Analytic study of pulse broadening in dispersive optical fibers

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It is demonstrated that for an optical pulse propagating along an optical fiber the rms pulse width varies parabolically with distance, irrespective of initial pulse form and frequency chirp variation. Furthermore, the result is true to arbitrary dispersive order and should prove a very useful tool in determining the information-carrying capability of long-distance optical-fiber transmission systems.

I. INTRODUCTION

An optical-fiber communication system for high-bit-rate transmission over long distances requires short pulses with small pulse broadening during propagation. Chromatic dispersion is an inherent property of optical fibers and leads to increasing pulse widths. This may severely limit the performance of the optical transmission system.

Thus, from the point of view of the transmission capability of an optical communication system, the variation of pulse width with distance of propagation is the most important single-pulse characteristic. Most analytical investigations of the effect of dispersion on pulse propagation in optical fibers have modeled the pulse envelope variation as Gaussian and have assumed a linearly varying chirp frequency.[1–4] This is an analytically convenient assumption which makes it possible to evaluate explicitly the resulting pulse form at the output end of the fiber. On the other hand, for initially non-Gaussian pulses resort is mostly taken to numerical computations.[5,6]

The purpose of the present work is to provide a shortcut to the problem of linear dispersive pulse broadening by concentrating on the variation of the pulse width, leaving the detailed form of the pulse envelope aside. It is possible to show that the pulse width, defined in the rms sense, has a parabolic variation with distance of propagation irrespective of dispersive order and initial pulse form, which only affects the propagation characteristics by determining the coefficients of the parabola. The analysis is carried through to third dispersive order to allow for the possibility of propagation at the zero-dispersion wavelength. Explicit expressions are given for super-Gaussian initial pulse forms with arbitrary rectangularity, index, and linewidth enhancement factors, cf. Ref. 6. The result should prove useful in determining the transmission characteristics of pulses of more realistic pulse forms than the conventionally assumed Gaussian form.

II. THE DISPERSIVE SCHRÖDINGER OPERATOR

The linear propagation properties of the optical wave pulse are determined by the dispersion relation, which relates to the wave frequency ω and the wave number k according to

$$k=k(\omega) \ . \tag{1}$$

We expand the function $k(\omega)$ around the wave carrier frequency ω_0 to obtain

$$k=\sum_{n=0}^{\infty}\frac{k_0^{(n)}}{n!}(\omega-\omega_0)^n \ , \tag{2}$$

where

$$k_0^{(n)}=\frac{d^n k}{d\omega^n}(\omega_0) \ . \tag{3}$$

Equation (2) can be rearranged to read

$$k-k_0=\sum_{n=1}^{\infty}\frac{k_0^{(n)}}{n!}(\omega-\omega_0)^n \ . \tag{4}$$

The correspondence

$$i(k-k_0)\leftrightarrow\frac{\partial}{\partial x}, \quad -i(\omega-\omega_0)\leftrightarrow\frac{\partial}{\partial t} \tag{5}$$

makes it possible to translate Eq. (4) into a differential equation for the slowly varying envelope $\psi(x,t)$ of the pulse, viz.,

$$i\frac{\partial\psi}{\partial x}=-\sum_{n=1}^{\infty}\frac{i^n k_0^{(n)}}{n!}\frac{\partial^n\psi}{\partial t^n} \ . \tag{6}$$

Finally, we introduce the retarded time τ, measured relative to the time it takes the pulse to travel the distance x, assuming a pulse velocity equal to the group velocity $v_g=(k_0')^{-1}$, i.e., $\tau=t-xk_0'$. Using τ as the time variable in Eq. (6) we obtain the generalized Schrödinger equation

$$i\frac{\partial\psi}{\partial x}=-\sum_{n=2}^{\infty}\frac{i^n k_0^{(n)}}{n!}\frac{\partial^n\psi}{\partial\tau^n}\equiv H\left[\frac{\partial}{\partial\tau}\right]\psi \ , \tag{7}$$

where an important property of the operator H is that it is self-adjoint. The subsequent analysis is carried through for a general self-adjoint operator H. In practice, the second (k_0'') dispersive order part of the operator dominates the pulse evolution, except for extremely short pulses and/or close to the zero-dispersion wavelength.

