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Uncertainty relations for light waves and the concept of photons

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A Lorentz-covariant localization for light waves is presented. The unitary representation for the electromagnetic four-potential is constructed for a monochromatic light wave. A model for covariant superposition is constructed for light waves with different frequencies. It is therefore possible to construct a wave function for light waves carrying a covariant probability interpretation. It is shown that the time-energy uncertainty relation $(\Delta t)(\Delta\omega)\simeq 1$ for light waves is a Lorentz-invariant relation. The connection between photons and localized light waves is examined critically.

I. INTRODUCTION

For light waves, the Fourier relation $(\Delta t)(\Delta\omega)$ was known before the present form of quantum mechanics was formulated.[1,2] However, the question of whether this is a Lorentz-invariant relation has not yet been properly addressed.[3] Let us consider a blinking traffic light. A stationary observer will insist on $(\Delta t)(\Delta\omega)\simeq 1$. An observer in an automobile moving toward the light will see the same blinking light. This observer will also insist on $(\Delta t^*)(\Delta\omega^*)\simeq 1$ on his or her coordinate system. However, these observers may not agree with each other because neither Δt nor $\Delta\omega$ is a Lorentz-invariant variable. The product of two noninvariant quantities does not always lead to an invariant quantity.

Let us assume that the automobile is moving in the negative z direction with velocity parameter β. Since both t and ω are the timelike components of four-vectors $(\mathbf{x},t)$ and $(\mathbf{k},\omega)$, respectively, a Lorentz boost along the z direction will lead to new variables

$$t^*=(t+\beta z)/(1-\beta^2)^{1/2},\quad \omega^*=(\omega+\beta k)/(1-\beta^2)^{1/2}\ , \tag{1}$$

where the light wave is assumed to travel along the z axis with $k=\omega$. In the above transformation, the light wave is boosted along the positive z direction. If the light passes through the point $z=0$ at $t=0$, then $t=z$ on the light front, and the transformations of Eq. (1) become

$$t^*=\left[\frac{1+\beta}{1-\beta}\right]^{1/2} t,\quad \omega^*=\left[\frac{1+\beta}{1-\beta}\right]^{1/2}\omega\ . \tag{2}$$

These equations will formally lead us to

$$(\Delta t^*)(\Delta\omega^*)=\left[\frac{1+\beta}{1-\beta}\right](\Delta t)(\Delta\omega)\ , \tag{3}$$

which indicates that the time-energy uncertainty relation is not Lorentz invariant, and that Planck's constant depends on the Lorentz frame in which the measurement is taken. This is not correct, and we need a better understanding of the transformation properties of Δt and $\Delta\omega$.

This problem is related to another fundamental problem in physics. We are tempted to say that the abovementioned Fourier relation is a time-energy uncertainty relation. However, in order that it be an uncertainty relation, the wave function for the light wave should carry a probability interpretation. This problem has a stormy history and is commonly known as the photon localization problem.[4–6] The traditional way of stating this problem is that there is no self-adjoint position operator for massless particles including photons.

In spite of this theoretical difficulty, it is becoming increasingly clear that single photons can be localized by detectors in laboratories. The question then is whether it is possible to construct the language of the photon localization which we observe through oscilloscopes. Throughout the history of this localization problem, the main issue has been and still is how to construct localized photon wave functions consistent with special relativity.

We do not propose to solve this difficult problem in this paper. We shall instead approach this problem by constructing covariant localized light waves and comparing them with photon field operators. First, we construct a unitary representation for Lorentz transformations for a monochromatic light wave. It is shown then that a Lorentz-covariant superposition of light waves is possible for different frequencies. After constructing the covariant light wave, we shall observe that there is a gap between the concept of photons and that of localized waves. From the physical point of view, this gap is not significant. However, there is a definite distinction between the mathematics of photons and that of light waves.

In approaching the problem of the covariant superposition of light waves, we shall start with the uncertainty relation applicable to nonrelativistic quantum mechanics. We shall then borrow the techniques from the covariant harmonic oscillator model which provides a quantification of the uncertainty relations observed in the relativistic quark model.[7–10] Since the uncertainty principle is universal, the uncertainty relation applicable to one specific physical example should be consistent with those for other physical phenomena.

In Sec. II, we start with the motion of free-particle wave packets in the Schrödinger picture of nonrelativistic quantum mechanics. For localized light waves, there is no difficulty in giving a probability interpretation if Lorentz boosts are not considered. It is pointed out that the basic problem for light waves is how to make the probability interpretation Lorentz covariant.

In Sec. III, we discuss Lorentz-transformation properties of the four-vector representation for photons. Section IV examines the time-energy uncertainty relation applicable to the relativistic quark model. It is noted that the uncertainty relation applicable to the time separation variable between the quarks confined in a hadron can be combined covariantly with the position-momentum uncertainty relation.

In Sec. V, based on the lessons we learned in Secs. II, III, and IV, we construct a model of Lorentz-covariant localization of light waves. Finally, in Sec. VI, we examine closely how the concept of photons can emerge from localized light waves.

