N-level atom and $N-1$ modes: Statistical aspects and interaction with squeezed light

A. M. Abdel-Hafez
Junior College and Mathematics and Science Center, Taif, Saudi Arabia

A.-S. F. Obada* and M. M. A. Ahmad
Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt

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A model is presented to investigate the problem of interaction between an N-level atom and $N-1$ modes of the field. The model includes detuning. Constants of motion are obtained. The evolution operator is calculated, and the probability distribution function for the photon numbers is computed for different initial atomic states. The characteristic functions are computed. Different statistical quantities concerning the photons or the atomic system are given. The case of a three-level atom and two modes is considered for its different configurations. The phenomenon of collapses and revivals is discussed for squeezed light, and the effect of squeezing is shown in this phenomenon.

I. INTRODUCTION

The interaction between electromagnetic (e.m.) fields and atoms lies at the heart of quantum optics,[1,2] some nonlinear phenomena,[3,4] laser theory,[5,6] and laser spectroscopy.[7,8] Some models have been presented to discuss different phenomena. The Jaynes-Cummings model[9] for the two-level atom has been investigated extensively to study emission, saturated absorption, dynamic Stark effect (see Refs. 2 and 8–11), and collapses and revivals.[12–17]

The three-level atom model has been introduced to study stepwise and two-photon excitations[11,18,19] which were first pointed out a long time ago,[20] nonlinear constants,[21,22] coherence trapping,[23–26] and two-photon lasers.[27,28] The semiclassical treatment of the problem may be found in Refs. 21–24, 29, and 30. Recently, the problem has been discussed in a full quantum-mechanical manner[31–36] to discuss dynamics of the system and collapses and revivals.[36] The multilevel atom has been considered to treat the Dicke model[37] for a system of two-level atoms in interaction with e.m. fields.[1,3,38–40] It has been used recently to discuss the multimode laser.[41]

In this article we present a model for the interaction between an N-level atom and $N-1$ modes of the radiation. This model includes a detuning parameter Δ. Its schematic representation is shown in Fig. 1, where we have an atom whose levels have energies $\omega_1>\omega_2>\cdots>\omega_N$. The lower level of energy ω_{k+1} is connected to the upper level with a photon of energy Ω_k. The detuning parameter Δ is given by

$$\Delta=\omega_j-\omega_{j+1}-\Omega_j,\quad j=1,2,\ldots,N-1\ . \tag{1.1}$$

The Hamiltonian for such a system in the rotating-wave approximation (RWA) can be written in the form[2–4,8,9,14–16,32–41]

$$\hat{H}=\sum_{j=1}^{N}\omega_j\hat{S}_{jj}+\sum_{j=1}^{N-1}\Omega_j\hat{a}_j^{\dagger}\hat{a}_j+\sum_{j=1}^{N-1}\lambda_j(\hat{S}_{1,j+1}\hat{a}_j+\hat{a}_j^{\dagger}\hat{S}_{j+1,1})\ . \tag{1.2}$$

Under the condition (1.1) which may be considered as a direct generalization to the two-photon resonance,[21,22] the Hamiltonian (1.1) can be cast in the form

$$\hat{H}=(\omega_1-\Delta)\hat{1}+\Delta\hat{S}_{11}+\sum_{j=1}^{N-1}\Omega_j(\hat{a}_j^{\dagger}\hat{a}_j-\hat{S}_{j+1,j+1})+\sum_{j=1}^{N-1}\lambda_j(\hat{S}_{1,j+1}\hat{a}_j+\hat{a}_j^{\dagger}\hat{S}_{j+1,1})\ , \tag{1.3}$$

where the operators $\hat{a}$ and $\hat{a}^{\dagger}$ are the boson operators for the photon fields; the $\hat{S}_{ij}$ the generators of the unitary group $U(N)$.[41] They satisfy the following commutation relations:[41]

$$[\hat{a}_i,\hat{a}_j^{\dagger}]=\delta_{ij},\quad [\hat{S}_{ij},\hat{S}_{kl}]=\hat{S}_{il}\delta_{jk}-\hat{S}_{kj}\delta_{il}\ ,$$
$$[\hat{a}_i,\hat{a}_j]=0,\quad [\hat{a}_k,\hat{S}_{ij}]=0,\quad [\hat{a}_k^{\dagger},\hat{S}_{ij}]=0\ . \tag{1.4}$$

In the rest of the paper we look for the constants of

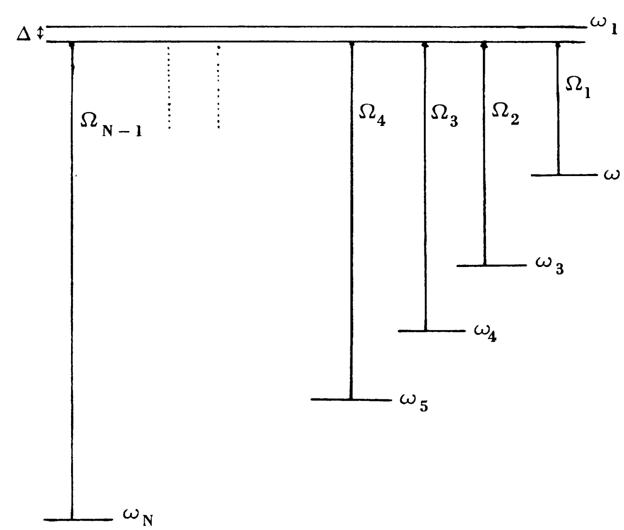

FIG. 1. Interaction scheme for the N-level atom and $N-1$ modes.

