Steady-state analysis of strongly colored multiplicative noise in a dye laser

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A new Langevin equation for dye-laser fluctuations below and near threshold is used to obtain the intensity distribution function in closed form. The effect of strongly colored multiplicative pump noise is incorporated by means of an ansatz of Hanggi *et al.* [Phys. Rev. A 32, 695 (1985)] for an effective diffusion coefficient at steady states. Excellent agreement with the measurements of Lett, Short, and Mandel [Phys. Rev. Lett. 52, 341 (1984)] is obtained.

I. INTRODUCTION

Recent work on dye-laser noise has emphasized the simultaneous presence of additive quantum noise and multiplicative pump noise. Moreover, it has become apparent that the pump noise is not white but colored with a relatively long correlation time. Two experimental approaches have been used to characterize the noise parameters. Steady-state measurements of the normalized variance of the intensity fluctuations and the intensity autocorrelation function,[1–4] and the first-passage-time technique have been used[5,6] to determine these parameters.

These measurements have produced refinements in the theoretical description of laser noise in terms of an augmented semiclassical laser model. The time evolution of the complex laser field is described by a Langevin equation containing both additive (spontaneous emission) noise[7–10] and multiplicative (pump) noise.[11–18] When additive white noise and strongly colored pump noise are simultaneously present it is necessary to perform numerical Monte Carlo simulations in order to deduce the predictions of the theory for comparison with measurements.[4–6] In this paper we will show that a detailed analysis of the steady-state measurements is possible which avoids time consuming simulations. This is possible because (a) we obtain a novel Langevin equation solely in terms of the laser intensity, which we prove fully describes the behavior of the intensity below, near, and somewhat above threshold, and (b) for steady state we invoke and justify the ansatz of Hanggi *et al.*[19] for treating Langevin equations containing strongly colored noise.

Lett, Short, and Mandel[4] observed a peak in the normalized variance of the intensity fluctuations as a function of mean intensity. They used the Monte Carlo procedure of Sancho *et al.*[14] as applied to the laser problem by Dixit and Sahni,[20] to fit their measurements. We have used our Langevin equation to obtain the steady-state probability distribution for the intensity with which we have also fit their data. This procedure is far simpler to implement than the Monte Carlo simulations. The agreement obtained among measurements, simulations, and our theory strongly supports both the Langevin intensity equation and the ansatz of Hanggi *et al.*[19] for colored noise in steady-state situations.

II. A NOVEL LANGEVIN EQUATION FOR LASER INTENSITY FLUCTUATIONS

The Langevin equation for the complex laser field E, as obtained from third-order Lamb theory, is[7–10]

$$\frac{dE}{dt}=a_0E-A\mid E\mid^2E+\tilde{q}(t)+\tilde{p}(t)E\ , \tag{1}$$

in which a_0 is the net gain, A represents saturation, $\tilde{q}(t)$ represents spontaneous emission noise, and $\tilde{p}(t)$ represents pump noise. The noise terms are assumed to be Gaussian and are characterized by their first two moments

$$\langle \tilde{q}_i(t)\rangle=0\ , \tag{2}$$

$$\langle \tilde{q}_i(t)\tilde{q}_j(s)\rangle=\delta_{ij}R\delta(t-s)\ , \tag{3}$$

$$\langle \tilde{p}_i(t)\rangle=0\ , \tag{4}$$

$$\langle \tilde{p}_i(t)\tilde{p}_j(s)\rangle=\delta_{ij}\frac{D}{2\tau}\exp\left[-\frac{\mid t-s\mid}{\tau}\right]\to\delta_{ij}D\delta(t-s)\ \text{ as }\tau\to 0\ , \tag{5}$$

in which the subscripts represent the real and imaginary components $(i,j=1,2)$, R is the additive noise strength, D is the multiplicative noise strength, and τ is the colored noise correlation time. In the limit of vanishing τ, the pump noise becomes white. In the case of purely multiplicative noise, the intensity $I=\mid E\mid^2$ satisfies

$$\frac{dI}{dt}=2a_0I-2AI^2+[\tilde{p}(t)+\tilde{p}^*(t)]I\ , \tag{6}$$

in which the multiplicative noise is clearly real. When additive noise is also present several researchers have studied an equation of the form

$$\frac{d}{dt}I=2a_0I-2AI^2+[\tilde{p}(t)+\tilde{p}^*(t)]I+[\tilde{q}(t)+\tilde{q}^*(t)]\sqrt{I}\ . \tag{7}$$