III. MOMENT EQUATIONS FOR THE GENERALIZED SCHRÖDINGER EQUATION

Consider the evolution of a pulse which satisfies the initial value problem:

$$i\frac{\partial\psi(x,\tau)}{\partial x}=H\left[\frac{\partial}{\partial\tau}\right]\psi(x,\tau)\ ,\qquad \psi(0,\tau)=\psi_0(\tau)\ , \tag{8}$$

where H is a self-adjoint operator. Instead of trying to find an explicit solution of Eq. (8) we characterize the solution in terms of its moments $I_n(x)$, defined as cf. Refs. 7 and 8,

$$I_n(x)\equiv\langle\tau^n\rangle=\int_{-\infty}^{+\infty}\psi^*(\tau,x)\tau^n\psi(\tau,x)d\tau\ . \tag{9}$$

If we introduce the commutator $[H,\tau^n]$ as

$$[H,\tau^n]=H\tau^n-\tau^nH \tag{10}$$

and the m-fold commutator $[H,\tau^n]_m$ as

$$[H,\tau^n]_{m+1}=[H,[H,\tau^n]_m]\ ,\qquad [H,\tau^n]_1=[H,\tau^n]\ , \tag{11}$$

it is straightforward to show by induction that

$$\frac{d^mI_n(x)}{dx^m}=i^m\langle[H,\tau^n]_m\rangle\ . \tag{12}$$

Furthermore, it can be shown (see the Appendix) that

$$\left[\frac{\partial^k}{\partial\tau^k},\tau^n\right]_m\equiv0\ \text{ if } n<m\ . \tag{13}$$

Thus, if $H(\partial/\partial\tau)$ is a self-adjoint differential operator with constant coefficients, cf. Eq. (7), i.e.,

$$H\left[\frac{\partial}{\partial\tau}\right]=\sum_{k=2}^{\infty}i^k\alpha_k\frac{\partial^k}{\partial\tau^k}\ , \tag{14}$$

we conclude that

$$\frac{d^{n+1}I_n(x)}{dx^{n+1}}=0\ , \tag{15}$$

i.e., $I_n(x)$ must be a polynomial of degree n:

$$I_n(x)=\sum_{k=0}^{n}a_{n,k}x^k\ . \tag{16}$$

The coefficients, $a_{n,k}$, are determined by the initial condition $\psi_0(\tau)$ according to

$$a_{n,k}=\frac{i^k}{k!}\langle[H,\tau^n]_k\rangle\,|_{x=0}=\frac{i^k}{k!}\int_{-\infty}^{+\infty}\psi_0^*(\tau)[H,\tau^n]_k\psi_0(\tau)d\tau\ . \tag{17}$$

From Eq. (16) we infer the following lowest-order moments:

$$\begin{aligned}I_0&=\int_{-\infty}^{+\infty}\psi^*(\tau,x)\psi(\tau,x)d\tau=\text{const}\ ,\\ I_1&=\int_{-\infty}^{+\infty}\psi^*(\tau,x)\tau\psi(\tau,x)d\tau=a_0+a_1x\ ,\\ I_2&=\int_{-\infty}^{+\infty}\psi^*(\tau,x)\tau^2\psi(\tau,x)d\tau=b_0+b_1x+b_2x^2\ ,\end{aligned} \tag{18}$$

where the coefficients a_j and b_j are determined by

$$\begin{aligned}a_0&=\int_{-\infty}^{+\infty}\psi_0^*(\tau)\tau\psi_0(\tau)d\tau\ ,\\ a_1&=i\int_{-\infty}^{+\infty}\psi_0^*(\tau)[H,\tau]\psi_0(\tau)d\tau\ ,\\ b_0&=\int_{-\infty}^{+\infty}\psi_0^*(\tau)\tau^2\psi_0(\tau)d\tau\ ,\\ b_1&=i\int_{-\infty}^{+\infty}\psi_0^*(\tau)[H,\tau^2]\psi_0(\tau)d\tau\ ,\\ b_2&=-\tfrac{1}{2}\int_{-\infty}^{+\infty}\psi_0^*(\tau)[H,\tau^2]_2\psi_0(\tau)d\tau\ .\end{aligned} \tag{19}$$

IV. APPLICATION TO DISPERSIVE PULSE PROPAGATION

In the case when $\psi(x,\tau)$ of Eq. (8) represents the slowly varying envelope function of an optical wave pulse, the moments of Eq. (18) have a well-known physical interpretation.

(i) I_0 represents the pulse energy which, without loss of generality, will be taken equal to unity.

