit the details. As shown in Ref. 4 a similar cancellation holds in the analogous problem of scattering in the presence of a low-frequency external plane-wave radiation field. The fact that cancellations of this type also occur in the calculation of infrared corrections was mentioned briefly in footnote 5 of Ref. 4.

[13] See, for example, Feshbach and Yennie, Ref. 2.

[14] See, for example, J. S. Blair and R. Anholt, Phys. Rev. A **25**, 907 (1982).