II. LIGHT WAVES AND WAVE PACKETS IN NONRELATIVISTIC QUANTUM MECHANICS

In this paper we are concerned with the possibility of constructing wave functions with quantum probability interpretation for relativistic massless particles. The natural starting point for tackling this problem is a free-particle wave packet in nonrelativistic quantum mechanics which we pretend to understand. Let us write down the time-dependent Schrödinger equation for a free particle moving in the z direction:

$$i\frac{\partial}{\partial t}\psi(z,t)=\frac{-1}{2m}\left[\frac{\partial}{\partial z}\right]^2\psi(z,t)\ . \tag{4}$$

The Hamiltonian commutes with the momentum operator. If the momentum is sharply defined, the solution of the above differential equation is

$$\psi(z,t)=\exp[i(pz-p^2t/2m)]\ . \tag{5}$$

If the momentum is not sharply defined, we have to take the linear superposition

$$\psi(z,t)=\int g(k)\exp[i(kz-k^2t/2m)]dk\ . \tag{6}$$

The width of the wave function becomes wider as the time variable increases, as is illustrated in Fig. 1(a). This is known as the wave-packet spread.

Let us study transformation properties of this wave function. Rotation and translation properties are trivial. In order to study the boost property within the framework of the Galilei kinematics, let us imagine an observer moving in the negative z direction. To this observer, the

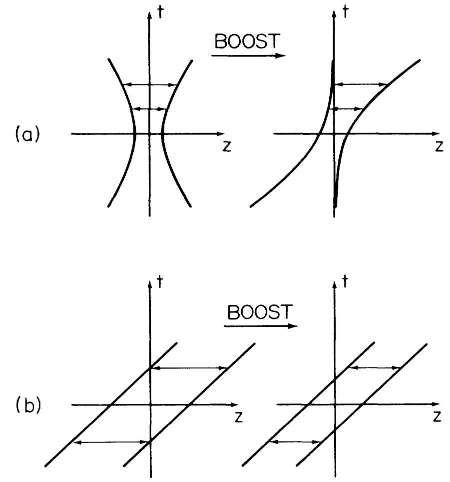

FIG. 1. The time dependence of the wave packets. (a) shows the spread of the Schrödinger wave function. (b) shows the behavior of the light wave which does not spread. However, for an observer moving in the negative z direction, the Schrödinger wave function is boosted according to the Galilei transformation. The quantum probability interpretation is consistent with the Galilean world. On the other hand, the light wave carries the burden of being consistent with the Lorentzian world.

center of the wave function moves along the positive z direction as is specified also in Fig. 1(b). The transformed wave function takes the form

$$\psi_v(z,t)=\exp[-im(vz-\tfrac{1}{2}v^2t)]\times\int g(k-mv)e^{i(kz-k^2t/2m)}dk\ , \tag{7}$$

where v is the boost velocity. This expression is different from the usual expression in textbooks by an exponential factor in front of the integral sign. The origin of this phase factor is well understood.[11]

In nonrelativistic quantum mechanics, $\psi(z,t)$ has a probability interpretation, and there is no difficulty in giving an interpretation for the transformed wave function in spite of the above-mentioned phase factor. The basic unsolved problem is whether the probabilistic interpretation can be extended into the Lorentzian regime. This has been a fundamental unsolved problem for decades, and we do not propose to solve all the problems in this paper. A reasonable starting point for approaching this problem is to see whether a covariant probability interpretation can be given to light waves.

For light waves, let us start with the usual expression

$$f(z,t)=\left[\frac{1}{2\pi}\right]^{1/2}\int g(k)e^{i(kz-\omega t)}dk\ . \tag{8}$$

Unlike the case of the Schrödinger wave, ω is equal to k, and there is no spread of wave packet. The velocity of propagation is always that of light, as is illustrated in Fig. 1(b). We might therefore be led to think that the problem for light waves is simpler than that for nonrelativistic Schrödinger waves. This is not the case for the following reasons.

(1) We like to have a quantal wave function for light waves. However, it is not clear which component of the Maxwell wave should be identified with the quantal wave whose absolute square gives a probability distribution. Should this be the electric or magnetic field, or should it be the four-potential?

(2) The expression given in Eq. (8) is valid in a given Lorentz frame. What form does this equation take for an observer in a different frame?

(3) Even if we are able to construct localized light waves, does this solve the photon localization problem?

(4) The photon has spin 1 either parallel or antiparallel to its momentum. The photon also has gauge degrees of freedom. How are these related to the above-mentioned problems?

Indeed, the burden on Eq. (8) is the Lorentz covariance. It is not difficult to carry out a spectral analysis and therefore to give a probability interpretation for the expression of Eq. (8) in a given Lorentz frame. However, this interpretation has to be covariant. This is precisely the problem we are addressing in the present paper.

III. UNITARY REPRESENTATION FOR FOUR-POTENTIALS

One of the difficulties in dealing with the photon problem has been that the electromagnetic four-potential could not be identified with a *unitary* irreducible representation of the Poincaré group.[12–15] The purpose of this section is to resolve this problem. In Ref. 15 we studied unitary transformations associated with Lorentz boosts along the direction perpendicular to the momentum. In this section we shall deal with the most general case of boosting along an arbitrary direction.

