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Numerical evaluation of path-integral solutions to Fokker-Planck equations. III. Time and functionally dependent coefficients

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A path-integral solution to truly nonlinear Fokker-Planck equations is derived. Such equations exhibit in the drift and diffusion coefficients a functional dependence on the distribution function. This type of implicit time dependence is shown to introduce terms into the propagator function of the exact same order in the time step τ, as does an explicit time dependence if the functional dependence is sufficiently smooth. A standard discrete lattice formulation of the path integral is then used to reproduce the appropriate, truly nonlinear Fokker-Planck equation. This discrete formulation provides a basis for an efficient numerical algorithm and is applied with excellent results to several example problems where exact solutions can be calculated.

I. INTRODUCTION

The dynamics of many nonequilibrium systems can often be described in terms of a Fokker-Planck equation,[1,2]

$$\frac{\partial}{\partial t}P(q,t)=-\frac{\partial}{\partial q_i}K^i(q)P(q,t)+\frac{1}{2}\frac{\partial^2}{\partial q_i\,\partial q_j}Q^{ij}(q)P(q,t)\ , \tag{1}$$

where the q_i $(i=1,2,\ldots,n)$ represent a set of state variables characterizing the system and $P(q,t)$ is the probability that the system is in a particular state at time t. The physical processes which determine the evolution of the system are then contained entirely in the drift force $K^i(q)$ and the diffusion tensor $Q^{ij}(q)$ together with a consistent set of initial and boundary conditions. A great deal of attention in the literature has been focused on Fokker-Planck coefficients which are nonlinearly dependent on q but independent of time.[1-3] However, in general, the forces driving the evolution of the system may be better described by time-dependent expressions. If the forces are such that they are determined by circumstances external to the system, this time dependence appears explicitly in the Fokker-Planck coefficients. Examples of such systems include lasers in which the input power is varied, fluid systems where the Rayleigh number can be changed, and chemical systems where the reaction rates depend on an externally controllable parameter such as temperature or reagent concentration.[3]

On the other hand, in many systems the driving forces depend on the state of the system itself. In a Fokker-Planck description, the drift vector and diffusion tensor then become functionals of the probability distribution, causing Eq. (1) to become a nonlinear partial differential equation. In this paper, we call these equations with a nonlinear dependence on the distribution function, $P(q,t)$, "truly nonlinear Fokker-Planck equations." This contrasts with the more conventional usage in the literature of "nonlinear Fokker-Planck equation," which refers to a nonlinear dependence of the drift and diffusion coefficients on the state variables q_i. The solution to such an equation, in contrast to that of a linear differential equation, does not allow for the choice of an arbitrary normalization. In other words, if $P(q,t)$ is a solution, an arbitrary constant times P is not. Hence, although $\int P(q,t)dq$ is independent of time (assuming so-called natural boundary conditions), it is not in general unity, implying that $P(q,t)$ cannot be interpreted as a probability distribution in the strictest sense of the term. However, as is more apparent after the development of Sec. II and elaborated on in the Appendix, the Markov property can be satisfied by such truly nonlinear Fokker-Planck equations if the functional dependence of the coefficients on $P(q,t)$ is evaluated only at the current time.

Several specific examples from the literature reveal two common classes of functional dependence. Systems such as nonequilibrium plasmas where collisions between like particles are important,[4] or nucleation processes where the density of nucleation sites determine the supply of nucleating particles[5] exhibit a dependence on the weighted averages of certain characteristic functions (such as the moments). This type of an integral dependence is encountered in mean-field theories of many-body systems where the driving forces acting on an individual member are determined by the long-range interaction (in state space) of the remaining population of similar members and would typically become important in regimes that could not be considered as dilute. Another class of functional dependence arises in systems which interact among themselves in a local manner (again in state space). An intuitive but simple example of this type would be the diffusion of a temperature profile through a medium possessing a temperature-dependent diffusion coefficient. The interaction of a photon gas with an electron gas via Compton scattering with the inclusion of induced phe-

nomena is another interesting but as yet unsolved example.[6,7] Also, a variety of prey-predator systems involving coupled nonlinear diffusion equations have been considered in the literature.[8]