motion and calculate the evolution operator in Sec. II. The probability distribution functions are given when the atomic system is supposed to be initially in one of its pure states. This is the subject of investigation of Sec. III. Some statistical aspects are given in Sec. IV related to the characteristic function and its utilization to compute different statistical averages of the photon number operators. The case of the three-level atom and two modes is investigated in some detail in Sec. V. We discuss the three different configurations for this system (Λ, ladder, and V configurations) and give some statistical quantities and their relations in the three configurations. Then the interaction with two squeezed modes is investigated. We specify different states of squeezing and coherence and study their effect on the time evolution for the photon number. We investigate the phenomenon of collapses and revivals and find it is observable in this model but with some difference in the detailed features from the case of two-level atom and a single mode. In Sec. VI we consider the case of multiphoton processes; we give the time evolution for the photon number in this case.

II. THE EVOLUTION OPERATOR

The Hamiltonian (1.2) under the conditions (1.1) cast in the form (1.3) helps in showing easily that the following operators

$$\hat{N}_j=\hat{a}^\dagger_j\hat{a}_j-\hat{S}_{j+1,j+1}=\hat{n}_j-\hat{S}_{j+1,j+1}\,, \quad j=1,2,\dots,N-1 \tag{2.1}$$

are constants of motion. They also commute with each other. Therefore the Hamiltonian becomes exactly solvable and breaks up into

$$\hat{H}=\hat{C}+\hat{D}\,, \tag{2.2a}$$

where

$$\hat{C}=\Delta\hat{S}_{11}+\sum_{j=1}^{N-1}\lambda_j(\hat{S}_{1,j+1}\hat{a}_j+\hat{a}^\dagger_j\hat{S}_{j+1,1}) \tag{2.2b}$$

and

$$\hat{D}=(\omega_1-\Delta)\mathbb{1}+\sum_{j=1}^{N-1}\Omega_j\hat{N}_j\,. \tag{2.2c}$$

The operators $\hat{C}$ and $\hat{D}$ commute with each other and they are constants of motion. Bearing these facts in mind we find that the evolution operator $U(t,0)=\exp(-i\hat{H}t)$ takes the following form:

$$\hat{U}(t,0)=\exp[-i(\omega_1-\Delta)t]\left[\prod_{j=1}^{N-1}\exp(-i\Omega_j\hat{N}_j)\right]\times\exp(-i\hat{C}t)\,. \tag{2.3}$$

The first two factors produce phases that will not affect the results that follow, while calculations of the third factor show that it takes the following compact matrix form:

$$\exp(-i\hat{C}t)=\exp(-\tfrac{1}{2}\Delta it)\begin{bmatrix}\hat{B}_0 & \hat{B}_1\\ -\hat{B}^\dagger_1 & \hat{B}_2\end{bmatrix}, \tag{2.4}$$

where $\hat{B}_0$ is the single element matrix $\{\hat{b}^{(0)}_{11}\}$ where

$$\hat{b}^{(0)}_{11}=\left[\cos(\hat{\mu}t)-\frac{i\Delta}{2\mu}\sin(\hat{\mu}t)\right], \tag{2.4a}$$

the matrix $\hat{B}_1$ is the $1\times N-1$ row matrix $[\hat{b}^{(1)}_{1,k}]$ where

$$\hat{b}^{(1)}_{1,k}=-i\frac{\sin(\hat{\mu}t)}{\hat{\mu}}\lambda_k\hat{a}_k\,,\quad k\in\{1,2,\dots,N-1\} \tag{2.4b}$$

and $\hat{B}^\dagger_1$ its Hermitian conjugate, and finally the matrix $\hat{B}_2$ is the $(N-1)\times(N-1)$ square matrix $[\hat{b}^{(2)}_{ij}]$ where

$$\hat{b}^{(2)}_{ij}=e^{(1/2)i\Delta t}\delta_{ij}+\lambda_i\hat{a}^\dagger_i\hat{\nu}^{-1}\left[\cos(\hat{\mu}t)-e^{(1/2)i\Delta t}+\frac{i\Delta}{2\mu}\sin(\hat{\mu}t)\right]\lambda_j\hat{a}_j\,,\quad i,j\in\{1,2,3,\dots,N-1\}\,. \tag{2.4c}$$

The operators $\hat{\mu}$ and $\hat{\nu}$ are given by

$$\hat{\mu}^2=\sum_{j=1}^{N-1}\lambda_j^2(\hat{n}_j+1)+\Delta^2/4=\hat{\nu}+\Delta^2/4\,. \tag{2.4d}$$

They satisfy the following relations:

$$\hat{a}^\dagger_k\hat{\mu}^2=\hat{\mu}^2_k\hat{a}^\dagger_k$$

and

$$\hat{\mu}^2\hat{a}_k=\hat{a}_k\hat{\mu}^2_k\,,$$

where

$$\hat{\mu}^2_k=\lambda_k^2\hat{n}_k+\sum_{\substack{j=1\\(j\neq k)}}^{N-1}\lambda_j^2(\hat{n}_j+1)=\hat{\nu}_k+\Delta^2/4\,. \tag{2.4e}$$

$\hat{U}^\dagger(t,0)$ is obtained by taking the Hermitian conjugate of $\hat{U}(t,0)$. It can be easily proved that $\hat{U}\hat{U}^\dagger=\hat{\mathbb{1}}$.