However, the correct equation in this case is

$$\frac{d}{dt}I=2a_0I-2AI^2+R+2\tilde{p}_r(t)I+2\tilde{q}_r(t)\sqrt{I}\ , \tag{8}$$

in which $\tilde{p}_r$ and $\tilde{q}_r$ are real noise terms and R is the strength parameter in (3). This equation was obtained as follows: from (1), the Fokker-Planck equation for the multiplicative white noise case in polar coordinate variables ($E = ye^{i\phi}$) is[6]

$$\frac{\partial}{\partial t}P(y,\phi,t) = -\frac{\partial}{\partial y}\left[\left[a_0 y - Ay^3 + \frac{R}{2y} + \frac{Dy}{2}\right]P(y,\phi,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[(R+Dy^2)P(y,\phi,t)] + \frac{1}{2}\left[D + \frac{R}{y^2}\right]\frac{\partial^2}{\partial\phi^2}P(y,\phi,t) . \tag{9}$$

Integration over ϕ produces the Fokker-Planck equation for the reduced distribution function $Q(y,t)$:

$$\frac{\partial}{\partial t}Q(y,t) = -\frac{\partial}{\partial y}\left[\left[a_0 y - Ay^3 + \frac{R}{2y} + \frac{Dy}{2}\right]Q(y,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[(R+Dy^2)Q(y,t)] . \tag{10}$$

It is remarkable that in this particular case this exact separation of the variables is obtained. Corresponding with this Fokker-Planck equation is the Langevin equation[21]

$$\frac{d}{dt}y = a_0 y - Ay^3 + \frac{R}{2y} + \tilde{q}_r(t) + \tilde{p}_r(t)y . \tag{11}$$

Using $I \equiv y^2$, we obtain (8) in which the R term of (8) results from the $(R/2y)$ term in (11). While it may initially seem that this term will produce only modest changes in the predictions of the theory, we show that it has a significant effect near and below threshold. This region is of special interest in the measurements of Lett *et al.*[4]

We note that in the case of purely additive noise, the presence of the $(R/2y)$ term in (11) was already implicit in the analysis of Arecchi *et al.*[22] This term played an important role in the recent first-passage-time analysis of laser fluctuations with both additive and multiplicative noise.[6]

III. STEADY-STATE FEATURES OF LASER INTENSITY FLUCTUATIONS

Comparison of experiment and theory involves determination of the mean intensity $\langle I \rangle$, and its normalized variance $\lambda(0) = \langle(\Delta I)^2\rangle/\langle I\rangle^2$, as functions of the net gain parameter a_0. In order to facilitate this comparison it is conventional to scale the laser field and time appropriately,[7] to obtain the scaled equation

$$\frac{d}{dt}E = aE - |E|^2 E + \tilde{q}(t) + \tilde{p}(t)E , \tag{12}$$

in which the correlation strength of the additive noise term has been scaled to $R = 2$. In addition, the scaled version of A is unity and the scaled value of a_0 is a. Our parameter D corresponds with $2Q$ in the paper of Lett *et al.* Since it is difficult to experimentally determine a, Lett *et al.* have plotted $\lambda(0)$ versus $\langle I \rangle$ where $\langle I \rangle$ is given in arbitrary units.

In the case of purely additive noise, the theory predicts a monotone curve for $\lambda(0)$ versus a. Measurements on the He-Ne laser, in which pump fluctuations are extremely small, have confirmed this prediction.[7–10] Far below threshold, $\lambda(0)$ is nearly unity; it decreases in a monotone fashion to extremely small values far above threshold. In contrast, for the dye laser, the observations of Lett *et al.* display a prominent peak with $\lambda(0) \gg 1$. Together with the earlier measurements of intensity correlation functions, these observations have indicated the necessity for colored pump noise in the laser equations. This means that in Eq. (8) we need to consider the effect of using colored noise for $\tilde{p}(t)$.