Path integrals have been shown by many authors[9–15] to provide formal solutions to Fokker-Planck equations with time-independent as well as explicitly time-dependent coefficients.[16,17] The authors have previously developed a numerical technique[18] based on the discrete equivalent, the path sum, and have extended it to include the effect of external boundary conditions.[19] In this paper, we will show that the path sum can also provide solutions to Fokker-Planck equations with functionally dependent coefficients. This includes, but is not restricted to the two previously mentioned classes of nonlinearities. In Sec. II, we propose a propagator function based on the prepoint discretization and prove that in the limit as the time step τ goes to zero, the path sum is a correct solution in the one-dimensional case. We do this by two different, albeit not independent arguments. The first recognizes that the functional dependence of the Fokker-Planck coefficients introduces terms into the propagator of the exact same order in τ as does an explicit time dependence. These terms are of order $O(\tau^2)$ and higher and therefore vanish in the path integral limit. The second argument uses this propagator function to reproduce the appropriate functionally dependent Fokker-Planck equation by a well-established discrete lattice formulation, demonstrating that the path sum is equivalent to the truly nonlinear Fokker-Planck equation in the limit $\tau \rightarrow 0$.

In Sec. III, we use a generalization of our previously developed numerical path sum algorithm to solve several specific one-dimensional examples as further evidence to the validity of our arguments. In the Appendix, an appropriate notation for the conditional probability is introduced in order to facilitate a brief discussion of the Markov property of such truly nonlinear systems.

II. PATH INTEGRAL SOLUTION

In order to keep notations to a minimum and streamline the development of the path integral, we will restrict the discussion to one-dimensional stochastic processes under natural boundary conditions for the remainder of the paper. The Fokker-Planck equation then becomes

$$\frac{\partial}{\partial t}P(q,t)=-\frac{\partial}{\partial q}K(q,t,[P])P(q,t)+\frac{1}{2}\frac{\partial^2}{\partial q^2}Q(q,t,[P])P(q,t)\ , \tag{2}$$

where the functional dependence of K and Q on the distribution function at time t is indicated by $[P]$.

The construction of a path integral solution to the Fokker-Planck equation (2) can proceed in several alternative ways. The traditional and most straightforward method approximates the continuous process by a discrete set of infinitesimally spaced lattice points in space and time. Although this does not lead to a uniquely covariant form, the various solutions obtained have all been shown to be equally valid solutions to Fokker-Planck equations with time-independent coefficients. Given certain assumptions regarding the nature of the functional and/or time dependence of the drift and diffusion coefficients, we will show the same to again be true. The second stage of the discrete lattice formulation of the path integral is the determination of a Green's function solution to Eq. (2) valid to order $O(\tau^{3/2})$, where τ is a short interval time. Often called the short-time propagator, $G(q,q_0,t_0,\tau)$ in this context, this function is interpreted as the transition probability from a state q_0 at time t_0 to another state q at time $t_0+\tau$ and hence must be normalized to unity. The simplest choice is obtained by assuming a Gaussian approximation to this transition probability. Then we may write

$$G(q,q_0,t_0,\tau)=\left[\frac{1}{2\pi\sigma^2}\right]^{1/2}\exp\left[-\frac{(q-\langle q\rangle)^2}{2\sigma^2}\right], \tag{3}$$

where $\langle q\rangle$ is the mean value and σ^2 is the variance which define such a Gaussian function. As has been previously shown by many authors, in order for Eq. (3) to be a solution of Eq. (2) to order $O(\tau^{3/2})$ for a general choice of time-independent Fokker-Planck coefficients, this mean value must be written as

$$\langle q\rangle = q_0+K(q_0)\tau+\cdots \tag{4a}$$

and this variance as[9–17]

$$\sigma^2=Q(q_0)\tau+\cdots\ . \tag{4b}$$

It is important to note that the Fokker-Planck coefficients are evaluated at the prepoint q_0. Although we do not consider them here, other equivalent non-Gaussian forms, valid to the same order in τ can also be developed, leading to evaluation points involving the midpoint, postpoint, or linear combinations of points.[11] Terms of order higher than τ in Eqs. (4) are easily shown to be unnecessary. Consider adding to Eq. (4a) the term $A\tau^2$, where A is an arbitrary constant. Then the prepoint propagator becomes

$$G(q,q_0,t_0,\tau)=\left[\frac{1}{2\pi Q(q_0)\tau}\right]^{1/2}\times\exp\left[-\frac{[q-q_0-\tau K(q_0,t_0)-A\tau^2]^2}{2Q(q_0)\tau}\right] \tag{5}$$

or

$$G(q,q_0,t_0,\tau)=\left[\frac{1}{2\pi Q(q_0)\tau}\right]^{1/2}\exp\left[-\frac{[q-q_0-\tau K(q_0t)]^2}{2Q(q_0)\tau}\right]\times\exp\left[\frac{2\tau A[q-q_0-\tau K(q_0,t_0)-A\tau^2]}{2Q(q_0)}\right]. \tag{6}$$