Let us consider a monochromatic light wave traveling along the z axis with four-momentum p. The four-potential takes the form

$$A^{\mu}(x)=A^{\mu}e^{i\omega(z-t)}\ , \tag{9}$$

with

$$A^{\mu}=(A_1,A_2,A_3,A_0)\ .$$

We use the metric convention $x^{\mu}=(x,y,z,t)$. The momentum four-vector in this convention is

$$p^{\mu}=(0,0,\omega,\omega)\ . \tag{10}$$

Among many possible forms of the gauge-dependent four-vector A^{μ}, we are interested in the eigenstates of the helicity operator

$$S_3=\begin{bmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{11}$$

The four-vectors satisfying this condition are

$$A^{\mu}_{\pm}=(1,\pm i,0,0)\ , \tag{12}$$

where the subscripts + and − specify the positive and negative helicity states, respectively. These are commonly called the photon *polarization vectors.*

In order that the four-vector be a helicity state, it is essential that the time-like and longitudinal components vanish:

$$A_3=A_0=0\ . \tag{13}$$

This condition is equivalent to the combined effect of the Lorentz condition

$$\frac{\partial}{\partial x^{\mu}}[A^{\mu}(x)]=0\ , \tag{14}$$

and the transversality condition

$$\nabla\cdot\mathbf{A}(x)=0\ . \tag{15}$$

As before, we call this combined condition the *helicity gauge.*[15]

While the Lorentz condition of Eq. (14) is Lorentz invariant, the transversality condition of Eq. (15) is not. However, both conditions are invariant under rotations and under boosts along the direction of momentum. We call these helicity preserving transformations. The boost along an arbitrary direction is illustrated in Fig. 2. This is not a helicity preserving transformation. However, according to Ref. 15, we can express this in terms of helicity preserving transformations preceded by a gauge transformation.

Let us consider in detail a boost along the arbitrary direction specified in Fig. 2. This boost will transform the momentum to $\mathbf{p}'$, as is illustrated in Fig. 2;

$$p'^{\mu}=B_{\phi}(\eta)p^{\mu}\ . \tag{16}$$

However, this is not the only way in which p can be transformed to p'. We can boost p along the z direction and rotate it around the y axis as is shown in Fig. 2. The application of the transformation $[R(\theta)B_z(\xi)]$ on the four-momentum gives the same effect as that of the application of $B_{\phi}(\eta)$. Indeed, the matrix

$$D(\eta)=[B_{\phi}(\eta)]^{-1}R(\theta)B_z(\xi) \tag{17}$$

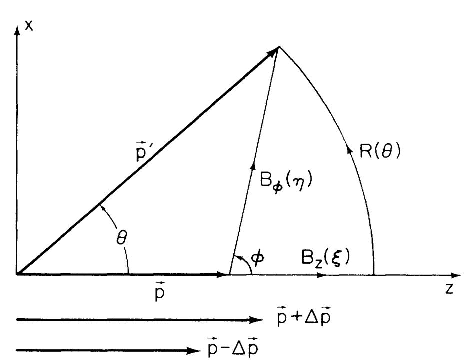

FIG. 2. Lorentz boost along an arbitrary direction of the light wave. The four-momentum can be boosted either directly by B_{ϕ} or through the rotation R_y preceded by B_z along the z direction. These operations produce two different four-vectors when applied to the polarization vector. However, they are connected by a gauge transformation.

leaves the four-momentum invariant, and is therefore an element of the $E(2)$-like little group for photons.

The effect of the above D matrix on the polarization vectors $A^{\mu}_{\pm}$ has been calculated in the Appendix, and the result is

$$D(\eta)A^{\mu}_{\pm}=A^{\mu}_{\pm}+(p^{\mu}/\omega)\mu(\eta,\theta)\ , \tag{18}$$

where

$$u(\eta,\theta)=\frac{-2\left[\sin\frac{\theta}{2}\cosh\frac{\eta}{2}\right]}{\left[\cos\frac{\theta}{2}\right]\left[\cos\frac{\eta}{2}\right]+\left[\left[\cos\frac{\theta}{2}\cosh\frac{\eta}{2}\right]^2-1\right]^{1/2}}\ .$$

Thus $D(\eta)$ applied to the polarization vector results in the addition of a term which is proportional to the four-momentum. $D(\eta)$ therefore performs a gauge transformation on $A^{\mu}_{\pm}$.[14,15]

With this preparation, let us boost the photon polarization vector

$$\tilde{A}^{\mu}_{\pm}=B_{\phi}(\phi)A^{\mu}_{\pm}\ . \tag{19}$$

The four-vector $\tilde{A}^{\mu}_{\pm}$ satisfies the Lorentz condition $p'^{\mu}\tilde{A}_{\pm\mu}=0$, but its fourth component will not vanish. The four-vector $\tilde{A}^{\mu}_{\pm}$ does not satisfy the helicity condition.

