Algebraic view of the optical propagation in a nonhomogeneous medium

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In this paper we propose an algebraic method to study the optical propagation in a nonhomogeneous medium with a quadratic profile of the refractive index. We use the Wei-Norman algebraic procedure establishing an analogy between the evolution operator and the optical propagation matrix.

I. INTRODUCTION

Algebraic methods have come into widespread use in quantum optics.[1] The evolution of two-level systems can be treated using the SU(2) algebra[2] and the two-photon dynamics is conveniently described by means of the SU(1,1) group.[3] Furthermore, unitary symmetries, already widely exploited in nuclear[4] and elementary-particle physics,[5] have been shown to be potentially powerful tools with which to understand the dynamical behavior of multilevel quantum systems and have displayed a number of previously unpredicted physical effects.[6]

To quote one more example, we recall that these algebraic tools have been applied also to the quantum harmonic oscillator with time-dependent frequency and have allowed the straightforward recovery of the adiabatic theorem, together with higher-order corrections.[7]

So far we have indicated quantum problems only, but we want to stress that some aspects of classical optics can also be modeled using algebraic methods. In fact, in Ref. 7 the relevance of the SU(1,1) group to the light propagation in self-focusing fibers[8] was indicated. This last point only touched upon in the papers quoted will be analyzed in further detail in the present paper, and the usefulness of algebraic concepts to the wave propagation in an inhomogeneous medium with a parabolic profile of the refractive index will be discussed.

Operatorial techniques in classical optics are, however, not new. Twenty years ago Vander Lugt[9] introduced the operator notation into coherent optics to simplify, for some special cases, the arising mathematical difficulties. This early suggestion was later elaborated on by Butterweck,[10] who extended the simple Vander Lugt notation into a more comprehensive set of system operators.

More recently, Stoler[11] has pointed out the structural similarity between physical optics and quantum mechanics, thus indicating the possibility of utilizing the wealth of mathematical techniques developed in quantum mechanics to treat the evolution of a light wave along an optical system. Later, Bacry and Cadilhac[12] showed that, in the paraxial approximation, lens transfer and free-space propagation are described by a set of operators belonging to the metaplectic group, isomorphic to the symplectic group of ray-transfer matrices representing the same systems in the framework of geometrical optics.

This paper develops both the points of view of Stoler and Bacry and Cadilhac. In fact, it will be shown that the wave propagation in a lenslike medium in the paraxial approximation can be described using a Schrödinger-like equation with a "time"-dependent harmonic potential. Furthermore, using the algebraic method of Ref. 7 we embed the "dynamical variables" of the problem to get SU(1,1) as a noninvariance group. Algebraic ordering methods are then used to describe the wave propagation in terms of an evolution operator. An interpretation of this operator is given, combining indeed both properties of the SU(1,1) generators and those of the propagation matrices, and its effect is understood as that of a beam expander, a thin lens, and a straight section. Finally, the well-known results of the propagation in a homogeneous medium are rederived as a particular case of the present formalism. Only the mathematical aspects will be emphasized in the present paper, while possible specific application will be discussed elsewhere.

II. THE HELMHOLTZ EQUATION FOR A REFRACTIVE INDEX WITH A PARABOLIC PROFILE AND THE SU(1,1) GROUP

The scalar Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{2.1}$$

for the wave propagation in an optically inhomogeneous medium with a refractive index exhibiting a parabolic dependence on the transverse (x,y) coordinates can be recast in the form[13]

$$\nabla^2 u + k_0(z)[k_0(z) - k_{2x}(z)x^2 - k_{2y}(z)y^2]u = 0 \; . \tag{2.2}$$

$k_0(z)$ is the wave number on the axis of propagation and $k_{2x,2y}$ are the expansion coefficients.