An expansion of the second exponential yields

$$\exp\left[\frac{A\tau}{Q}\left[q-q_0-\tau K-\frac{\tau^2 A}{2}\right]\right]=1+\frac{\tau A}{Q}\left[q-q_0-\tau K-\frac{\tau^2 A}{2}\right]+\cdots\ . \tag{7}$$

Recalling[15] that $|q-q_0| \sim \tau^{1/2}$, we see that under the requirement that $A=O(\tau^{-\alpha})$ with $\alpha \leq 0$, no additional contributions of order less than $O(\tau^{3/2})$ are made to G.

We may now generalize these arguments to processes with time and/or functionally dependent coefficients by writing as a short time propagator:

$$G(q,q_0,t_0,\tau)=\left[\frac{1}{2\pi\tau Q(q_0,t')}\right]^{1/2} \times \exp\left[-\frac{[q-q_0-\tau K(q_0,t')]^2}{2\tau Q(q_0,t')}\right], \quad (8)$$

where $t'=t_0+\eta\tau$ with $0\leq\eta\leq 1$ and the functional dependence of K and Q on the distribution function $P(q_0,t')$ is implied. Since t' is a time intermediate between the initial and final times we may expand

$$K(q_0,t')=K(q_0,t_0)+\eta\tau\frac{\partial K}{\partial t_0}+\eta\tau\int\frac{\delta K}{\delta P}\frac{\partial P(q_1,t_0)}{\partial t}dq_1 , \quad (9)$$

where $\delta K/\delta P$ is the functional derivative. A similar expansion may be written for $Q(q_0,t')$. If

$$\frac{\partial K}{\partial t_0}+\int\frac{\delta K}{\delta P}\frac{\partial P(q_1,t_0)}{\partial t_0}dq_1=O(\tau^{\beta-1}) \quad (10)$$

holds where $\beta>0$, and a similar condition applies to the time dependence of Q, then, as we have just shown, the time derivatives contribute terms to the propagator of Eq. (8) only of order $O(\tau^{3/2})$ and higher. This observation then suggests that functionally dependent Fokker-Planck coefficients present a case not unlike that of explicitly time-dependent coefficients. Indeed, for the explicitly time-dependent case, Eq. (9) requires changes in the coefficients to be on a time scale longer than the path integral time step. Since τ is to be taken to zero in the path-integral limit, such a restriction precludes only truly random variables as coefficients. On the other hand, the time scale for changes in functionally dependent coefficients is determined by $\partial P/\partial t$, the time scale for changes to the system itself which, of course, must be longer than τ for a path integral description to be valid. Such time-scale issues arise again in the numerical implementation of the path integral when one must choose a small but nonzero time step.

Up to this point, we have only shown that functionally dependent coefficients contribute higher-order terms in the propagator of Eq. (8) which vanish as $\tau\rightarrow 0$. To prove the validity of the path integral solution for such cases, we must also show that in the limit that $\tau\rightarrow 0$, the appropriate Fokker-Planck equation can be reproduced from the short-time propagator. The starting point is the discrete path sum for a single iteration

$$P(q,t_0+\tau)=\int G(q,q_0,t_0,\tau)P(q_0,t_0)dq_0 . \quad (11)$$

Subtracting $P(q,t_0)$ from Eq. (11), multiplying by an analytic but otherwise arbitrary function $R(q)$, and integrating over all q results in

$$\int dq\,R(q)[P(q,t_0+\tau)-P(q,t_0)] = \int dq\int dq_0\,R(q)G(q,q_0,t_0,\tau)P(q_0,t_0) - \int dq\,R(q)P(q,t_0) . \quad (12)$$

We may then expand the arbitrary function about q_0 such that

$$R(q)=R(q_0)+\sum_{n=1}^{\infty}\frac{(q-q_0)^n}{n!}\frac{\partial^n R(q_0)}{\partial q_0^n}+\cdots . \quad (13)$$