On the other hand, if we boost the four-vector $A^{\mu}_{\pm}$ after performing the gauge transformation $D(\eta)$,

$$\begin{aligned} A'^{\mu}_{\pm}&=B_{\phi}(\eta)A^{\mu}_{\pm}\\ &=B_{\phi}(\eta)\{[B_{\phi}(\eta)]^{-1}R(\theta)B_z(\xi)\}A^{\mu}_{\pm}\\ &=R(\theta)B_z(\xi)A^{\mu}_{\pm}\ . \end{aligned} \tag{20}$$

Since $B_z(\xi)$ leaves $A^{\mu}_{\pm}$ invariant, we arrive at the conclusion that

$$A'^{\mu}_{\pm}=R(\theta)A^{\mu}_{\pm}\ . \tag{21}$$

This means

$$A'^{\mu}_{\pm}=B_x(\eta)D(\eta)A^{\mu}_{\pm}=(\cos\theta,\pm i,-\sin\theta,0)\ , \tag{22}$$

which satisfies the helicity condition

$$A'^{0}_{\pm}=0$$

and (23)

$$\mathbf{p}'\cdot\mathbf{A}'_{\pm}=0\ .$$

The Lorentz boost $B(\eta)$ on $A^{\mu}_{\pm}$ preceded by the gauge transformation $D(\eta)$ leads to the pure rotation $R(\theta)$. This rotation is a finite-dimensional unitary transformation.

The above result indicates, for a monochromatic wave, that all we have to know is how to rotate. If, however, the photon momentum has a distribution, we have to deal with a linear superposition of waves with different momenta. The photon momentum can have both longitudinal and transverse distributions. In this paper we shall *assume* that there is only longitudinal distribution. This, of course, is a limitation of the model we present. However, our apology is limited in view of the fact that laser beams these days can go to the moon and come back after reflection.[16]

With this point in mind, we note first that the above-mentioned unitary transformation preserves the photon polarization. This means that we can drop the polarization index from A^{μ} assuming that the photon has either positive or negative polarization. $A^{\mu}(x)$ can now be replaced by $A(x)$.

Next, the transformation matrices discussed in this section depend only on the direction and the magnitude of the boost but *not* on the photon energy. This is due to the fact that the photon is a massless particle.[17] Indeed, the matrices in Sec. III remain invariant even if ω in Eq. (9) is replaced by a different value. This means that for the superposition of two different frequency states,

$$A(x)=A_1e^{i\omega_1(z-t)}+A_2e^{i\omega_2(z-t)}\ , \tag{24}$$

a Lorentz boost along an arbitrary direction results in a rotation preceded by a boost along the z direction. Since neither the rotation nor the boost along the z axis changes the magnitude of $A_i(i=1,2)$, the quantity

$$|A|^2=|A_1|^2+|A_2|^2 \tag{25}$$

remains invariant under the Lorentz transformation. This result can be generalized to the superposition of many different frequencies:

$$A(x)=\sum_k A_ke^{i\omega_k(z-t)}\ , \tag{26}$$

with

$$|A|^2=\sum_k|A_k|^2\ . \tag{27}$$

The norm $|A|^2$ remains invariant under the Lorentz transformation in the sense that it is invariant under rotations and is invariant under the boost along the z direction.

Can this sum be transformed into an integral form of Eq. (8)? From the physical point of view, the answer should be yes. Mathematically, the problem is how to construct a Lorentz-invariant integral measure. It is not difficult to see that the norm of Eq. (27) remains invariant under rotations, which perform unitary transformations on the system. The problem is how to construct a measure invariant under the boost along the z direction.

For this purpose, we shall borrow the techniques developed for the covariant harmonic-oscillator formalism which has been very effective in explaining the basic relativistic features in the quark model,[10,18–20] and which enables us to combine covariantly Heisenberg's position-momentum uncertainty relations and the c-number time-energy uncertainty relation.[3,9,21]

IV. LOCALIZATION PROBLEMS IN THE RELATIVISTIC QUARK MODEL

We shall discuss in this section the aspects of the covariant harmonic-oscillator formalism which are useful in converting the sum of Eq. (26) into an integral form while preserving the invariance of the sum of Eq. (27) under the Lorentz boost along the z direction. The covariant oscillator formalism has been extensively discussed in the literature. What we need is here to review its property under the boost along the z direction.

Let us use x_a and x_b for the space-time coordinates of the two quarks bound together in a hadron. Then it is more convenient to use the four-vectors

$$X=(x_a+x_b)/2,\quad x=(x_a-x_b)/2\sqrt{2}\ . \tag{28}$$

It is not difficult to write down the uncertainty relations for space-time separation variables and to define the region within which the hadronic wave function is localized in the Lorentz frame where the hadron is at rest. However, the crucial question is how these uncertainty relations appear to an observer in the laboratory frame.