Once the evolution operator $\hat{U}(t)$ is obtained, the time evolution for any operator $\hat{O}$ can be easily calculated through the equation

$$\hat{O}(t)=\hat{U}(t,0)\hat{O}(0)\hat{U}^\dagger(t,0)\,. \tag{2.5a}$$

The expectation value of any operator and its time dependence can be easily obtained through

$$\langle\hat{O}(t)\rangle=\mathrm{Tr}\rho(t)\hat{O}(0)\,, \tag{2.5b}$$

where $\hat{\rho}(t)$ is the density operator of the system.

The density operator $\hat{\rho}(t)$ is initially assumed to have the following form:

$$\hat{\rho}(0)=\hat{\rho}_A(0)\otimes\hat{\rho}_F(0)\,, \tag{2.6}$$

which means that the atomic system and the fields were decoupled at time $t=0$. For $t>0$ the field density operator $\hat{\rho}_F(t)$ is given by

$$\hat{\rho}_F(t)=\mathrm{Tr}_A\hat{\rho}(t)\ , \tag{2.7}$$

where the trace is taken over the atomic states.

Once this operator is calculated, the probability distribution function for finding n_i photons in the mode i is given by

$$P(\{n_j\})=\langle n_1,n_2,\ldots,n_{N-1}|\hat{\rho}_F(t)|n_1,n_2,\ldots,n_{N-1}\rangle$$
$$=P(n_1,n_2,\ldots,n_{N-1})\ . \tag{2.8}$$

In the following section we calculate this function for different initial atomic states.

III. THE PROBABILITY DISTRIBUTION FUNCTION

We start by calculating the function of (2.8) by considering the atom initially in the state of energy ω_{k+1} ($k\neq 0$), and then when the atom is initially in its upper state of energy ω_1.

A. Atom in the state with energy ω_{k+1} ($k\neq 0$)

We assume the atom to be in this state. In this case equation (2.6) becomes

$$\hat{\rho}^{k+1}(0)=\hat{\rho}_F(0)\otimes\hat{S}_{k+1,k+1}\ . \tag{3.1}$$

Taking this into account we find the probability distribution function

$$P_\Delta^{k+1}(\{n_j\},t)=\lambda_k^2(n_k+1)\frac{\sin^2(\mu t)}{\mu^2}P(n_1,n_2,\ldots,n_k+1,\ldots,n_{N-1})$$
$$+|e^{(1/2)i\Delta t}+A_k|^2P(\{n_j\})+\lambda_k^2(n_k+1)\sum_{\substack{j=1\\(j\neq k)}}^{N-1}|A_j|^2P(n_1,n_2,\ldots,n_j-1,\ldots,n_k+1,\ldots,n_{N-1})\ , \tag{3.2a}$$

where

$$A_j=\frac{\lambda_j}{\nu_j}\sqrt{n_j}\left[\cos(\mu_j t)-e^{(1/2)i\Delta t}+\frac{i\Delta}{2\mu_j}\sin(\mu_j t)\right]\ , \tag{3.2b}$$

and $P(\{n_j\})$ is the initial value for the probability distribution function of the fields.

B. Atom in its upper excited state

In this case the density operator $\hat{\rho}(0)$ is given by

$$\hat{\rho}^{(1)}(0)=\hat{\rho}_F(0)\otimes\hat{S}_{11}\ . \tag{3.3}$$

The probability distribution function takes the form

$$P_\Delta^{(1)}(\{n_j\},t)=\left[\cos^2(\mu t)+\frac{\Delta^2}{4\mu^2}\sin^2(\mu t)\right]P(\{n_j\})+\sum_{j=1}^{N-1}\lambda_j^2 n_j\frac{\sin^2(\mu_j t)}{\mu_j^2}P(n_1,n_2,\ldots,n_j-1,\ldots,n_{N-1})\ . \tag{3.4}$$

The formulas (3.2) and (3.4) give the time evolution for the probability distribution function when the atomic system starts initially in one of its pure states. If the atom is not in a pure state, but it has probability distribution p_j for the atom in the state of energy ω_j. Thus the probability distribution function in the general case

$$P(n_1,n_2,\ldots,n_{N-1},t)=\sum_{j=1}^{N}p_jP^j(n_1,n_2,\ldots,n_{N-1},t). \tag{3.5}$$

In the following section we compute the characteristic function and calculate some statistical quantities.

IV. STATISTICAL ASPECTS

The characteristic function operator is defined by[34]

$$\hat{\chi}(\beta_1,\beta_2,\ldots,\beta_{N-1})=\exp\left[i\sum_{j=1}^{N-1}\beta_j\hat{n}_j\right]\ , \tag{4.1}$$

for $N-1$ modes. For a single mode we may define the following characteristic function:

$$\hat{\chi}_j(\beta)=\exp(i\beta\hat{n}_j)\ . \tag{4.2}$$

Their development in time can be obtained when multiplied by $\hat{\rho}(t)$ and traced over the atomic and photon states. Thus we obtain, using the definitions of the probabilities,

$$\chi(\beta_1,\beta_2,\ldots,\beta_{N-1})=\sum_{\{n_j\}}\exp\left[i\sum_{j=1}^{N-1}\beta_jn_j\right]P_\Delta(\{n_j\},t) \tag{4.3}$$

and

$$\chi_j(\beta)=\sum_{\{n_j\}}\exp(i\beta n_j)P_\Delta(\{n_j\},t)\;, \tag{4.4}$$

where $P(\{n_j\},t)$ is given by one of the expressions (3.2), (3.4), or (3.5).