Next, divide Eq. (12) by τ, substitute the expansion (13) and take the limit $\tau\rightarrow 0$ resulting in

$$\int dq\,R(q)\frac{\partial P(q,t_0)}{\partial t}=\lim_{\tau\rightarrow 0}\left[\frac{1}{\tau}\int dq\int dq_0\,R(q_0)G(q,q_0,t_0,\tau)P(q_0,t_0)\right]-\lim_{\tau\rightarrow 0}\left[\frac{1}{\tau}\int dq\,R(q)P(q,t_0)\right] +\lim_{\tau\rightarrow 0}\left[\sum_{n=1}^{\infty}\frac{1}{\tau n!}\int dq_0\left[\int dq\,(q-q_0)^n G(q,q_0,t_0,\tau)\right]P(q_0,t_0)\frac{\partial^n R(q_0)}{\partial q_0^n}\right] . \quad (14)$$

The first two terms cancel in the limit if G is properly normalized. By applying integration by parts repeatedly, the differentiation in the third term may be shifted resulting in

$$\int dq\,R(q)\frac{\partial P(q,t_0)}{\partial t}=\lim_{\tau\rightarrow 0}\left[\sum_{n=1}^{\infty}\int dq_0\,R(q_0)\frac{\partial^n}{\partial q_0^n}[A_n(q_0,\tau)P(q_0,t_0)]\right] , \quad (15)$$

where

$$A_n(q_0,\tau)=\frac{(-1)^n}{n!}\frac{1}{\tau}\int dq\,(q-q_0)^n G(q,q_0,t_0,\tau) \quad (16)$$

and assuming natural boundary conditions on $P(q,t)$. Utilizing the propagator of Eq. (8), the first few integrals are to order $O(\tau^2)$,

$$\begin{aligned} A_1(q_0,\tau)&=-K(q_0,t') ,\\ A_2(q_0,\tau)&=\tfrac{1}{2}Q(q_0,t')+\tfrac{1}{2}\tau K^2(q_0,t') ,\\ A_3(q_0,\tau)&=-\tfrac{1}{6}[3\tau Q(q_0,t')K(q_0,t')+\tau^2K^3(q_0,t')] ,\\ A_3(q_0,\tau)&=\tfrac{1}{24}[3\tau Q(q_0,t')+6\tau^2Q(q_0,t')K^2(q_0t') +\tau^3K^3(q_0,t')] , \end{aligned} \quad (17)$$

and all other A_n are of order $O(\tau^2)$ or higher. Since $t'=t_0+\eta\tau$, we may expand $K(q_0,t')$ and $Q(q_0,t')$ as in Eq. (9). If the time and functional dependences satisfy Eq. (10) then

$$\lim_{\tau\to 0} A_n(q_0,\tau)=\begin{cases} -K(q_0,t_0), & n=1 \\ \frac{1}{2}Q(q_0,t_0'), & n=2 \\ 0, & n\geq 3 \ . \end{cases} \tag{18}$$

Since the integration limits are the same on both sides of Eq. (15), upon interchange of q and q_0, the Fokker-Planck equation (2) is reproduced from Eq. (11). For the finite time relaxation, the path integral solution is given by

$$P(q,t)=\lim_{\substack{N\to\infty \\ \tau\to 0}} \prod_{i=1}^{N} \int \cdots \int G(q_{i+1},q_i,t_i,\tau)P(q_0,t_0)dq_i \ , \tag{19}$$

where the product $N\tau=t-t_0$ is fixed. Since the errors introduced by assuming the propagator Eq. (8) are of order τ^2 for each iteration and, importantly, the product $N\tau^2$ goes to zero, the long-time solution to the Fokker-Planck equation is reproduced in this limit.

Although the notation in Eq. (19) suppresses it, the short-time propagator $G(q_{i+1},q_i,t_i,\tau)$ is functionally dependent on $P(q,t_i)$. Thus Eq. (19) is not the discrete version of a functional integral in the conventional sense since each integral over q_i must be evaluated in the order of increasing i to keep all quantities known. Rather, Eq. (19) is related to a conventional functional integral in the same way as a nonlinear Volterra integral equation is related to the conventional Riemannian integral. However, in a numerical scheme, such issues are not of concern as long as the correct ordering is preserved.