With colored noise for $\tilde{p}(t)$ in (8) the equation is mathematically intractable, and a solution for the steady-state distribution cannot be obtained in closed form. Lett *et al.* performed Monte Carlo simulations of Eq. (12) for comparison with experiment. In this paper we instead use the steady-state distribution function obtained from an effective Fokker-Planck equation for colored pump noise. This distribution function differs from the distribution function for the white-noise case in that an effective diffusion constant D' replaces the white-noise diffusion constant D in accord with the ansatz of Hanggi *et al.*[19]

$$D' = \frac{D}{1 + 2\tau A\langle I \rangle} . \tag{13}$$

A justification and discussion of the domain of validity for this ansatz is presented in the Appendix.

IV. COMPARISON OF MEASUREMENTS, SIMULATIONS, AND THEORY

The steady-state distribution for the multiplicative white-noise case is obtained from (8) using the parameter scaling introduced in (12). The corresponding Fokker-Planck equation in this case is

$$\frac{\partial}{\partial t}P(I,t) = -\frac{\partial}{\partial I}[(2a_0 I - 2AI^2 + R)P(I,t)] + 2R\frac{\partial}{\partial I}\sqrt{I}\frac{\partial}{\partial I}\sqrt{I}P(I,t) + 2D\frac{\partial}{\partial I}I\frac{\partial}{\partial I}IP(I,t) . \tag{14}$$

Its steady-state distribution P_s is

$$P_s(I) = N(R + DI)^{\gamma}\exp\left[-\frac{A}{D}I\right] , \tag{15}$$

in which

$$\gamma = \frac{a_0 - D}{D} + \frac{AR}{D^2} , \tag{16}$$

and N is the normalization factor. It should be noted that $\langle I \rangle$ and $\lambda(0)$, as functions of a_0, can be expressed in terms of the incomplete gamma function.

In contrast, the steady-state distribution obtained for Eq. (7) is

$$\bar{P}_s(I)=\bar{N}\frac{1}{\sqrt{I}}(R+DI)^{\gamma+1/2}\exp\left[-\frac{A}{D}I\right], \tag{17}$$

and was previously obtained by Schenzle and Brand.[11]

In Fig. 1 we plot $\langle I \rangle$ versus a (we have converted the parameters to their scaled values $R=2$, $A=1$, and $a_0 \rightarrow a$ for both distributions). In Fig. 2 we reproduce the measured data of Lett *et al.* and their Monte Carlo simulation curve. For their simulations they used the parameter values $Q=300$ and $\Gamma=5$ which correspond to the values $D=600$ and $\tau=0.2$ in our notation. Remember that their $\langle I \rangle$ units are arbitrary, while for us D and τ predetermine $\langle I \rangle$ absolutely; no scale factor has been used. In addition we plot the results implied by the distributions in (15) and (17). These curves were obtained by numerical integration, a far less time consuming procedure then the Monte Carlo simulations. The parameter values used were (in scaled form) $D=150$ and $\tau=1.0$. For these values the curves do not match the curve of Lett *et al.* However, as we discuss below, application of the ansatz of Hanggi *et al.*[19] improves comparison dramatically. This is a consequence of the effect of colored noise and such good agreement cannot be obtained without its use.

Implementation of the ansatz is not trivial because the value of D' depends upon that of $\langle I \rangle$ which in turn is determined from the D' value. This requires that we use an iterative numerical procedure to locate self-consistent