Once the characteristic and probability distribution functions are known, it is easy to compute the statistical moments $\langle n_k^m\rangle$ of the photon numbers in the jth modes and their correlations $\langle n_1^q(t)n_2^r(t)\cdots n_{N-1}^l(t)\rangle$. This can be done by using the relations

$$\langle n_k^m(t)\rangle=\sum_{\{n_j\}}n_k^mP(\{n_j\})=\left.\frac{\partial^m\chi_k}{\partial(i\beta)^m}\right|_{\beta=0}$$

and

$$\begin{aligned}\langle n_1^q(t)\cdots n_k^s(t)\cdots n_{N-1}^l(t)\rangle &=\sum_{\{n_j\}}n_1^q\cdots n_k^s\cdots n_{N-1}^lP(\{n_j\})\\ &=\left.\frac{\partial^{q+\cdots+s+\cdots+l}\chi(\{\beta_j\})}{\partial(i\beta_1)^q\cdots\partial(i\beta_k)^s\cdots\partial(i\beta_{N-1})^l}\right|_{\{\beta_j\}=\{0\}}\;. \end{aligned}\tag{4.5}$$

In particular, we find in the following the characteristic functions and first and second moments for the photon number in the jth mode, and the correlation function. We now look for these quantities when the atom starts in one of its states other than the upper state.

A. For the state with energy ω_{k+1}

The single-mode characteristic functions can be calculated in this case for the modes $j\neq k$. We find

$$\chi_{j-\Delta}^{(k+1)}(\beta)=\sum_{\{n_l\}}P(\{n_l\})\exp(i\beta n_j)\times[1+(e^{i\beta}-1)\lambda_j^2(n_j+1)\,|\,A_k\,|^2]\;, \tag{4.6a}$$

where $j\neq k$.

The characteristic function for the kth mode is given by

$$\chi_{k-\Delta}^{(k+1)}(\beta)=\sum_{\{n_j\}}P(\{n_j\})\exp(i\beta n_k)\left[1+(e^{-i\beta}-1)\left[\lambda_k^2n_k\frac{\sin^2(\mu_kt)}{\mu_k^2}+(\nu_k-\lambda_k^2n_k)\,|\,A_k\,|^2\right]\right]\;. \tag{4.6b}$$

The multimode characteristic function is given by

$$\chi_\Delta^{(k+1)}(\{\beta_j\})=\sum_{\{n_j\}}P(\{n_j\})\exp\left[i\sum_{j=1}^{N-1}\beta_jn_j\right]\left[1+(e^{-i\beta_k}-1)\lambda_k^2n_k\frac{\sin^2(\mu_kt)}{\mu_k^2}+\sum_{\substack{j=1\\(j\neq k)}}^{N-1}(e^{i\beta_j-i\beta_k}-1)\lambda_j^2(n_j+1)\,|\,A_k\,|^2\right]\;. \tag{4.6c}$$

The expectation value can be calculated from (4.6). Thus we find the photon number expectation values in the jth mode

$$\langle\hat n_j(t)\rangle_\Delta^{(k+1)}=\left.\frac{\partial\chi_j^{(k+1)}(\beta)}{\partial(i\beta)}\right|_{\beta=0}=\bar n_j+\sum_{\{n_i\}}P(\{n_i\})\lambda_j^2(n_j+1)\,|\,A_k\,|^2\;,$$

where $j\neq k$. It is always greater than its initial value The kth mode has the photon number expectation value

$$\langle\hat n_k(t)\rangle_\Delta^{(k+1)}=\left.\frac{\partial\chi_k^{(k+1)}(\beta)}{\partial(i\beta)}\right|_{\beta=0}=\bar n_k-\sum_{\{n_j\}}P(\{n_j\})\left[\lambda_k^2n_k\frac{\sin^2(\mu_kt)}{\mu_k^2}+(\nu_k-\lambda_k^2n_k)\,|\,A_k\,|^2\right]\;. \tag{4.7b}$$

This shows that the photon number in the kth mode is always less than its initial value. When we put $\Delta=0$ we get the results of Ref. 41.

The second moments are given by

$$\langle\hat n_j^2(t)\rangle_\Delta^{(k+1)}=\left.\frac{\partial^2\chi_j^{(k+1)}(\beta)}{\partial(i\beta)^2}\right|_{\beta=0}=\overline{n_j^2}+[\langle\hat n_j(t)\rangle_\Delta^{(k+1)}-\bar n_j]+2\sum_{\{n_i\}}P(\{n_i\})\lambda_j^2(n_j-1)^2\,|\,A_k\,|^2\;, \tag{4.8a}$$

where $j\neq k$. For the mode k we find

$$\langle \hat{n}_k^2(t)\rangle_\Delta^{(k+1)} = \left.\frac{\partial^2 \chi_k^{(k+1)}(\beta)}{\partial(i\beta)^2}\right|_{\beta=0}$$

$$= \overline{n_k^2} - [\langle \hat{n}_k(t)\rangle_\Delta^{(k+1)} - \bar{n}_k] - 2\sum_{\{n_j\}} n_k P(\{n_j\}) \left[\lambda_k^2 n_k \frac{\sin^2(\mu_k t)}{\mu_k^2} + (\nu_k - \lambda_k^2 n_k)|A_k|^2\right]. \quad (4.8b)$$