III. NUMERICAL EXAMPLES

A previously developed numerical technique based on the path integral provides further evidence as to the validity of the ideas presented in Sec. II. The general algorithm is based on the path sum, Eq. (19), and a numerical simulation of the path integral limit, i.e., a small time step τ, and a large number of iterations, N. Most of the details of the numerical method for time-independent coefficients remain unchanged when applied to problems involving time-dependent coefficients in one dimension. However, there are two notable differences. First, since the nodal point spacing Δq_i was previously required to be equal to $[Q(q_i)\tau]^{1/2}$, it must now be time dependent. The second and more important point is that because of this fact and also because the propagator function is now time dependent, the transition matrix must be recalculated after every iteration. This adds significantly to the required computing time but otherwise is not an impediment. More details of the numerical technique are available elsewhere.[18,19]

A. Ornstein-Uhlenbeck process with explicitly time-dependent coefficients

Van Kampen has given the solution in one dimension of the Fokker-Planck equation with the arbitrarily time-dependent coefficients

$$K(q,t)=qA(t) \tag{20}$$

and

$$Q(q,t)=B(t) \tag{21}$$

as a Gaussian function of the form of Eq. (5) but with the mean value given as[20,21]

$$\langle q\rangle = q_0 e^{-S(t)} \tag{22}$$

and the covariance as

$$\sigma^2(t)=e^{-2S(t)}\int_0^t B(t')e^{2S(t')}dt' \ , \tag{23}$$

where

$$S(t)=-\int_0^t A(t')dt' \ . \tag{24}$$

As a particular example, which can be solved analytically for purposes of comparison, consider

$$A(t)=at$$

and

$$B(t)=bt+c \ , \tag{25}$$

where a, b, and c are constant. Then the mean and covariance become

$$\langle q\rangle = q_0 e^{-at^2/2}$$

and

$$\sigma^2=\frac{c}{2}\sqrt{\pi/a}\,e^{at^2}\mathrm{erf}(\sqrt{a}\,t)-\frac{b}{2a}(1-e^{at^2}) \ . \tag{26}$$

Figure 1 shows this solution (solid circles) compared with the numerical path sum solution (solid line). The parameters a, b, and c were chosen such that the Fokker-Planck coefficients varied on a time scale faster than the distribution function. Agreement is quite good although the numerical solution is slightly lower near the peak value and consequently slightly broader in the "wings." This type and amount of error is typical of this procedure.[18]

B. Ornstein-Uhlenbeck process with functionally dependent coefficients

Analytic solutions to Fokker-Planck equations whose coefficients depend on the probability distribution itself are generally difficult to find in the time-dependent case. However, in many instances, one may reduce the steady-state equation to a solvable first-order nonlinear differential equation. As an example, consider the following generalized Ornstein-Uhlenbeck process:

$$\begin{aligned} K(q,t)&=-[a-bP(q,t)]q \ , \\ Q(q,t)&=Q \ , \end{aligned} \tag{27}$$

where a, b, and Q are constants. The stationary solution is given by

$$\frac{d}{dq}\left[\frac{Q}{2}\frac{dP(q)}{dq}+[a-bP(q)]qP(q)\right]=0 \ . \tag{28}$$

The expression in the large parentheses is simply the net probability current at the point q, and is obviously

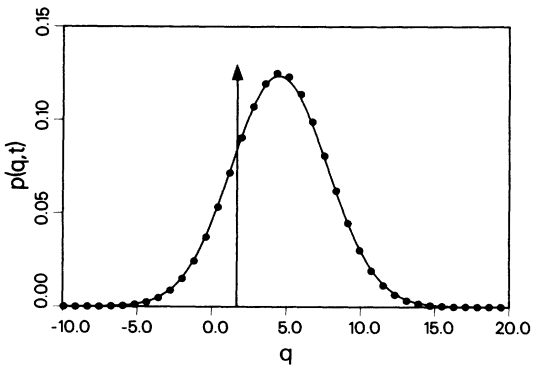

FIG. 1. Comparison of the exact solution and the numerical path sum solution using a trapezoidal representation of the probability function for the Fokker-Planck equation with $K(q,t)=aqt$ and $Q(q,t)=bt+c$ at time $t=2.0$ with $a=0.05$, $b=1.0$, $c=0.02$. The time step τ was chosen at 0.05. The initial condition is a delta function represented by the vertical arrow. The exact solution is illustrated by the closed circles for comparison.

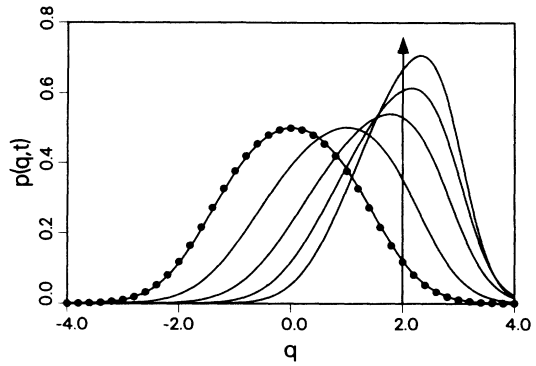

FIG. 2. Comparison of the exact steady-state solution and the numerical path sum solution of the process defined by $K(q,t)=-[a-bP(q,t)]q$ and $Q(q,t)=2$ with $a=b=1$ at various times. The time step τ was chosen as 0.02. The exact solution is illustrated by the solid circles for comparison.

