PHYSICAL REVIEW A VOLUME 35, NUMBER 4 FEBRUARY 15, 1987

Degrees of freedom of turbulence

Giovanni Paladin* and Angelo Vulpiani*
Centre Européen de Calcul Atomique et Moléculaire, Batiment 506, Université de Paris-Sud, Orsay, France

(Received 5 November 1986)

We compute in the framework of a multifractal model for three-dimensional fully developed turbulence the number of degrees of freedom N as function of the Reynolds number R. N depends on the whole spectrum of singularities h related to the anomalous scaling of the velocity differences. On the other hand, we have also considered what the total number N_T^* of equations needed in a computer simulation is since N just has theoretical relevance. We stress, however, that the main features of intermittency can be described by an effective number $\tilde{N}^*$, which is much smaller than N_T^* because $\tilde{N}^*$ neglects very improbable events. We show that $N_T^* \propto R^3$ while we get from a fit of experimental data that $\tilde{N}^* \propto R^{2.3}$ is of the same order of $N \propto R^{2.2}$.

The numerical simulations of turbulent incompressible flows at high Reynolds number R require taking into account a large number of degrees of freedom. This implies that with the present technology there is no realistic hope of reproducing the small-scale behavior of physically interesting flows which can be found in geophysical, astrophysical, or engineering sciences. It is, however, a problem interesting in itself to understand how many degrees of freedom are necessary for describing the fully developed turbulence in three dimansions. The first estimate was given by Landau and Lifshitz[1] in the framework of the Kolmogorov theory[2] (K41).

It is clear that a satisfactory description of turbulent fluids needs a resolution up to a scale of the same order of the dissipative Komogorov length η at which the molecular friction is able to compete with the nonlinear transfer. One sees

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} , \tag{1}$$

where ε is the rate of the energy dissipation for unit mass and time (assumed to be constant in K41) and ν is the kinematic viscosity.

If L is the system characteristic length at which the external energy input is pumped, then the dimensional ratio $R = (\varepsilon L^4)^{1/3}/\nu$ is the Reynolds number. The number of grid points per unit volume necessary to obtain a resolution up to η is thus

$$N(R) \sim (L/\eta)^3 \propto R^{9/4} . \tag{2}$$

This argument hides the central assumption that all the fluid is "active," i.e., that the energy dissipation density field is smoothly distributed on a three-dimensional region. On the contrary, experimental and numerical evidence indicates that spatial intermittency is generally present.[3] Roughly speaking, if we look at a scale l just a percentage $\alpha(l) \propto l^{3-D_F}$ of the fluid is "active."

Let us recall that for definition of fractal dimension D_F the number $n(l)$ of cubes of edge $l \rightarrow 0$ necessary to cover an object scales as l^{-D_F}.

It is, however, simple to repeat the Landau-Lifshitz argument under the hypothesis that the energy dissipation $\varepsilon(x)$ is concentrated on a homogeneous fractal[4] with noninteger dimension $D_F < 3$ (β model of Frisch, Sulem, and Nelkin[5]). In this case the velocity differences scale for $r \rightarrow 0$ as

$$\Delta u_{\mathbf{x}}(r) = | \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}) | \propto r^h , \tag{3}$$

where h does not depend on $\mathbf{x}$:

$$h = (D_F - 2)/3 . \tag{4}$$

The Kolmogorov theory is, of course, recovered in the limit $D_F = 3$ and $h = \frac{1}{3}$.

The dissipation scale η can be now determined by imposing that the Reynolds number related to an eddy of length scale l is of order 1,

$$\eta \Delta u(\eta)/\nu \sim O(1) . \tag{5}$$

This is equivalent to the requirement that the dissipative (linear) term of the Navier Stokes equations is able to compete with the nonlinear transfer term.

Inserting Eq. (3) in Eq. (5) we obtain

$$\eta \sim \frac{L}{R^{1/(1+h)}} . \tag{6}$$

It follows that

$$N(R) \sim \left(\frac{L}{\eta} \right)^{D_F} \propto R^{3D_F/(1+D_F)} . \tag{7}$$

The above formula has been derived by Kraichnan.[6]

Let us remark that some other variables are also necessary to describe the nonactive regions of the fluid, but their number is not a function of R. If the β-model assumptions were correct, Eq. (7) would give, in principle, the scaling of $N(R)$. This is not the case, as the implications of the homogeneous fractal model for the structure function scaling

$$\langle | \Delta u(r) |^p \rangle \propto r^{\zeta_p} , \tag{8}$$

with

$$\zeta_p = \frac{D_F - 2}{3} p + (3 - D_F) ,$$

do not seem to be satisfied. In Eq. (8) $\langle (\cdots) \rangle$ means spa-

 © 1987 The American Physical Society

tial average. Experimental results[7] show that ζ_p is a nonlinear function of p for $p \gtrsim 8$, indicating that there are different singularity values h. It was therefore suggested that the energy dissipation is concentrated on an inhomogeneous fractal without global scaling invariance.[8] Let us introduce the set $S(h)$ of the points $\mathbf{x}$ for which

$$\Delta u_{\mathbf{x}}(r) \propto r^h \ , \tag{9}$$

assuming $h_{\min} \leq h \leq h_{\max}$. The probability $P(l)$ that a point $\mathbf{x}$ belongs to $S(h)$ scales as

$$P(l) \propto l^{-D(h)+3} \ , \tag{10}$$

where $D(h) \leq D_F$ is the fractal dimension of the set $S(h)$.