(ii) I_1 determines the "center of mass" velocity of the pulse since $\langle\tau\rangle=\langle t-xk_0'\rangle=a_0+a_1x$ implies that

$$\frac{1}{v_c}\equiv\left\langle\frac{dt}{dx}\right\rangle=k_0'+a_1=\frac{1}{v_g}+a_1\ . \tag{20}$$

(iii) I_2 determines the rms pulse width σ defined according to

$$\begin{aligned}\sigma^2&=\langle(\tau-\langle\tau\rangle)^2\rangle\\ &=\langle\tau^2\rangle-\langle\tau\rangle^2\\ &=b_0-a_0^2+(b_1-2a_0a_1)x+(b_2-a_1^2)x^2\ .\end{aligned} \tag{21}$$

The coefficients a_1, b_1, and b_2 include contributions from arbitrarily high dispersive orders, but in practice, except for extremely short pulses, the coefficients are determined by the second- and third-order dispersive terms in the Schrödinger equation. To third dispersive order we find, using the operator identities derived in the Appendix,

$$\begin{aligned}a_1&=k_0''\,\mathrm{Im}\int_{-\infty}^{+\infty}\psi_0\frac{\partial\psi_0^*}{\partial\tau}d\tau+\frac{k_0'''}{2}\int_{-\infty}^{+\infty}\left|\frac{\partial\psi_0}{\partial\tau}\right|^2d\tau\ ,\\ b_1&=2k_0''\,\mathrm{Im}\int_{-\infty}^{+\infty}\psi_0\tau\frac{\partial\psi_0^*}{\partial\tau}d\tau+k_0'''\int_{-\infty}^{+\infty}\tau\left|\frac{\partial\psi_0}{\partial\tau}\right|^2d\tau\ ,\\ b_2&=(k_0'')^2\int_{-\infty}^{+\infty}\left|\frac{\partial\psi_0}{\partial\tau}\right|^2d\tau\\ &\quad-\frac{k_0''k_0'''}{2}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{\partial\psi_0^*}{\partial\tau}\frac{\partial^2\psi_0}{\partial\tau^2}d\tau\\ &\quad+\left[\frac{k_0'''}{2}\right]^2\int_{-\infty}^{+\infty}\left|\frac{\partial^2\psi_0}{\partial\tau^2}\right|^2d\tau\ .\end{aligned} \tag{22}$$

The results given by Eqs. (21) and (22) provide a generalization of previous works of ours, see Refs. 7 and 8.

V. APPLICATION TO CHIRPED GAUSSIAN PULSES

The above results, Eqs. (20)—(22), make it possible to evaluate to third dispersive order the pulse velocity as well as the rms pulse width variation for pulses of arbitrary initial form. In order to demonstrate the analysis on a well-known example, we investigate the pulse broadening of chirped Gaussian pulses of the form

$$\psi_0(\tau)=A\exp\left[-\tfrac{1}{2}(1-i\alpha)\left[\frac{\tau}{a}\right]^2\right]. \tag{23}$$

Using Eq. (23) in Eqs. (20)—(22) we find

$$\frac{1}{v_c}=k_0'+\frac{k_0'''}{4a^2}(1+\alpha^2) \tag{24}$$

and

$$\frac{\sigma^2}{\sigma_0^2}=1-\frac{\alpha k_0''}{\sigma_0^2}x+\frac{1+\alpha^2}{4\sigma_0^4}\left[(k_0'')^2+\frac{(1+\alpha^2)}{8\sigma_0^2}(k_0''')^2\right]x^2, \tag{25}$$

where $\sigma_0=a\sqrt{2}$ denotes the initial rms pulse width. Equation (25) reduces to well-known results in the limits of vanishing second- or third-order dispersion, cf. Refs. 1—4, 8, and 9. However, as has recently been pointed out,[6,7] the Gaussian pulse approximation is analytically convenient, but is not always very realistic. The following flexible ansatz was recently[6] suggested as a more useful model,

$$\psi_0(\tau)=A\exp\left[-\tfrac{1}{2}(1-i\alpha)\left[\frac{\tau}{a}\right]^{2m}\right]. \tag{26}$$

The parameter m determines the degree of rectangularity. For $m=1$ we regain the usual Gaussian pulse with a linearly varying chirp frequency. However, for increasing m, the corresponding super-Gaussian envelope functions become successively more rectangular and the chirping becomes concentrated to the leading and trailing edges of the pulse. Both features are characteristic of pulses from directly modulated semiconductor lasers.[6]