Making the usual transformation

$$u(x,y,z) = \exp\left[-i\int_{z_0}^{z} k_0(z')dz'\right]\psi(x,y,z) \; , \tag{2.3}$$

and assuming that $\psi(x,y,z)$ is a slowly varying function of z, we get from (2.2),

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\psi-2ik_0(z)\frac{\partial}{\partial z}\psi-i\left[\frac{d}{dz}k_0(z)\right]\psi$$
$$-k_0(z)[k_{2x}(z)x^2+k_{2y}(z)y^2]\psi=0\ . \quad (2.4)$$

We can further simplify the above equation assuming variables separation, which yields the following harmonic-oscillator Schrödinger-type equation,

$$\left[\frac{1}{2k_0(z)}\frac{\partial^2}{\partial\eta^2}-\frac{k_2(z)}{2}\eta^2\right]f(\eta,z)=i\frac{\partial}{\partial z}f(\eta,z)\ , \quad (2.5)$$

where

$$\eta=x,y \quad k_2(z)=k_{2x,2y}(z)\ ,$$
$$\psi(x,y,z)=\left[\frac{k_0(z)}{k_0(z_0)}\right]^{-1/2}f(x,z)f(y,z)\ . \quad (2.6)$$

The problem of solving the Helmholtz equation, in the actual conditions, has therefore been reduced to that of solving an equation which formally resembles a Schrödinger one with a harmonic-oscillator potential with "time"-dependent mass and frequency not necessarily being real functions.

These are not major drawbacks and we tackle the solution of Eq. (2.5) using the technique adopted in Ref. 7 to treat the evolution of quantum states ruled by a time-dependent harmonic-oscillator Hamiltonian. In that paper it was shown that by suitably embedding the harmonic-oscillator dynamical variables one can get an SU(1,1) algebraic structure.

In fact, the operators

$$\hat{k}_+=\frac{i}{2}\eta^2\ ,\quad \hat{k}_-=-\frac{i}{2}\frac{\partial^2}{\partial\eta^2},\quad \hat{k}_3=\tfrac{1}{2}\left[\eta\frac{\partial}{\partial\eta}+\tfrac{1}{2}\right]\ , \quad (2.7)$$

with the relations of commutation

$$[\hat{k}_+,\hat{k}_-]=-2\hat{k}_3,\quad [\hat{k}_3,\hat{k}_\pm]=\pm\hat{k}_\pm \quad (2.8)$$

are immediately recognized as the SU(1,1)-group generators.

We can rewrite Eq. (2.5) in the following, more compact, operatorial form,

$$i\frac{\partial f}{\partial z}=\hat{H}f\ , \quad (2.9)$$

where the "Hamiltonian" operator $\hat{H}$ is a "time"-dependent linear combination of SU(1,1) generators, namely

$$\hat{H}=\frac{i}{k_0(z)}\hat{k}_-+ik_2(z)\hat{k}_+\ . \quad (2.10)$$

The reason we characterize $\hat{H}$ as a "Hamiltonian" operator is due to the fact that it is not necessarily Hermitian, since $k_{0,2}(z)$ can be a real or a complex function of z. The possible non-Hermitian nature of H will not create additional troubles in solving (2.5) and we will utilize the already widely exploited algebraic methods,[1] with a few minor changes only.

III. WEI-NORMAN-TYPE SOLUTION

Equation (2.5), although with the above-mentioned non-Hermiticity problems, has been recognized as being Schrödinger-like. We can therefore look for its solution exploiting the well-established techniques of quantum mechanics.

We can indeed introduce an appropriate evolution operator, not necessarily unitary, according to what has already been discussed, such that

$$f(z)=\hat{\mathcal{U}}(z,z_0)f(z_0)\ , \quad (3.1)$$

and obeying the equation

$$i\frac{\partial}{\partial z}\hat{\mathcal{U}}(z,z_0)=\hat{H}(z)\hat{\mathcal{U}}(z,z_0)\ ,$$
$$\hat{\mathcal{U}}(z_0,z_0)=\mathbb{1}\ . \quad (3.2)$$

A straightforward solution of (3.2) is hampered by the "time" dependence of the "Hamiltonian," but according to its SU(1,1) structure the underlying ordering problems can be overcome by means of the Wei-Norman method,[14] thus giving[15]

$$\hat{\mathcal{U}}(z,z_0)=\exp(2u\hat{k}_3)\exp(v\hat{k}_+)\exp(-w\hat{k}_-)\hat{\mathbb{1}}\ , \quad (3.3)$$

where the functions u,v,w obey the following system of differential equations (the prime means d/dz),

$$u'=\frac{1}{k_0(z)}ve^{2u}\ ,$$
$$v'=k_2(z)e^{-2u}-vu'\ , \quad (3.4)$$
$$w'=-\frac{1}{k_0(z)}e^{2u},\quad u(z_0)=v(z_0)=w(z_0)=0\ ,$$

whose solution depends on that of a single Riccati equation, known as the characteristic equation (CE), namely

$$h'+h^2+\left[\frac{d}{dz}\ln[k_0(z)]\right]h+\frac{k_2(z)}{k_0(z)}=0\ ,$$
$$[h=-u',\ h(z_0)=0]\ . \quad (3.5)$$