The structure functions can then be computed as an average taken over a measure $d\mu(h)$ concentrated on the different sets $S(h)$:

$$\langle \Delta u(r)^p \rangle \sim \int d\mu(h) l^{hp+3-D(h)} \ . \tag{11}$$

A simple saddle-point calculation of (11) in the limit $l \rightarrow 0$ shows that

$$\zeta_p = \min_h [hp - D(h)] + 3 \ . \tag{12}$$

Equation (12) is a Legendre transform: For a given p value, ζ_p peaks up to a particular $D(h)$. The knowledge of observable moments, in fact, allows (under certain general conditions) the building up of the probability distribution. The number of degrees of freedom necessary for describing such a multifractal picture of turbulence must be defined with much more care. In fact, for each singularity h a different dissipative length $\eta(h)$ is peaked up by condition (6): $\eta(h) \propto R^{-1/(1+h)}$.

Since the number of eddies of scale l with singularity h is proportional to $l^{-D(h)}$, one sees that the number of grid points which have to be considered for resolving the set $S(h)$ is

$$N_h(R) \sim \left(\frac{L}{\eta(h)} \right)^{D(h)} \propto R^{D(h)/(1+h)} \ . \tag{13}$$

We can thus get the total number of degrees of freedom by integrating (13) over h:

$$N(R) = \int d\mu(h) N_h(R) \propto R^{\delta} \ , \tag{14}$$

where δ can be estimated by the steepest descent method in the limit of large R:

$$\delta = \max_h [D(h)/(1+h)] \ . \tag{15}$$

A fit of the experimental data of Ref. (7) gives the value $\delta \simeq 2.2$, which is close to the value given by Eq. (7). The results (14) and (7) are, nevertheless, quite different from a conceptual point of view.

We must stress that the estimate (15) just has a theoretical relevance, since it is rather difficult in a computer simulation to locate the grid points on the sets $S(h)$ (which also evolves in time). Indeed, one usually works with a fixed grid or with a pseudospectral method.[9] It follows that the only relevant parameter is the minimal scale $l_{\min}$ considered which is bounded from below by the dissipative length related to the strongest singularity:

$$l_{\min} \sim \eta(h_{\min}) \propto R^{-1/(1+h_{\min})} \ . \tag{16}$$

The estimate $l_{\min} = \eta(h_{\min})$ assures that all the sets $S(h)$ (i.e., even very improbable events) are taken into account.

The number of equations which enables such a fully accurate description is thus

$$N_T^* \sim \left(\frac{L}{l_{\min}} \right)^3 \propto R^{3/(1+h_{\min})} \ . \tag{17}$$

Equation (17) is in agreement with rigorous bounds.[10] On the other hand, if one decides to neglect the rare events a sufficient resolution is $\tilde{l} \gg l_{\min}$, on which just the relevant features of turbulence are reproduced losing some details. In this case the number of equations is reduced to

$$\tilde{N}^* \sim \left(\frac{L}{\tilde{l}} \right)^3 \ .$$

This scale $\tilde{l}$ can be estimated by the dissipative length $\eta(\tilde{h})$ related to an effective singularity $\tilde{h}$.

Let us define an "effective"-mass dimension $\tilde{D}$ of the object on which the energy dissipation is concentrated by

$$\tilde{h} = \frac{\tilde{D} - 2}{3} \ .$$

Mandelbrot[11] has, e.g., assumed $\tilde{D} = D_I$ the information dimension, and from the data of Ref. 7 one has $D_I = 2.87$. By inverting the Legendre transform (12), this assumption corresponds to selecting $\tilde{h} = (d\zeta_p/dp)_{p=3} \simeq 0.29$, and, roughly speaking, $\tilde{l}$ is thus the smallest scale on which *in average* active eddies are still present.

On the other hand, some heuristic arguments,[4,5] as well as a fit of experimental data,[7] indicate $h_{\min} = 0$. It follows that

$$N_T^* \propto R^3 \text{ and } \tilde{N}^* \propto R^{3/(1+\tilde{h})} \sim R^{2.3} \ .$$

Let us finally emphasize that N_T^* is much greater than $\tilde{N}^*$, which is close to the estimates (2), (7), and (14) of the number of degrees of freedom obtained, respectively, in the K41, in the β-model, and in the framework of the multifractal approach.

This work was done during our residence at Centre Européen de Calcul Atomique et Moléculaire of Orsay for the workshop Dynamics on Fractals and Hierarchies of Critical Exponents. We thank Dr. C. Moser for this opportunity.

---

*Permanent address: Dipartimento di Fisica, Università "La Sapienza" piazzale Aldo Moro 2, 00185 Roma, Italy.

[1]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, MA, 1959).

[2]A. N. Kolmogorov, C. R. (Dokl.) Acad. Sci. URSS **30**, 301 (1941).

[3]A. C. Monin and A. M. Yaglon, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2, Sec. 25.

[4]B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[5]U. Frisch, P. L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).
[6]R. H. Kraichnan, Phys. Fluids **28**, 10 (1985).
[7]F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[8]U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical and Fluid Dynamics,* edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 71; R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984); G. Paladin and A. Vulpiani, in *Fractals in Physics,* edited by L. Pietronero and E. Tosatti (Elsevier, New York, 1986), p. 447.
[9]G. S. Patterson and S. Orszag, Phys. Fluids **14**, 2538 (1971).
[10]D. Ruelle, Commun. Math. Phys. **87**, 287 (1982); P. Costantin, G. Foias, O. P. Manley, and R. Teman, J. Fluid Mech. **150**, 427 (1985).
[11]B. B. Mandelbrot, in *Turbulence and Navier-Stokes Equations,* Lecture Notes in Mathematics, Vol. 565, edited by R. Teman (Springer, Berlin, 1976), p. 121.