Using the ansatz (26) in Eqs. (20)—(22) we obtain for the pulse velocity

$$\frac{1}{v_c}=k_0'+\frac{k_0'''}{8}\frac{1+\alpha^2}{\sigma_0^2}\frac{\Gamma\left[\frac{3}{2m}\right]\Gamma\left[2-\frac{1}{2m}\right]}{\Gamma^2\left[1+\frac{1}{2m}\right]}, \tag{27}$$

where $\Gamma(x)$ denotes the gamma function and $\sigma_0^2=a^2\Gamma(3/2m)/\Gamma(1/2m)$ is the initial rms pulse width. The variation of the pulse width is determined by

$$\sigma^2/\sigma_0^2=1-\frac{\alpha k_0''}{\sigma_0^2}x+\frac{1+\alpha^2}{4\sigma_0^4}\left[H(m)k_0''^2+G(m)[K_0(m)+K_1(m)\alpha^2]\frac{(k_0''')^2}{\sigma_0^2}\right], \tag{28}$$

where

$$H(m)=\frac{\Gamma\left[\frac{3}{2m}\right]\Gamma\left[2-\frac{1}{2m}\right]}{\Gamma^2\left[1+\frac{1}{2m}\right]},$$

$$G(m)=\frac{\Gamma^2\left[\frac{3}{2m}\right]}{16\Gamma^4\left[1+\frac{1}{2m}\right]},$$

$$K_0(m)=\frac{\Gamma\left[\frac{1}{2m}\right]\Gamma\left[3-\frac{1}{2m}\right]\Gamma\left[2-\frac{3}{2m}\right]}{\Gamma\left[1-\frac{1}{2m}\right]}-\Gamma^2\left[2-\frac{1}{2m}\right], \tag{29}$$

$$K_1(m)=\Gamma\left[\frac{1}{2m}\right]\Gamma\left[4-\frac{3}{2m}\right]-\Gamma^2\left[2-\frac{1}{2m}\right],$$

which is seen to reduce to Eq. (25) when $m=1$. A previously made[7] comparison between the second-order dispersive result, according to Eqs. (28) and (29), and the numerical computations of Ref. 6 showed complete agreement.

VI. APPLICATION TO TRUNCATED EXPONENTIAL CHIRP-FREE PULSES

As a further illustration of the usefulness and the flexibility of the result given by Eqs. (28) and (29), we will apply it to a recently suggested model for chirp-free optical pulses,[5] viz.,

$$\psi_0(\tau)=A\exp(-\tau/2\tau_0+i\Delta\omega t)\Theta(\tau), \tag{30}$$

where $\Theta(\tau)$ is the Heaviside step function and $\Delta\omega$ models the frequency deviation due to injection current modulation in a semiconductor laser. In the case of an envelope variation, as in Eq. (30), $\Delta\omega$ is constant since $\Delta\omega\sim d\ln|\psi^2|/d\tau$.[5]

Using Eq. (30) in Eqs. (28) and (29) we find

$$\frac{\sigma^2(x)}{\sigma_0^2}=1+\left[\frac{k_0''}{2\sigma_0^2}\right]^2x^2. \tag{31}$$

Minimizing the output pulse with respect to the input pulse for a given distance of propagation x, we obtain

$$\sigma_{\min}=(x\,|\,k_0''\,|\,)^{1/2}, \tag{32}$$

which compares favorably with the numerical result obtained in Ref. 5 [$\sigma_{\min}\simeq 0.9(x\,|\,k_0''\,|\,)^{1/2}$]: The difference should be due to the different definition of the rms pulse width used in Ref. 5 (defined with respect to the powers squared).

VII. INFLUENCE OF FIBER LOSS

If fiber loss is included into the generalized Schrödinger equation, Eq. (8), we obtain

$$i\frac{\partial\psi}{\partial x}=-i\gamma\psi+H\psi\ . \tag{33}$$

The loss term $(-i\gamma\psi)$ can be transformed away by introducing $\psi=\tilde{\psi}\exp(-i\gamma x)$. The resulting equation for $\tilde{\psi}$ is the same as Eq. (8). This implies that the evolution of the moments $I_n(x)$ will not be affected by fiber loss since

$$I_n(x)\equiv\frac{\int_{-\infty}^{+\infty}\psi^*(\tau,x)\tau^n\psi(\tau,x)d\tau}{\int_{-\infty}^{+\infty}\psi^*(\tau,x)\psi(\tau,x)d\tau}=\frac{\int_{-\infty}^{+\infty}\tilde{\psi}^*(\tau,x)\tau^n\tilde{\psi}(\tau,x)d\tau}{\int_{-\infty}^{+\infty}\tilde{\psi}^*(\tau,x)\tilde{\psi}(\tau,x)d\tau}\ . \tag{34}$$

The normalized definition of $I_n(x)$ in Eq. (34) is consistent with definition (9) complemented with the condition $\langle 1\rangle=1$. Thus all our previous results for pulse broadening are still valid when fiber loss is included.