constant. However, for natural boundaries, the current is zero there, and hence must be zero everywhere in the interior, resulting in the nonlinear first-order ordinary differential equation

$$\frac{Q}{2}\frac{dP}{dq}+(a-bP)qP=0 . \tag{29}$$

This is of the simplest type of Riccati equations, with the solution given as[22]

$$P(q)=\frac{1}{\frac{b}{a}+Ce^{aq^2}/Q} , \tag{30}$$

where C is positive constant determined by the normalization condition

$$\int_{-\infty}^{\infty} P(q)dq=N_0 . \tag{31}$$

Note that in the limit as $b\to 0$, Eq. (30) reduces to a stationary Gaussian solution. By choosing $a=b$ and $C=1$ we may calculate this integral, finding

$$N_0=\sqrt{Q/a}\int_0^{\infty}\frac{x^{-1/2}}{e^x+1}dx=\sqrt{Q/a}\,(1-\sqrt{2})\Gamma(\tfrac{1}{2})\zeta(\tfrac{1}{2}) ,$$

where ζ is the Riemann-zeta function and Γ is the gamma function. Evaluating these functions yields

$$N_0=1.15162\sqrt{Q/a} .$$

In Fig. 2, we show the numerical path-integral solution to Eq. (27) obtained by choosing this normalization and the values $a=b=1$ and $Q=2$. The initial condition was chosen to be a δ function at $q=2.0$. The calculated steady-state solution is seen to be in close agreement with the analytic steady-state solution, Eq. (30), which is represented by the solid circles. In addition, interesting transient behavior is also revealed. The peak and mean values of the distribution function are initially displaced towards the right before relaxing to the left to form the symmetric steady-state distribution.

This transient is explained by realizing that wherever $P(q,t)>a/b$ the drift force is repulsive. Initially, this is indeed the case as the δ function relaxes to a narrow and tall distribution. However, as time progresses, diffusion flattens the distribution and reverses this condition. Hence, the nonlinearity of Eq. (27) produced this qualitatively different feature of reversing the direction of the average drift.

C. Ornstein-Uhlenbeck process with coefficients dependent on the integral $\int P(q,t)dq$

Consider as an example of a process where $P(q,t)$ appears in an integral dependence the following generalized Ornstein-Uhlenbeck process with

$$K(q,t)=-aq+b\int_0^q P(q',t)dq'$$

and

$$Q(q)=2D , \tag{32}$$

where a, b, c, and D are constants. Then the steady-state solution under natural boundary conditions is determined by

$$P(q)\left[-aq+b\int_0^q P(q')dq'\right]-D\frac{dP}{dq}=0 . \tag{33}$$

This integro-differential equation may be differentiated to yield the nonlinear second-order ordinary differential equation

$$\frac{d^2P}{dq^2}=\frac{1}{P(q)}\left[\frac{dP}{dq}\right]^2+\frac{b}{D}[P(q)]^2+\frac{a}{D}P(q) . \tag{34}$$

Given a specified value of $P(q)$ at a particular value of q, this equation may be numerically integrated by standard Runge-Kutta techniques. In Fig. 3, we show this steady-state solution (again as solid circles) for the values

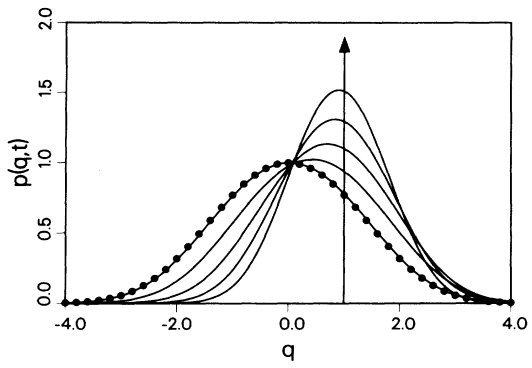

FIG. 3. Comparison of the numerical path sum solution at various times and a direct numerical integration of the steady-state equation for the process given by $K(q,t)=-aq+b\int_0^q P(q',t)dq'$ and $Q=2$ with $a=1$ and $b=0.5$. The time step was chosen as 0.02. The exact solution is again illustrated by the solid circles.