The uncertainty principle applicable to the space-time separation of quarks in the hadronic rest frame is the same as the currently accepted form based on the existing theories and observations. The usual Heisenberg uncertainty relation holds for each of the three spatial coordinates. The time-separation variable is a c number and therefore does not cause quantum excitations.[3,21] The question is then whether this peculiar time-energy uncertainty can be combined with Heisenberg's position-momentum uncertainty relation to a covariant form.[9] Such a combination is possible within the framework of the covariant harmonic-oscillator formalism which can explain the basic hadronic features including the mass spectrum,[18] proton form factors,[19] parton picture,[9,22] and jet phenomenon.[20]

We assume throughout this section that the hadron moves along the z axis, and ignore and x and y coordinates which are not affected by the boost along the z direction. If we consider only the ground-state wave function, then the localization dictated by the uncertainty relations associated with both space and time separation variables will lead to a distribution centered around the origin in the hadron-rest frame with the z^* and t^* variables. The ground-state harmonic-oscillator wave function takes the form

$$\psi(z,t)=(1/\sqrt{\pi})\exp\{-[(z^*)^2+(t^*)^2]/2\}\ . \tag{29}$$

The question then is how this region appears to the laboratory—frame observer, while the coordinates of the two different frames are related by the Lorentz transformation

$$\begin{aligned} z&=(z^*+\beta t^*)/(1-\beta^2)^{1/2}\ ,\\ t&=(t^*+\beta z^*)/(1-\beta^2)^{1/2}\ , \end{aligned} \tag{30}$$

where β is the velocity parameter of the hadron.

In order to approach this problem, let us employ Dirac's form of Lorentz transformation. In his 1949 paper,[23] Dirac introduced the light-cone coordinate system in which the coordinate variables are

$$z_+=(z+t)/\sqrt{2},\quad z_-=(z-t)/\sqrt{2}\ . \tag{31}$$

In terms of these variables, the Lorentz transformation of Eq. (30) takes the form

$$z_+=\left[\frac{1+\beta}{1-\beta}\right]^{1/2} z_+^*,\quad z_-=\left[\frac{1-\beta}{1+\beta}\right]^{1/2} z_-^*\ . \tag{32}$$

z_+ and z_- are called the light-cone variables, and the product z_+z_- remains invariant under the boost:

$$z_+z_-=z_+^*z_-^*\ . \tag{33}$$

In the light-cone coordinate system, the ground-state wave function of Eq. (29) takes the form

$$\psi(z,t)=\frac{1}{\sqrt{\pi}}\exp\left[-\left[\frac{1-\beta}{1+\beta}z_+^2+\frac{1+\beta}{1-\beta}z_-^2\right]\right]\ . \tag{34}$$

This wave function or the probability density is localized in a circular region centered around the origin in the z_+z_- plane when $\beta=0$. As the hadron moves, the region becomes elliptic. This elliptic deformation property is illustrated in Fig. 3.

Let us next consider the momentum-energy wave function. If the quarks have four-momenta p_a and p_b, respec-

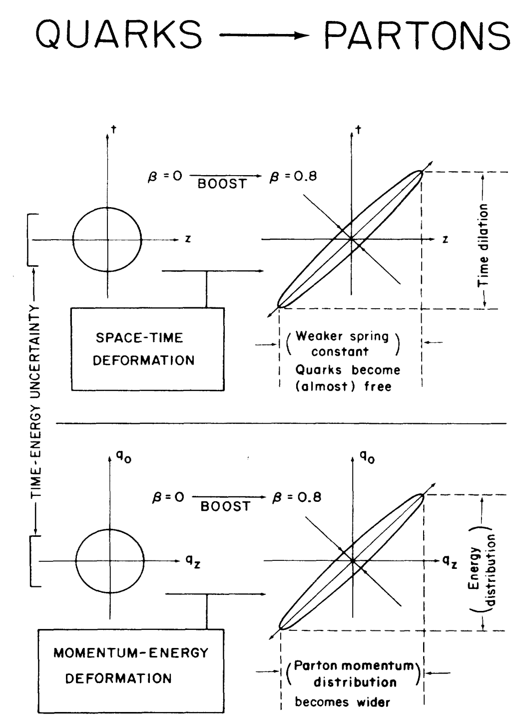

FIG. 3. The Lorentz deformation of a relativistic extended hadron in both the space-time and momentum-energy coordinate systems. Because the Lorentz transformation property of momentum-energy four-vector is the same as that of the space-time four-vector, the Lorentz deformation in the momentum-energy plane is expected to be the same as that in the zt plane. This figure summarizes the content of the earlier paper on the parton picture. (Refs. 9 and 10) in which the hadron, while being a bound state of quarks in its rest frame, appears as a collection of partons to an observer who moves with a speed close to that of light.

tively, then the standard procedure is to introduce P and q

$$P=p_a+p_b, \quad q=\sqrt{2}(p_a-p_b) , \tag{35}$$

where P is the four-momentum of the hadron, and q is the momentum-energy separation between the quarks. We are concerned here with the uncertainty relations between the x variables of Eq. (28) and the above q variables. The momentum-energy wave function is

$$\begin{aligned}\phi(q_z,q_0)&=\left[\frac{1}{2\pi}\right]\int \exp[i(q_z z-q_0 t)]\psi(z,t)dt\,dz\\&=(1/\sqrt{\pi})\exp\{-[(q_z^*)^2+(q_0^*)^2]/2\} ,\end{aligned} \tag{36}$$

where q_z and q_0 are the momentum and energy separation variables, respectively. Their Lorentz transformation property is the same as that for z and t. The form of this wave function is identical to that of the space-time wave function. In terms of the light-cone variables:

$$q_+=(q_z+q_0)/\sqrt{2}, \quad q_-=(q_z-q_0)/\sqrt{2} , \tag{37}$$

the momentum wave function of Eq. (36) takes the form

$$\phi(q_z,q_0)=(1/\sqrt{\pi})\exp\left[\frac{1}{2}\left[\frac{1-\beta}{1+\beta}q_+^2+\frac{1+\beta}{1-\beta}q_-^2\right]\right] . \tag{38}$$