VIII. CONCLUSION

The maximum bit rate of long-distance optical-fiber communication systems is determined by the dispersive broadening of the input pulse during propagation. Our investigation results in explicit analytical expressions for the rms pulse width variation in terms of a second-order polynomial in x—the distance of propagation. The coefficients of the polynomial are determined by the properties of the input pulse. We emphasize that the result is true to any dispersive order and applicable to input pulses of arbitrary form. The result provides a more direct and convenient way of determining the transmission capacity of an optical-fiber communication system than previous cumbersome numerical procedures based on Fourier-transform analysis.

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APPENDIX

The crucial point in the presented analysis is the result given in Eq. (13), viz.,

$$\left[\frac{\partial^k}{\partial\tau^k},\tau^n\right]_m=0\ \text{ if } n<m\ . \tag{A1}$$

In order to prove this we need the following operator identity, which is straightforward, to verify using induction:

$$\left[\frac{\partial^k}{\partial\tau^k},e^{s\tau}\right]_m=e^{s\tau}\left[\left[\frac{\partial}{\partial\tau}+s\right]^k-\frac{\partial^k}{\partial\tau^k}\right]^m . \tag{A2}$$

The left-hand side of Eq. (A2) can be expanded as

$$\left[\frac{\partial^k}{\partial\tau^k},e^{s\tau}\right]_m=\left[\frac{\partial^k}{\partial\tau^k},\sum_{n=0}^{\infty}\frac{(s\tau)^n}{n!}\right]_m=\sum_{n=0}^{\infty}\frac{s^n}{n!}\left[\frac{\partial^k}{\partial\tau^k},\tau^n\right]_m . \tag{A3}$$

The right-hand side (rhs) of Eq. (A3) is rewritten as

$$e^{s\tau}\left[\left[\frac{\partial}{\partial\tau}+s\right]^k-\frac{\partial^k}{\partial\tau^k}\right]^m=\sum_{n=0}^{\infty}\frac{(s\tau)^n}{n!}\left[\sum_{r=0}^{k}\binom{k}{r}s^r\frac{\partial^{k-r}}{\partial\tau^{k-r}}-\frac{\partial^k}{\partial\tau^k}\right]^m=\sum_{n=0}^{\infty}\frac{(s\tau)^{n+m}\tau^n}{n!}\left[\sum_{r=1}^{k}\binom{k}{r}s^{r-1}\frac{\partial^{k-r}}{\partial\tau^{k-r}}\right]^m . \tag{A4}$$

Identification of powers of s in Eqs. (A3) and (A4) yields Eq. (A1) for the vanishing powers, and the lowest-order nonvanishing powers yield the following useful commutator identities:

$m=1$,

$$s:\quad\left[\frac{\partial^k}{\partial\tau^k},\tau\right]=k\frac{\partial^{k-1}}{\partial\tau^{k-1}}\ ,$$

$$s^2:\quad\left[\frac{\partial^k}{\partial\tau^k},\tau^2\right]=2k\tau\frac{\partial^{k-1}}{\partial\tau^{k-1}}+k(k-1)\frac{\partial^{k-2}}{\partial\tau^{k-2}}\ , \tag{A5}$$

$m=2$,

$$s^2:\quad\left[\frac{\partial^k}{\partial\tau^k},\tau^2\right]_2=2k^2\frac{\partial^{2k-2}}{\partial\tau^{2k-2}}\ .$$

The following commutator results will also prove useful:

(i) If $H=H_0+H_1$, then

$$[H,\tau^2]_2=[H_0,\tau^2]_2+[H_1,\tau^2]_2+2[H_0,[H_1,\tau^2]]\ . \tag{A6}$$

(ii)

$$\left[\frac{\partial^2}{\partial\tau^2},\left[\frac{\partial^3}{\partial\tau^3},\tau^2\right]\right]=12\frac{\partial^3}{\partial\tau^3}\ . \tag{A7}$$

[1]D. Marcuse, Appl. Opt. **20**, 3573 (1981).

[2]D. Anderson, M. Lisak, and P. Anderson, Opt. Lett. **10**, 134 (1985).

[3]F. Koyama and Y. Suematsu, IEEE J. Quantum Electron. **QE-21**, 292 (1985).

[4]K. Tayima and K. Washio, Opt. Lett. **10**, 460 (1985).

[5]K. Petermann, Electron. Lett. **21**, 1143 (1985).

[6]G. P. Agrawal and M. J. Potasek, Opt. Lett. **11**, 318 (1986).

[7]D. Anderson and M. Lisak, Opt. Lett. **11**, 569 (1986).

[8]D. Anderson and J. Askne, Proc. IEEE **62**, 1518 (1974).

[9]M. J. Potasek, G. P. Agrawal, and S. C. Pinault, J. Opt. Soc. Am. B **3**, 205 (1986).