It is, however, more convenient for our purposes to introduce the functions

$$U=\exp(-u),\quad W=w\exp(-u)\ , \quad (3.6)$$

which are linear independent solutions of the second-order differential equation

$$\xi''+\left[\frac{d}{dz}\ln[k_0(z)]\right]\xi'+\frac{k_2(z)}{k_0(z)}\xi=0\ , \quad (3.7)$$

with initial conditions

$$U(z_0)=1,\quad \left.\frac{dU}{dz}\right|_{z=z_0}=0\ ,$$
$$W(z_0)=0,\quad \left.\frac{dW}{dz}\right|_{z=z_0}=-\frac{1}{k_0(z_0)}\ . \quad (3.8)$$

It is worth stressing that this new CE equation is identical to that of the ray paraxial propagation[13] in a medium of the type considered. An analytical solution of (3.7) and (3.5) depends on the functional form of k_0 and k_2 functions. Following a standard procedure,[16] a class of $k_{0,2}$ functions allowing analytical solutions can be indicated. This problem, together with its physical meaning, will be discussed elsewhere.

IV. CONCLUSIONS

So far we have indicated the possible impact of an algebraic solution technique on the study of the wave propagation in an inhomogeneous medium. The results of the preceding section are more relevant to the structure of "evolution" operator, which within this framework might be better identified as a propagation operator (PO).

In this section we will discuss a possible interpretation of PO's, restricting ourselves to the case in which $k_{0,2}$ are real functions of z. To clarify the physical meaning of PO's, we follow the suggestion put forward in Refs. 7,11, and 12. We consider therefore, a Gaussian beam obeying the propagation equation

$$\left[\frac{\partial^2}{\partial\eta^2}-2i\bar{k}\frac{\partial}{\partial z}\right]f_n(\eta,z)=0\ , \tag{4.1}$$

where

$$\bar{k}_0=k_0(0)\ ,$$
$$f_n(\eta,z)=\left[\frac{\sqrt{2}}{\sqrt{\pi}2^n n!}\right]^{1/2}\frac{1}{\sqrt{\omega(z)}}H_n\left[\sqrt{2}\frac{\eta}{\omega(z)}\right]\exp\left[-i\bar{k}_0\left[z+\frac{1}{2q(z)}\eta^2\right]\right]\exp\left[i(n+\tfrac{1}{2})\tan^{-1}\left[\frac{\lambda z}{\pi\omega_0^2}\right]\right]\ , \tag{4.2}$$

with $H_n()$ Hermite polynomials and

$$\omega_0=\omega(0),\quad \omega^2(z)=\omega_0^2[1+(z/z_R)^2]\ ,$$
$$z_R=\frac{\pi\omega_0^2}{\lambda},\quad \lambda=\frac{2\pi}{\bar{k}_0},\quad \frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi\omega^2(z)}\ , \tag{4.3}$$
$$R(z)=z\left[1+\left[\frac{z_R}{z}\right]^2\right]\ .$$

A rather immediate interpretation of the operator $\exp(-w\hat{k}_-)$ follows from (4.1); indeed,

$$\exp(-w\hat{k}_-)=\exp\left[w\left[\frac{i}{2}\frac{\partial^2}{\partial\eta^2}\right]\right]=\exp\left[-w\left[\bar{k}_0\frac{\partial}{\partial z}\right]\right]\ , \tag{4.4}$$

and its action on $f_n(\eta,0)$ is immediately understood to be

$$\exp(-w\hat{k}_-)f_n(\eta,0)=f_n(\eta,-w\bar{k}_0)\ , \tag{4.5}$$

and thus interpreted as a shifting operator on a straight section of length $-w\bar{k}_0$ with the following simple ray matrix form (usually adopted in the paraxial approximation):[13]

$$\begin{bmatrix}1 & -w\bar{k}_0\\ 0 & 1\end{bmatrix}\ . \tag{4.6}$$

The action of the operator $\exp(v\hat{k}_+)$ on $f_n(\eta,z)$ is straightforward,