The Lorentz deformation of this wave function is also illustrated in Fig. 3.[24]

The basic advantage of using the light-cone variables is that the coordinate system remains orthogonal, and z_+ and z_- do not become linearly mixed when the system is boosted. The Fourier relations between the space-time and momentum-energy coordinates are

$$q_-=-i(\partial/\partial z_+), \quad q_+=-i(\partial/\partial z_-) . \tag{39}$$

This means that the major and minor axes of the momentum-energy coordinates are the Fourier conjugates of the minor and major axes of the space-time coordinates, respectively. Thus we have the following Lorentz-invariant uncertainty relations.[25]

$$\begin{aligned}&(\Delta z_+)(\Delta q_-)=(\Delta z_+^*)(\Delta q_-^*)\simeq 1 ,\\&(\Delta z_-)(\Delta q_+)=(\Delta z_-^*)(\Delta q_+^*)\simeq 1 .\end{aligned} \tag{40}$$

These uncertainty relations are well understood when the hadron is at rest with $\beta=0$. On the other hand, the limit $\beta\to 1$ can teach us many interesting lessons. The connection between this limit and Feynman's original form of the parton model[22] has been discussed repeatedly in the literature.[9,10] As far as the localization of massless particles is concerned, the distribution along one of the light-cone axes becomes so widespread that it loses its localization along the axis.

In Sec. V, we shall "give up" the localization along one of the light-cone axes in order to study photons and light waves. In so doing, we will have the problem of normalizing the wave function by integration. The integration measure dz_+dz_- is a boost-invariant quantity. This means that the normalization integral,

$$\int |\psi(z,t)|^2 dz_+dz_- , \tag{41}$$

is independent of β. However, dz_- or dz_+ alone is not. The integration over z_+ gives the factor $[(1+\beta)/(1-\beta)]^{1/2}$, and this factor is compensated by its inverse $[(1-\beta)/(1+\beta)]^{1/2}$ coming from the z_- integration.

We used in this section the Gaussian form of the wave function purely for convenience. The above reasoning is valid for all forms of distributions having the same space-time boundary condition as that of the Gaussian function. Indeed, if we give up the localization along the z_+ axis, then the integration measure along the z_- axis should be compensated by the contraction or elongation along the z_+ direction. If the system is boosted along the z direction, dz_+ and dz_- are transformed as

$$dz_+\to\left[\frac{1+\beta}{1-\beta}\right]^{1/2}dz_+, \quad dz_-\to\left[\frac{1-\beta}{1+\beta}\right]^{1/2}dz_- . \tag{42}$$

We can give the same reasoning for the momentum-energy measures dq_+ and dq_-. This transformation property will play an important role in constructing localized light waves.

V. COVARIANT LOCALIZATION OF LIGHT WAVES

We discussed in Sec. IV the relativistic quark model in which the overall hadron four-momentum is well defined, and the internal coordinate system has a momentum-energy distribution. In the case of light waves, the frequency or momentum is not sharply defined, but has a distribution. In this case, we can take the average value of the momentum, and the distribution around this average value. We can treat the average momentum like the hadronic momentum, and the distribution around the average value like the internal momentum distribution.

With this point in mind, let us rewrite Eq. (8) as

$$f(z,t)=\left[\frac{1}{2\pi}\right]^{1/2}\int g(k)e^{i(kz-\omega t)}dk . \tag{43}$$

We shall approach this problem using Dirac's light-cone coordinate system discussed in Sec. IV. For convenience, we shall define here the light-cone variables as

$$s=(z+t)/2, \quad u=(z-t) . \tag{44}$$

s and u are different from z_+ and z_- of Eq. (31) by a factor of $\sqrt{2}$. but their Lorentz transformation property remains the same. We shall also define the new momentum variables as

$$k_u=(k+\omega)/2, \quad k_s=(k-\omega) . \tag{45}$$

In the case of light waves, k_s vanishes and k_u becomes k or ω. In terms of the light-cone variables, the expression of Eq. (43) becomes

$$f(u)=\left[\frac{1}{2\pi}\right]^{1/2}\int g(k)e^{iku}dk . \tag{46}$$

For a massive particle, the most convenient Lorentz frame is the frame in which the particle is at rest, as was noted in Sec. IV. For a massless particle, as our study in Sec. III suggests,[26] we can start with a specific Lorentz frame in which the photon momentum has a given magnitude along the z direction. In this Lorentz frame, we assume that the average photon frequency is Ω_0

$$\Omega_0=\frac{1}{N}\int k\,|g_0(k,\Omega_0)|^2 dk\ , \tag{47}$$

with

$$N=\int |g_0(k,\Omega_0)|^2 dk\ . \tag{48}$$

It is important to note that the introduction of this specific Lorentz frame does not cause any loss of generality.[26] We can obtain the most general form by boosting the photon along the z direction. If we use β as the boost parameter, the new photon frequency is