The correlation function for the $N-1$ modes is given by

$$\langle \hat{n}_1(t)\cdots \hat{n}_k(t)\cdots \hat{n}_{N-1}(t)\rangle_\Delta^{(k+1)}$$

$$= \left.\frac{\partial^{(N-1)}\chi^{(k+1)}(\beta_1,\ldots,\beta_{N-1})}{\partial(i\beta_1)\cdots\partial(i\beta_k)\cdots\partial(i\beta_{N-1})}\right|_{\{\beta_j\}=\{0\}}$$

$$= \overline{\left[\prod_{j=1}^{N-1} n_j\right]} - \sum_{\{n_j\}} P(\{n_j\}) \left[\left[\prod_{j=1}^{N-1} n_j\right]\lambda_k^2 \frac{\sin^2(\mu_k t)}{\mu_k^2} + \sum_{\substack{j=1\\(j\neq k)}} \left[\prod_{\substack{r=1\\(r\neq j)}}^{N-1} n_r\right] \lambda_j^2 (n_j+1)(n_j+1-n_k)|A_k|^2 n_k^{-1}\right]. \quad (4.8c)$$

It is noted that $\langle \hat{n}_j(t)\rangle \geq \bar{n}_j$ for $j\neq k$, while $\langle \hat{n}_k(t)\rangle \leq \bar{n}_k$. The system can develop once the mode k is not in a vacuum state.

B. For the upper state with energy ω_1

The single-mode characteristic function is given in this case by

$$\chi_{\Delta-j}^{(1)}(\beta) = \sum_{\{n_j\}} P(\{n_j\})\exp(i\beta n_j)\left[1 + (e^{i\beta}-1)\lambda_j^2(n_j+1)\frac{\sin^2(\mu t)}{\mu^2}\right], \quad (4.9a)$$

where $j=1,2,\ldots,N-1$. The multimode characteristic function is given by

$$\chi_\Delta^{(1)}(\{\beta_j\}) = \sum_{\{n_j\}} P(\{n_j\})\exp\left[i\sum_{j=1}^{N-1}\beta_j n_j\right]\left[1 + \sum_{j=1}^{N-1}(e^{i\beta_j}-1)\lambda_j^2(n_j+1)\frac{\sin^2(\mu t)}{\mu^2}\right]. \quad (4.9b)$$

The expectation value for the photon number in the mode j can be calculated by using equations (4.9), and we find

$$\langle \hat{n}_j(t)\rangle_\Delta^{(1)} = \left.\frac{\partial \chi_{\Delta-j}^{(1)}(\beta)}{\partial(i\beta)}\right|_{\beta=0} = \bar{n}_j + \sum_{\{n_j\}} P(\{n_j\})\lambda_j^2(n_j+1)\frac{\sin^2(\mu t)}{\mu^2}, \quad (4.10a)$$

where $j=1,2,\ldots,N-1$. It is found in this case that for any mode $\langle \hat{n}_j(t)\rangle_\Delta^{(1)} \geq \bar{n}_j$. Even if all modes start initially in vacuum the system can develop and we find

$$\langle \hat{n}_j(t)\rangle_\Delta^{(1)} = \bar{n}_j + \frac{\lambda_j^2}{\lambda^2}\sin(\lambda t)$$

where

$$\lambda^2 = \sum_{j=1}^{N-1}\lambda_j^2 + \frac{\Delta^2}{4}.$$

The second moments are given by

$$\langle \hat{n}_j^2(t)\rangle_\Delta^{(1)} = \left.\frac{\partial^2 \chi_{\Delta-j}^{(1)}(\beta)}{\partial^2(i\beta)^2}\right|_{\beta=0} = \overline{n_j^2} - [\langle \hat{n}_j(t)\rangle_\Delta^{(1)} - \bar{n}_j] + 2\sum_{\{n_j\}} P(\{n_j\})\lambda_j^2(n_j+1)^2\frac{\sin^2(\mu t)}{\mu^2} \quad (4.10b)$$

where $j=1,2,\ldots,N-1$, while the expectation value for the multiplication of photon numbers is

$$\langle \hat{n}_1(t)\cdots \hat{n}_k(t)\cdots \hat{n}_{N-1}(t)\rangle_\Delta^{(1)} = \left.\frac{\partial^{(N-1)}\chi_\Delta^{(1)}(\beta_1,\ldots,\beta_k,\ldots,\beta_{N-1})}{\partial(i\beta_1)\cdots\partial(i\beta_k)\cdots\partial(i\beta_{N-1})}\right|_{\{\beta_j\}=\{0\}}$$

$$= \overline{\left[\prod_{j=1}^{N-1} n_j\right]} + \sum_{\{n_j\}}^{N-1} P(\{n_j\})\left[\sum_{j=1}^{N-1}\left[\prod_{\substack{r=1\\(r\neq j)}}^{N-1} n_r\right]\lambda_j^2(n_j+1)\frac{\sin^2(\mu t)}{\mu^2}\right]. \quad (4.10c)$$

C. The atomic state occupation number

These numbers can be computed easily from the constants of motion of (2.1). This equation means that the difference between the occupation number in the state with energy ω_{j+1} and the photon number in the mode j that connects this energy level with the upper level of energy ω_1, is a constant through the time development of the system.