$$\exp(v\hat{k}_+)f_n(\eta,z)=\left[\frac{\sqrt{2}}{\sqrt{\pi}2^n n!}\right]^{1/2}\frac{1}{\sqrt{\omega(z)}}H_n\left[\sqrt{2}\frac{\eta}{\omega(z)}\right]\exp(-\bar{k}_0 z)$$
$$\times\exp\left[i(n+\tfrac{1}{2})\tan^{-1}\left[\frac{\lambda z}{\pi\omega_0^2}\right]\right]\exp\left[-\frac{i}{2}\bar{k}_0\left[\frac{1}{q(z)}-\frac{v}{\bar{k}_0}\right]\eta^2\right]\ , \tag{4.7}$$

and secures an interpretation of the operator as a thin lens with focal length $f=\bar{k}_0/v$ or, in matrix form,

$$\begin{bmatrix}1 & 0\\ -v/\bar{k}_0 & 1\end{bmatrix}\ . \tag{4.8}$$

Finally, since

$$\exp(2uk_0)f_n(\eta,z)=e^{u/2}f_n(\eta e^u,z)\ , \tag{4.9}$$

the following matrix representation for this last operator can be used:

$$\begin{bmatrix}e^{-u} & 0\\ 0 & e^u\end{bmatrix}\ . \tag{4.10}$$

We can therefore rewrite PO's in the more suggestive form

$$U=\begin{bmatrix} e^{-u} & 0 \\ 0 & e^{u} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{v(z)}{\bar{k}_0} & 1 \end{bmatrix}\begin{bmatrix} 1 & -w(z)\bar{k}_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} U & -W\bar{k}_0 \\ -\dfrac{V}{\bar{k}_0} & \dfrac{(WV+1)}{u} \end{bmatrix}, \tag{4.11}$$

where

$$V=e^{u}v=-k_0(z)\frac{dU}{dz} . \tag{4.12}$$

The relation (4.11) provides a rather immediate interpretation of the propagation operator as the effect of a beam expander, a thin lens, and a straight section. Let us also stress that a useful by-product of the above-stated analogy is an almost direct application of the ABCD law, which in this framework yields the following expression for the complex beam radius:[13]

$$q(z)=\frac{Uq_0-\bar{k}_0 W}{(-v/\bar{k}_0)q_0+\dfrac{1+VW}{U}} . \tag{4.13}$$

Just to recover a well-known result, we have considered the case in which both $k_{0,2}$ are constants; in this connection we get

$$\begin{aligned} U &= \cos\left[\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} z\right], \\ W &= -\frac{1}{(k_2\bar{k}_0)^{1/2}}\sin\left[\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} z\right], \\ V &= (k_2\bar{k}_0)^{1/2}\sin\left[\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} z\right], \end{aligned} \tag{4.14}$$

thus getting, for $q(z)$ a very familiar expression,

$$q(z)=\left\{\cos\left[\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} z\right]q_0+\left[\frac{\bar{k}_0}{k_2}\right]^{1/2}\sin\left[\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} z\right]\right\}\Big/\left\{-\sin\left[\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} z\right]\left[\frac{k_2}{\bar{k}_0}\right]^{1/2} q_0+\cos\left[\left[\frac{k_2}{\bar{k}_0}\right]z\right]\right\} . \tag{4.15}$$

Let us finally discuss how PO transforms an input Gaussian beam, namely,

$$\begin{aligned} \hat{\mathscr{U}}(z,0)f_n(\eta,0) &= \left[\frac{\sqrt{2}}{\sqrt{\pi}2^n n!}\right]^{1/2}\frac{1}{[U\omega(-w\bar{k}_0)]^{1/2}}H_n\left[\sqrt{2}\frac{\eta}{U\omega(-w\bar{k}_0)}\right] \\ &\quad\times\exp(iw\bar{k}_0^{\,2})\exp\left[-i(n+\tfrac{1}{2})\tan^{-1}\left[\frac{2w(z)}{\omega_0^2}\right]\right] \\ &\quad\times\exp\left[-i\frac{\bar{k}_0\eta^2}{2U^2}\left[\frac{1}{q(-w\bar{k}_0)}+\frac{k_0(z)}{\bar{k}_0}UU'\right]\right] . \end{aligned} \tag{4.16}$$

The results of this section complete our preliminary analysis of the relevance of the algebraic methods to wave propagation in a nonhomogeneous medium. In a future paper we will discuss the relevance of the present results to $k_{0,2}$ complex functions of z to account for eventual gain or losses.

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