$$\Omega=\left[\frac{1+\beta}{1-\beta}\right]^{1/2}\Omega_0\ , \tag{49}$$

and this frequency should be the average value calculated from the new and most general distribution function $g(k,\Omega)$:

$$\Omega=\frac{1}{N}\int k\,|g(k,\Omega)|^2 dk\ , \tag{50}$$

where N is given in Eq. (48) of $g_0(k,\Omega_0)$. N is a Lorentz-invariant quantity. In order that unitarity be preserved, $g(k,\Omega)$ should satisfy the normalization condition

$$N=\int |g(k,\Omega)|^2 dk\ . \tag{51}$$

The basic problem here is that the integral measure dk is not Lorentz invariant. One form for $g(k,\Omega)$ which meets the invariance requirement is

$$g(k,\Omega)=(1/\Omega)^{1/2}a(k-\Omega)\ , \tag{52}$$

where $a(k-\Omega)$ is a scalar function depending only on $(k-\Omega)$. The $(1/\Omega)^{1/2}$ factor is proportional to $[(1-\beta)/(1+\beta)]^{1/4}$, and assures the Lorentz invariance of the normalization integral. This makes $(1/Q)dk$ a Lorentz-invariant measure.

Let us next consider the left-hand side of Eq. (8). If we insist on the Lorentz invariance of the normalization integral

$$\int |f(u)|^2 du\ , \tag{53}$$

then $(\Omega/\Omega_0)du$ is a Lorentz-invariant measure, and $f(u)$ can take the form

$$f(u)=(\Omega/\Omega_0)^{1/2}A(u)\ , \tag{54}$$

where $A(u)$ is a scalar quantity. The integral form of Eq. (26) is

$$(\Omega/\Omega_0)^{1/2}A(u)=(\tfrac{1}{2}\pi\Omega)^{1/2}\int a(k-\Omega)e^{iku}dk\ , \tag{55}$$

with

$$(\Omega/\Omega_0)\int_{-\infty}^{\infty}|A(u)|^2 du=(1/\Omega)\int_{-\infty}^{\infty}|a(k-\Omega)|^2 dk\ . \tag{56}$$

Indeed, in Eq. (56), we have to multiply and divide the right- and left-hand sides, respectively, by $\sqrt{Q}$. We did this in order to make the system covariant. The net result is that

$$f(u)=\left[\frac{1+\beta}{1-\beta}\right]^{1/4}A(u) \tag{57}$$

and

$$g(k)=\left[\frac{1-\beta}{1+\beta}\right]^{1/4}(1/\Omega_0)^{1/2}a\left[k-\left[\frac{1+\beta}{1-\beta}\right]^{1/2}\Omega_0\right]\ . \tag{58}$$

The velocity parameter β is zero when the average photon frequency is Ω_0.

Let us examine the problem using a concrete form of $g(k,\Omega)$. The covariant oscillator model discussed in Sec. IV suggests the following normalized Gaussian form for the frequency distribution:

$$g(k)=\left[\frac{1}{\pi b}\right]^{1/4}\left[\frac{1}{\Omega_0}\right]^{1/2}\left[\frac{1-\beta}{1+\beta}\right]^{1/4}\times\exp\left[\frac{-1}{2b}(k-\Omega)^2\right]\ , \tag{59}$$

where b is a constant and specifies the width of the distribution. The above form describes the distribution in k or k_u around Ω, and there is no localization in the k_s variables. In view of the discussion given in Sec. IV, it is not difficult to understand the origin of the factor $[(1-\beta)/(1+\beta)]^{1/2}$ in Eq. (59). The space-time wave function $f(u)$ takes the form

$$f(u)=\left[\frac{b}{\pi}\right]^{1/4}\left[\frac{1+\beta}{1-\beta}\right]^{1/4}\exp\left[i\Omega(z-t)-\frac{b}{2}(z-t)^2\right]\ . \tag{60}$$

This function has a distribution along the $u=(z-t)$ axis, but has no localization along the s axis.

Let us go back to the question mentioned in Sec. I. Is the time-frequency uncertainty relation a Lorentz-invariant relation? The wave function of Eq. (59) and Eq. (60) constitute the quantification of the Lorentz-invariant uncertainty relation

$$(\Delta u)(\Delta k)\simeq 1\ . \tag{61}$$

From the definition given in Eq. (45), $\Delta k=\Delta\omega$. From Eq. (44), $\Delta u=-\Delta t$ for a fixed value of z. This relation becomes $\Delta u=\Delta t$ when the symbol Δ means the width of distribution. Thus the time-frequency relation $(\Delta\omega)(\Delta t)$ is a Lorentz-invariant relation.

VI. THE CONCEPT OF PHOTONS

We discussed in this paper Lorentz-covariant wave functions for light waves. It is possible to construct a localized wave function for light waves with a Lorentz-invariant normalization. The mathematics of this procedure is not complicated. We are then led to the question of why the photon localization is so difficult, while it is possible to produce photons in states narrowly confined in space and time.[27]

Let us see how the mathematics for the light-wave localization is different from that of quantum electrodynamics (QED) where photons acquire a particle interpretation through second quantization. In QED, we start with the Klein-Gordon equation with its normalization procedure. As a consequence, we use the expression[28]

$$g(k)=(1/\sqrt{k})a(k)\ , \tag{62}$$

instead of the form given in Eq. (52). The Lorentz-transformation property of this quantity is the same as that for $g(k)$ of Eq. (52).