$a=1$, $b=0.5$, $D=1$, $P(0)=1$. The normalization of this distribution was then calculated and used in the numerical path integral algorithm to solve the time-dependent equation assuming an initial δ function at $q=1.0$. This solution, represented by the solid curves of Fig. 3, agrees well in the long-time limit with the numerical solution of Eq. (34). The functional dependence of the drift coefficient of Eq. (31) shows how arbitrary boundary conditions do not always lead to well posed problems. For instance, the solution to Eq. (34) with $P(0)=1$ would diverge as $q\rightarrow\pm\infty$ for $b>a$. For $b=a$, the solution is $P(q)=1$ everywhere. Only for $b<a$ does the solution satisfy the natural boundary condition $P(\pm\infty)=0$. Hence only this choice can be consistent with the zero current requirement implicitly assumed in Eq. (33). The other possibilities obviously lead to contradiction.

IV. DISCUSSION

We have presented a new methodology of integrating Fokker-Planck equations with functionally dependent coefficients. The derivation and final form are reminiscent of a traditional discrete lattice formulation of the path integral solution to conventional Fokker-Planck equations. The proof presented relied on the argument that a sufficiently smooth functional dependence of the Fokker-Planck coefficients on the distribution function as defined by Eq. (10) introduced terms of a similar nature to those of an explicit time dependence when constructing approximate solutions valid to order $O(\tau^2)$. Then following traditional arguments, the appropriate truly nonlinear Fokker-Planck equation may be reproduced.

The fact that the formal integration of such a large class of truly nonlinear Fokker-Planck equations is so strikingly similar to that of the linear version is a remarkable one. In general, nonlinear equations are not covered by the powerful existence and uniqueness theorems so often applied to linear equations.[22,23] In addition, many the basic properties associated with conventional linear Fokker-Planck equations are not as easily connected to the truly nonlinear Fokker-Planck equations of this discussion. As previously mentioned, the distribution function cannot always be described as a probability density since the normalization $\int P(q,t)dq$ is not always arbitrary. This can sometimes lead to ill posed problems. For example, in certain problems,[6,7] a unique stationary state exists where the normalization is a fixed quantity. Since the Fokker-Planck operator in the absence of external boundary conditions or volume sources is conservative, the initial conditions cannot be arbitrary but must be subject to a normalization consistent with the stationary state. Boundary conditions must also be considered with care as Sec. III C illustrated.

An important question arises with regard to the Markov property of stochastic processes described by truly nonlinear Fokker-Planck equation. We find that the Markov property is not altered by the appearance of the distribution function in the Fokker-Planck coefficients. As discussed in the Appendix, this follows from the observation that these nonlinearities do not affect the knowledge about the system in question prior to the initial conditions.

The generalization of Eq. (19) to a functional integral form for such truly nonlinear Fokker-Planck equations is straightforward. Although the nonlinearities make such an expression even less amenable to further analytical reduction, the conceptual value of the path integral is greatly enhanced. From a numerical standpoint, however, Eq. (19) still provides the basis for an efficient algorithm which we have shown to accurately reproduce several known solutions.

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APPENDIX: THE MARKOV PROPERTY FOR FUNCTIONALLY DEPENDENT FOKKER-PLANCK EQUATIONS

By requiring that the Fokker-Planck coefficients depend on the distribution function only at the current time t, one would intuitively expect the Markov property to be valid for systems described by such nonlinear equations. However, because the distribution function is not necessarily a probability density distribution, a precise mathematical definition of the Markov property is not as straightforward as for the linear case. Consider, for example, a distribution function $W(q,t)$ which is not normalized to unity (which could be the solution to such a truly nonlinear Fokker-Planck equation). Then one can denote the probability of finding a hypothetical "particle" in the state q at the time t given that the system as a whole is described by W as

$$P(q,t,[W])=\frac{W(q,t)}{\int W(q,t)}\ , \tag{A1}$$

where we have denoted a functional dependence on this distribution as $[W]$. In a certain sense, this is a type of conditional probability relating a particular state to the status of all the other states. Although clearly in Eq. (A1) this functional dependence is only on the normalization, the concept is useful when defining the more conventional conditional probability. This is then the probability of a particle being in the state q_2 at time t_2 given that it was in the state q_1, at the earlier time t_1, and that the entire system was described at that time by W_1. Symbolically, we may write this as $P(q_2,t_2\,|\,q_1,t_1,[W_1])$. However, since W is not a random variable, a unique choice of W_1 implies a unique W_2, the state of the system at time t_2. Hence one might equally as well write $P(q_2,t_2,[W_2]\,|\,q_1,t_1,[W_1])$ without restricting this conditional probability in any way. This suggests two levels of conditioning: the condition on the particle in question and the condition on the system as a whole. For a longer sequence of times ($t_n>\cdots>t_k\cdots>t_1$) we can generalize the notation to