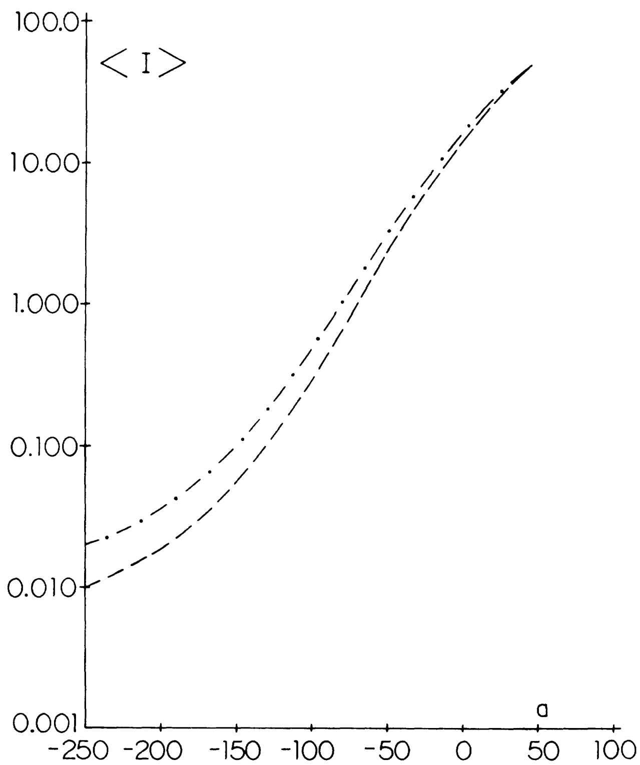

FIG. 1. Variation of the mean intensity $\langle I \rangle$ with a, the pump parameter. —·—·—·, corresponding to Eq. (15); — — —, corresponding to Eq. (17). $D=150$ for both curves.

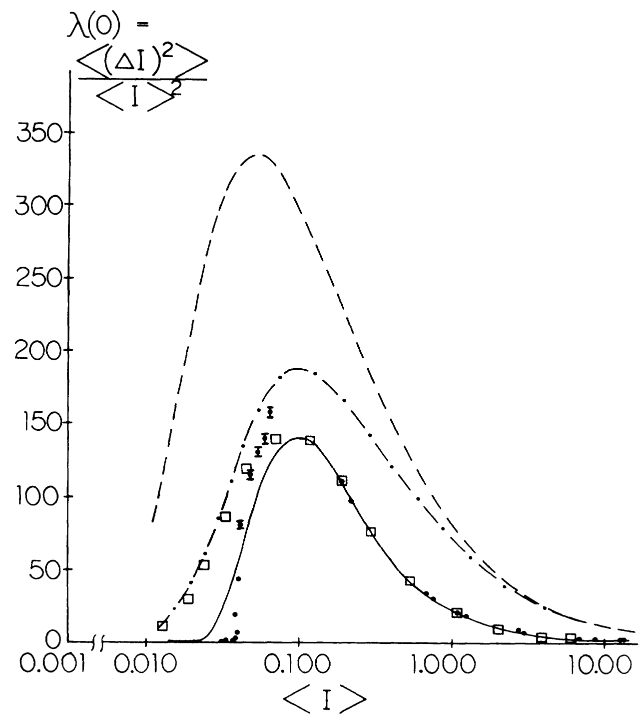

FIG. 2. The normalized variance $\lambda(0)$ vs the mean intensity $\langle I \rangle$. —·—·—·, from Eq. (15), $D=150$; — — —, from Eq. (17), $D=150$; ——, computer simulations by Lett *et al.* (Ref. 4); the experimental data is that of Lett *et al.*; □, modification of the curve from Eq. (15) when the ansatz of Hanggi *et al.* (Ref. 19) is implemented with $D=150$ and $\tau=1.0$.

values. For most points of the curve, the iteration converges rapidly to very stable fixed points. A certain range of the curve consists of unstable fixed points; however, we are still able to determine the self-consistent values by a simple numerical procedure. The results from the application of the ansatz are also shown in Figs. 1 and 2.

While this comparison clearly shows the virtues of both the new intensity equation (8) and the ansatz of Hanggi *et al.*[19] [Eq. (13)] for steady-state phenomena, we have found that for first-passage-time analysis of the transient dynamics of laser intensity it is still necessary to perform Monte Carlo simulations of the laser field equation (1). Time-dependent, transient distribution functions for the intensity or for first passage times which account for the colored nature of the pump noise do not exist in closed form. Approximate treatments which adequately describe strongly colored noise are also not available. Monte Carlo simulations of the laser field equation (1) work well, whereas such simulations of Eq. (8) require an extremely small step size for accurate implementation at small intensities. This feature is perhaps not obvious in (8) but it is clearly the result of the small denominator in the equivalent R term in Eq. (11). Nevertheless, Eq. (8) may be conveniently used to determine correlation functions of the intensity at steady state by Monte Carlo simulations. Finally, for transient phenomena, such as first-passage-time dynamics, the ansatz of Hanggi *et al.*[19] may not be valid (see Appendix).