Thus we can get the time development for the occupation number in the level of energy ω_{j+1} from the following:

$$\langle \hat{S}_{j+1,j+1}(t)\rangle = \langle \hat{n}_j(t)\rangle - \langle \hat{N}_j\rangle \quad j\in\{1,2,\ldots,N-1\} . \tag{4.11}$$

Therefore we find the following formulas for the occupation numbers in the atomic states.

1. Atom initially in state of energy ω_{k+1}

When the atom is initially in one of its states other than its upper state, let it initially occupy the state with energy level ω_{k+1} where $k=1,2,\ldots,N-1$. Using (4.7a) and (4.7b), we obtain

$$\langle \hat{S}_{j+1,j+1}(t)\rangle^{(k+1)} = [\langle \hat{n}_j(t)\rangle^{(k+1)} - \bar{n}_j] = \sum_{\{n_j\}} P(\{n_j\})\lambda_j^2(n_j+1)\,|A_k|^2 , \tag{4.12a}$$

where $j\neq k$, while for the level with energy ω_{k+1} we get

$$\begin{aligned}\langle \hat{S}_{k+1,k+1}(t)\rangle^{(k+1)} &= \langle \hat{n}_k(t)\rangle^{(k+1)} - (\bar{n}_k - 1)\\ &= 1 - \sum_{\{n_j\}} P(\{n_j\})\left[\lambda_k^2 n_k \frac{\sin^2(\mu_k t)}{\mu_k^2} + (\nu_k - \lambda_k^2 n_k)\,|A_k|^2\right] .\end{aligned} \tag{4.12b}$$

The occupation number in the upper-most state of energy ω_1 is given by

$$\langle \hat{S}_{11}(t)\rangle^{(k+1)} = \sum_{\{n_j\}} P(\{n_j\})\lambda_k^2 n_k \frac{\sin^2(\mu_k t)}{\mu_k^2} . \tag{4.12c}$$

2. Atom initially in its uppermost state (of energy ω_1)

When the atom starts initially in its uppermost state with energy ω_1, we use (4.10a) to obtain the following:

$$\langle \hat{S}_{j+1,j+1}(t)\rangle^{(1)} = \langle \hat{n}_j(t)\rangle^{(1)} - \bar{n}_j = \sum_{\{n_j\}} P(\{n_j\})\lambda_j^2(n_j+1)\frac{\sin^2(\mu t)}{\mu^2} \tag{4.13a}$$

for $j=1,2,\ldots,N-1$. The occupation number in the initially occupied level ω_1 is given by

$$\langle \hat{S}_{11}(t)\rangle^{(1)} = \sum_{\{n_j\}} P(\{n_j\})\left[\cos^2(\mu t) + \frac{\Delta^2}{4\mu^2}\sin^2(\mu t)\right] . \tag{4.13b}$$

In what follows we restrict the number of levels to three. This means we consider a three-level atom in interaction with two modes.

V. INTERACTION OF A THREE-LEVEL ATOM WITH TWO MODES

In this section we consider the three-level atom and two-mode system. We first give the average photon numbers when initial conditions are stated. We consider next the interaction with two squeezed modes of light.

When we restrict the numbers of levels to be three in the foregoing study, we get the so-called Λ (or inverted) configuration. However, there are two other configurations, namely, the V and ladder or cascade configuration.. These configurations are shown in Fig. 2.

A. The Λ configuration

For this we have the Hamiltonian (1.2) with $N=3$ while the condition (1.1) becomes

$$\Delta = \omega_1 - \omega_2 - \Omega_1 = \omega_1 - \omega_3 - \Omega_2 , \tag{5.1}$$

which is the two-photon resonance,[21] and the constants of motion of (2.1) are now[33,34]

$$\hat{N}_1 = \hat{n}_1 - \hat{S}_{22}, \quad \hat{N}_2 = \hat{n}_2 - \hat{S}_{33} . \tag{5.2}$$

We look now for the expectation values for $\hat{n}$ and $\hat{n}^2$ which are the special values of equations (4.5) in this case.

1. Atom initially in its ground state (of energy ω_3)

The expectation values are given by

$$\langle \hat{n}_1(t)\rangle_\Lambda^{\mathrm{gr}} = \bar{n}_1 + \sum_{n_1,n_2}^{\infty} P(n_1,n_2)\lambda_1^2(n_1+1)\,|A_2|^2 \tag{5.3a}$$

and

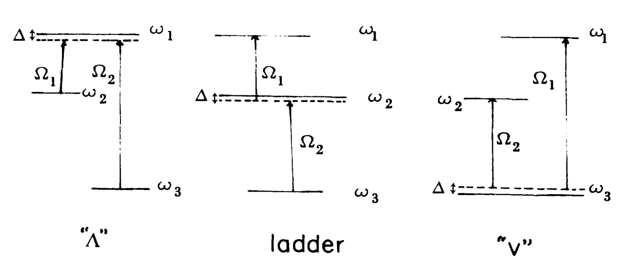

FIG. 2. The different configurations for the three-level atom and two-mode system.