$$P(q_n,t_n;q_{n-1},t_{n-1};\ldots;q_{k+1},t_{k+1}\,|\,q_k,t_k;q_{k-1},t_{k-1};\ldots;q_1,t_1,[W_1])\ ,$$

which means the probability of a particle following the particular path denoted by the sequence to the left of the slash mark given that it has already followed the path denoted by the sequence to the right of the slash mark and that the entire system was described by W_1 at the initial time t_1. As before, since W is not a random variable, we may rewrite the past sequence in this expression resulting in

$$P(q_n,t_n;q_{n-1},t_{n-1};\ldots;q_{k+1},t_{k+1}\,|\,q_k,t_k,[W_k];q_{k-1},t_{k-1},[W_{k-1}];\ldots;q_1,t_1,[W_1])\ ,$$

without changing the meaning of this conditional probability. The reader will note that we could similarly rewrite the future sequence but that is not necessary for our present purpose. Instead we may now define the Markov property as follows: the probability of a particle following the future path shall be determined solely by the most recent conditions.[2] This includes both the condition on the particle in question as well as the condition on the system as a whole. This conditional probability then takes the form

$$P(q_n,t_n;q_{n-1},t_{n-1};\ldots;q_{k+1},t_{k+1}\,|\,q_k,t_k,[W_k])\ .$$

We have arrived at this result by assuming that a unique choice of W_1 uniquely determines W_k. One might ask if the converse is true, namely does a unique knowledge of W_k uniquely determine W_1. If so, the Markov property would be violated as the distant past would also directly influence the future sequence. However, such an inconsistency does not arise when one considers that the Fokker-Planck operator is of parabolic character. As is well known, parabolic differential equations are asymmetric when subject to a time reversal. For nonnegative diffusion coefficients, causality conditions prohibit solutions for negative time increments. Hence although W_1 uniquely determines W_k, a precise knowledge of W_k alone reveals nothing about W_1 regardless of the nonlinearity of the Fokker-Planck operator. Therefore an expression like the one above contains no hidden information at times prior to t_k and is entirely consistent with the Markov property.

[1]H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).
[2]C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1983).
[3]H. Haken, *Synergetics, An Introduction*, 3rd ed. (Springer-Verlag, Berlin, 1983).
[4]F. L. Hinton, in *Handbook of Plasma Physics*, edited by M. N. Rosenbluth and R. Z. Sagdeev (North-Holland, Amsterdam, 1983), Vol. 1.
[5]M. F. Wehner and W. G. Wolfer, Philos. Mag. A **52**, 189 (1985).
[6]J. S. Chang and G. Cooper, J. Comput. Phys. **6**, 1 (1970).
[7]D. L. Youngs, Atomic Weapons Research Establishment Report No. 44/92/28, 1976 (unpublished).
[8]*Partial Differential Equations and Dynamical Systems*, edited by W. E. Fitzgibbon III (Pitman, London, 1984).
[9]H. Haken, Z. Phys. B **24**, 321 (1976).
[10]R. Graham, Z. Phys. B **26**, 281 (1976).
[11]C. Wissel, Z. Phys. B **35**, 185 (1979).
[12]H. Dekker, Physica A **85**, 363 (1976).
[13]H. Grabert and M. S. Green, Phys. Rev. A **19**, 1747 (1979).
[14]H. Dekker, Phys. Rev. A **19**, 2102 (1979).
[15]F. Langouche, D. Roekaerts, and E. Tirapegui, Nuovo Cimento B **53**, 135 (1979).
[16]T. Morita and H. Hara, Physica A **125**, 607 (1984).
[17]F. Langouche, D. Roekaerts, and E. Tirapegui, Physica A **101**, 301 (1980).
[18]M. F. Wehner and W. G. Wolfer, Phys. Rev. A **27**, 2663 (1983).
[19]M. F. Wehner and W. G. Wolfer, Phys. Rev. A **28**, 3003 (1983).
[20]N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[21]N. G. Van Kampen, Can. J. Phys. **39**, 551 (1961).
[22]H. L. Davis, *Introduction to Nonlinear Integral and Differential Equations* (Dover, New York, 1962).
[23]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1985), Part 1.