V. DISCUSSION

While our fit to experiment and simulation is seen to be generally excellent, there is a clear difference at the very low mean intensity and of the curves. We conjecture that both the measurements and the simulations are subject to a possible artifact which only manifests itself at very low mean intensities. This possibility results from the relatively long times it takes for fluctuations to occur in the pump noise which produce sizable intensity spikes. These spikes would contribute to enhanced $\lambda(0)$ values. Sampling times may have to be increased significantly more than was done in either the measurements or simulations.

Even though our parameter values ($D=150$, $\tau=1.0$) yield an excellent fit, they differ from those obtained by Lett *et al.* ($D=600$, $\tau=0.2$). In each case, a definitive best fit has not been sought. This feature results in part from the use of arbitrary $\langle I\rangle$ units in the steady-state analysis.[4] A comparison of steady-state and transient analysis may be necessary to provide a unified picture with no arbitrary scale factors.

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APPENDIX

The ansatz of Hanggi *et al.*[19] arises in the study of Langevin equations with colored noise, e.g.,

$$\frac{dx}{dt}=W(x)+g(x)\tilde{f}(t)\ , \tag{A1}$$

in which both $W(x)$ and $g(x)$ may be nonlinear, and the noise term $\tilde{f}(t)$ is Gaussian with zero mean and correlation given by

$$\langle\tilde{f}(t)\tilde{f}(s)\rangle=\frac{D}{\tau}\exp\left[-\frac{|t-s|}{\tau}\right]. \tag{A2}$$

In two recent papers using a functional calculus approach[23] we showed that the distribution function for $x(t)$ satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}P(y,t)=&-\frac{\partial}{\partial y}[W(y)P(y,t)]\\&+\frac{D}{\tau}\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}g(y)\int_0^t ds'\exp\left[-\frac{|t-s'|}{\tau}\right]\int\!\!\!\int\mathcal{D}f\,P[f]\delta[y-x(t)]\\&\times\exp\left[\int_{s'}^t ds\left[W'(x(s))-\frac{g'(x(s))}{g(x(s))}W(x(s))\right]\right],\end{aligned} \tag{A3}$$

in which $P[f]$ is the Gaussian functional for the noise and W' and g' denote the x derivatives of W and g, respectively. In those two papers it was shown that for sufficiently small τ, (A3) goes over into an effective Fokker-Planck equation for weakly colored noise.

Here, however, we want to consider strongly colored noise in the steady-state regime. Thus, even though τ is large, we argue that at steady state the last exponential in (A3) is approximated by

$$\begin{aligned}&\exp\left[\int_{s'}^t ds\left[W'(x(s))-\frac{g'(x(s))}{g(x(s))}W(x(s))\right]\right]\\&\qquad\cong\exp\left[(t-s')\left[W'(x_s)-\frac{g'(x_s)}{g(x_s)}W(x_s)\right]\right],\end{aligned} \tag{A4}$$

where x_s denotes the steady-state value of $\langle x(t)\rangle$.[24] Inserting this approximation into (A3) and performing the remaining s' integral as well as the path integral over $P[f]$ yields the effective Fokker-Planck equation for strongly colored noise at steady state:

$$\begin{aligned}\frac{\partial}{\partial t}P(y,t)=&-\frac{\partial}{\partial y}[W(y)P(y,t)]\\&+\frac{D}{1-\tau\left[W'(x_s)-\frac{g'(x_s)}{g(x_s)}W(x_s)\right]}\\&\times\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}g(y)P(y,t)\ ,\end{aligned} \tag{A5}$$

in which the effective diffusion constant D' is given by

$$D'=\frac{D}{1-\tau\left[W'(x_s)-\frac{g'(x_s)}{g(x_s)}W(x_s)\right]}\ , \tag{A6}$$

which is precisely the ansatz of Hanggi *et al.*

For the laser problem, the colored multiplicative noise term is $2\tilde{p}(t)I=\sqrt{2}\tilde{f}(t)I$, so that we have $W(I)=2a_0I-2AI^2$ and $g(I)=\sqrt{2}I$. Thus,

$$D'=\frac{D}{1+2\tau A\langle I\rangle}\ , \tag{A7}$$

where $\langle I \rangle$ is the steady-state value.

We note here that the ansatz of Hanggi *et al.* has also been invoked for transient processes such as in first-passage-time analysis.[23] In this context there is reason to doubt its validity as was recently shown by Masoliver *et al.*[25] The argument in favor of the ansatz here is clearly restricted to the steady state.

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