$$\langle \hat{n}_2(t)\rangle_\Lambda^{\rm gr}=\bar{n}_2-\sum_{n_1,n_2}P(n_1,n_2)\left[\lambda_2^2 n_2\frac{\sin^2(\mu_2 t)}{\mu_2^2}+\lambda_1^2(n_1+1)\,|A_2|^2\right], \tag{5.3b}$$

where

$$A_j=\frac{\lambda_i}{\nu_j}\sqrt{n_j}\left[\cos(\mu_j t)-e^{i(1/2)\Delta t}+\frac{i\Delta}{2\mu_j}\sin(\mu_j t)\right] \tag{5.4a}$$

and

$$\mu_1^2=\nu_1+\Delta^2/4,\quad \nu_1=\lambda_1^2 n_1+\lambda_2^2(n_2+1) \tag{5.4b}$$

$$\mu_2^2=\nu_2+\Delta^2/4,\quad \nu_2=\lambda_1^2(n_1+1)+\lambda_2^2 n_2\ . \tag{5.4c}$$

The occupation numbers for the ground state, intermediate state, and upper state in this case are given respectively by

$$\langle \hat{S}_{33}(t)\rangle_\Lambda^{\rm gr}=1+[\langle \hat{n}_2(t)\rangle_\Lambda^{\rm gr}-\bar{n}_2]\ , \tag{5.5a}$$

$$\langle \hat{S}_{22}(t)\rangle_\Lambda^{\rm gr}=\langle \hat{n}_1(t)\rangle_\Lambda^{\rm gr}-\bar{n}_1\ , \tag{5.5b}$$

and

$$\langle \hat{S}_{11}(t)\rangle_\Lambda^{\rm gr}=\sum_{n_1,n_2}^{\infty}\lambda_2^2 n_2 P(n_{1,n_2})\frac{\sin^2(\mu_2 t)}{\mu_2^2}\ . \tag{5.5c}$$

The time evolution for $\hat{n}_1^2$, $\hat{n}_2^2$, and $\hat{n}_1\hat{n}_2$ is given as follows:

$$\langle \hat{n}_1^2(t)\rangle_\Lambda^{\rm gr}=\overline{n_1^2}+[\langle \hat{n}_1(t)\rangle_\Lambda^{\rm gr}-\bar{n}_1]+2\sum_{n_1,n_2}\lambda_1^2 n_1(n_1+1)P(n_1,n_2)\,|A_2|^2\ , \tag{5.6a}$$

$$\langle \hat{n}_2^2(t)\rangle_\Lambda^{\rm gr}=\overline{n_2^2}-[\langle \hat{n}_2(t)\rangle_\Lambda^{\rm gr}-\bar{n}_2]-2\sum_{n_1,n_2}P(n_1,n_2)\left[\lambda_2^2 n_2^2\frac{\sin^2(\mu_2 t)}{\mu_2^2}+\lambda_1^2 n_2(n_1+1)\,|A_2|^2\right], \tag{5.6b}$$

and

$$\langle \hat{n}_1(t)\hat{n}_2(t)\rangle_\Lambda^{\rm gr}=\overline{n_1 n_2}-\langle \hat{n}_1(t)\rangle_\Lambda^{\rm gr}+\sum_{n_1,n_2}P(n_1,n_2)[\lambda_1^2(n_1+1)n_2\,|A_2|^2+n_1\,|e^{(1/2)i\Delta t}+\lambda_2\sqrt{n_2}A_2|^2]\ . \tag{5.6c}$$

It is observed that if we take the second mode to be in vacuum, i.e., we assume $P(n_1,n_2)=P(n_1)\delta_{n_2 0}$ the system never develops in time whatever the nature of the mode one.

If the system starts with the mode one in vacuum, it will stay without any change and never develop in time.

2. *Atom initially in its intermediate state (of energy ω_2)*

In this special case we find the following

$$\langle \hat{n}_1(t)\rangle_\Lambda^{\rm in}=\langle \hat{n}_2(t)\rangle_\Lambda^{\rm gr} \tag{5.7a}$$

of (5.4b), but with 1↔2; and

$$\langle \hat{n}_2(t)\rangle_\Lambda^{\rm in}=\langle \hat{n}_1(t)\rangle_\Lambda^{\rm gr} \tag{5.7b}$$

of (5.4a), but with 1↔2. The occupation numbers for the three states are given by

$$\langle \hat{S}_{33}(t)\rangle_\Lambda^{\rm in}=\langle \hat{n}_2(t)\rangle_\Lambda^{\rm in}-\bar{n}_2\ , \tag{5.8a}$$

$$\langle \hat{S}_{22}(t)\rangle_\Lambda^{\rm in}=1+\langle \hat{n}_1(t)\rangle_\Lambda^{\rm in}-\bar{n}_1\ , \tag{5.8b}$$

and

$$\langle \hat{S}_{11}(t)\rangle_\Lambda^{\rm in}=\sum_{n_1,n_2}\lambda_1^2 n_1 P(n_1,n_2)\frac{\sin^2(\mu_1 t)}{\mu_1^2}\ , \tag{5.8c}$$

which is the same as (5.5c) with 1↔2. For the expectation values of $\hat{n}_1^2$, $\hat{n}_2^2$, and $\hat{n}_1\hat{n}_2$ they are the same as in the ground state but with 1↔2.