However, the basic difference between the above expression and that of Eq. (52) is that the kinematical factor in front of $a(k)$ is $(1/\sqrt{k})$ in Eq. (62), while that for Eq. (52) is $(1\sqrt{Q})$. There is no concept of the average momentum in quantum field theory, while it was essential for the localized light wave discussed in Sec. V. Numerically, the above expression becomes equal to $g(k)$ of Sec. V when $a(k)$ of Eq. (62) represents a sharp distribution around a fixed value of k. This is why the photon can appear as a light pulse on oscilloscope screens.

On the other hand, the normalization property of Eq. (62) is quite different. In quantum field theory, it is possible to give a particle interpretation in terms of creation and annihilation operators by second-quantizing $a(k)$. In the light-wave normalization, it is very difficult, if not impossible, to give a particle interpretation. This means that, from the mathematical point of view, the gap between photons and localized light waves is real and very serious.

As for the space-time distribution $f(u)$ of Eq. (46), we use the form

$$f(u)=A(u) \tag{63}$$

in QED, without the factor $(\Omega/\Omega_0)^{1/2}$ discussed in Sec. V. As a consequence, the normalization condition is that the integral

$$i\int\left[A^*(u)\frac{\partial}{\partial t}A(u)-A(u)\frac{\partial}{\partial t}A^*(u)\right]dz \tag{64}$$

be Lorentz invariant. If we use this form of normalization, the total probability is not always positive.[28]

We can summarize the discussion of this section in Table I. There definitely is a gap between the concept of localized waves and that of photons. Numerically this gap is not serious. However, we have to cross this gap when we make the transition from localized Maxwell waves to photons through second quantization.

TABLE I. Light waves and photons. Light waves can be localized, but we still do not know how to localize photons. The difference between these two cases is not serious from the physical point of view. The mathematical difference is still a serious problem.

	Probability interpretation	Particle interpretation
Light waves	yes	no
Photons	no	yes

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APPENDIX: UNITARY TRANSFORMATIONS OF PHOTON POLARIZATION VECTORS

Let us work out the kinetics of Fig. 2. If we use the four-vector convention $x^\mu=(x,y,z,t)$, the matrix which boosts p to p' is

$$B_\phi(\eta)=\begin{bmatrix} c^2+s^2(\cosh\eta) & 0 & sc(\cosh\eta-1) & s(\sinh\eta) \\ 0 & 1 & 0 & 0 \\ sc(\cosh\eta-1) & 0 & s^2+c^2(\cosh\eta) & c(\sinh\eta) \\ s(\sinh\eta) & 0 & c(\sinh\eta) & \cosh\eta \end{bmatrix}, \tag{A1}$$

where $c=\cos\phi$ and $s=\sin\phi$. The parameters η and ϕ specify the magnitude and direction of the boost, respectively. On the other hand, we can achieve the same purpose on the four-momentum by boosting p along the z direction first and rotating the boosted four-momentum as is specified in Fig. 2. The boost matrix takes the form

$$B_z(\xi)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\xi & \sinh\xi \\ 0 & 0 & \sinh\xi & \cosh\xi \end{bmatrix}, \tag{A2}$$

with $e^\xi=\cosh\eta+(\cos\phi)\sinh\eta$. The rotation matrix is

$$R(\theta)=\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A3}$$

The rotation angle θ is related to the boost parameters η and ϕ by

$$\sin\theta=\frac{(\sin\phi)[(\cosh\eta-1)\cos\phi+\sinh\eta]}{\cosh\eta+(\cos\phi)\sinh\eta}\ ,$$
$$\cos\theta=\frac{1+(\cos\phi)\sinh\eta+(\cos\phi)^2(\cosh\eta-1)}{\cosh\eta+(\cos\phi)\sinh\eta}\ . \tag{A4}$$

The key question then is what is the difference between these two transformations which produce the same result on the four-momentum. The best way to examine this problem is to examine the closed-loop transformation

$$D(u)=[B_\phi(\eta)]^{-1}R(\theta)B_z(\xi)\ . \tag{A5}$$

The matrix algebra is somewhat complicated, but is straightforward. The result is

$$D(u)=\begin{bmatrix}1 & 0 & -u & u\\ 0 & 1 & 0 & 0\\ u & 0 & 1-u^2/2 & u^2/2\\ u & 0 & -u^2/2 & 1+u^2/2\end{bmatrix}, \tag{A6}$$

where

$$u=\frac{-(\sin\phi)\sinh\eta}{\cosh\eta+(\cos\phi)\sinh\eta} .$$

We can now write $\sin\phi$ and $\cos\phi$ in terms of θ, and arrive at the expression given in Eq. (18).

Similar calculations exist in the literature, but the previous calculations are only for specified values of ϕ. In Ref. 15 the calculation was made for $\phi=90°$. In a recent paper by Han *et al.*[17] a similar calculation was carried out for the angle which will make $\xi=0$. Here we carried out the calculation for the most general case.

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