3. *Atom initially in its upper state (of energy ω_1)*

For this case we find the evolution in time for the photon number operators of the two modes is given by the formulas

$$\langle \hat{n}_1(t)\rangle_\Lambda^{u}=\bar{n}_1+\sum_{n_1,n_2}\lambda_1^2(n_1+1)P(n_1,n_2)\frac{\sin^2(\mu t)}{\mu^2}\ , \tag{5.9a}$$

$$\langle \hat{n}_2(t)\rangle_\Lambda^{u}=\bar{n}_2+\sum_{n_1,n_2}\lambda_2^2(n_2+1)P(n_1,n_2)\frac{\sin^2(\mu t)}{\mu^2}\ , \tag{5.9b}$$

where

$$\mu^2=\nu+\Delta^2/4,\quad \nu=\lambda_1^2(n_1+1)+\lambda_2^2(n_2+1)\ . \tag{5.9c}$$

The occupation numbers in this case for the ground, intermediate, and upper state are given by

$$\langle \hat{S}_{33}(t)\rangle_\Lambda^{u}=\langle \hat{n}_2(t)\rangle_\Lambda^{u}-\bar{n}_2\ , \tag{5.10a}$$

$$\langle \hat{S}_{22}(t)\rangle_\Lambda^{u}=\langle \hat{n}_1(t)\rangle_\Lambda^{u}-\bar{n}_1\ , \tag{5.10b}$$

and

$$\langle \hat{S}_{11}(t)\rangle_{\Lambda}^{u}=1-\sum_{n_1,n_2} P(n_1,n_2)\nu\frac{\sin^2(\mu t)}{\mu^2}\ , \qquad (5.10c)$$

respectively. Finally, the time dependence for the expectation values of $\hat{n}_1^2$, $\hat{n}_2^2$, and $\hat{n}_1\hat{n}_2$ is given by the formulas

$$\langle \hat{n}_1^2(t)\rangle_{\Lambda}^{u}=\overline{n}_1^2+[\langle \hat{n}_1(t)\rangle_{\Lambda}^{u}-\overline{n}_1] +2\lambda_1^2\sum_{n_1,n_2} n_1(n_1+1)P(n_1,n_2)\frac{\sin^2(\mu t)}{\mu^2}\ , \qquad (5.11a)$$

$\langle \hat{n}_2^2(t)\rangle_{\Lambda}^{u}$ is the same formula as above but with $1\rightarrow 2$ and

$$\langle \hat{n}_1(t)\hat{n}_2(t)\rangle_{\Lambda}^{u} =\overline{n_1n_2}+\sum_{n_1,n_2} P(n_1,n_2)[\lambda_1^2(n_1+1)n_2 +\lambda_2^2(n_2+1)n_1]\frac{\sin^2(\mu t)}{\mu^2}\ . \qquad (5.11b)$$

In this case the system develops in time even if the two modes start from vacuum states. From these formulas the different correlation functions can be calculated easily[10] once the initial conditions are stated.

B. The V configuration

We now look at another configuration, namely, the V configuration (see Fig. 2). In this model the Hamiltonian in RWA is given by

$$\hat{H}=\sum_{j=1}^{3}\omega_j\hat{S}_{jj}+\sum_{j=1}^{2}\Omega_j\hat{a}_j^{\dagger}\hat{a}_j+\lambda_1(\hat{S}_{13}\hat{a}_1+\hat{a}_1^{\dagger}\hat{S}_{31}) +\lambda_2(\hat{S}_{23}\hat{a}_2+\hat{a}_2^{\dagger}\hat{S}_{32})\ . \qquad (5.12a)$$

The detuning parameter Δ is given by

$$\Delta=\omega_1-\omega_3-\Omega_1=\omega_2-\omega_3-\Omega_2 \qquad (5.12b)$$

in contrast to (5.2) in the Λ configuration. The constants of motion in this configuration corresponding to (5.2) are given by

$$\hat{N}_1=\hat{n}_1+\hat{S}_{11},\ \ \hat{N}_2=\hat{n}_2+\hat{S}_{22}\ . \qquad (5.12c)$$

This means that the summation (rather than the difference in the Λ case) of the occupation number in a level and the photon number in the mode connecting it with the ground state is a constant of motion of the Hamiltonian (5.12a) under the conditions of exact two-photon resonance (5.12b).

1. Atom initially in its ground state (of energy ω_3)

The expectation value for the photon number operators in this case are given by

$$\langle \hat{n}_1(t)\rangle_{V}^{gr}=\overline{n}_1-\sum_{n_1,n_2}\lambda_1^2 n_1P(n_1,n_2)\frac{\sin^2(\mu' t)}{\mu'^2}\ , \qquad (5.13a)$$

and

$$\langle \hat{n}_2(t)\rangle_{V}^{gr}=\overline{n}_2-\sum_{n_1,n_2}\lambda_2^2 n_2P(n_1,n_2)\frac{\sin^2(\mu' t)}{\mu'^2}\ , \qquad (5.13b)$$

where

$$\mu'^2=\nu'+\Delta^2/4,\ \ \nu'=\lambda_1^2n_1+\lambda_2^2n_2\ . \qquad (5.14)$$

They may be compared with (5.9) of the Λ configuration where the stimulated emission process is apparent.[33] The occupation numbers for the levels of energy ω_3, ω_2, and ω_1 are given

$$\langle \hat{S}_{33}(t)\rangle_{V}^{gr}=1-\sum_{n_1,n_2}\frac{\nu'\sin^2(\mu' t)}{\mu'^2}P(n_1,n_2)\ , \qquad (5.15a)$$

$$\langle \hat{S}_{22}(t)\rangle_{V}^{gr}=\overline{n}_2-\langle \hat{n}_2(t)\rangle_{V}^{gr}\ , \qquad (5.15b)$$

and

$$\langle \hat{S}_{11}(t)\rangle_{V}^{gr}=\overline{n}_1-\langle \hat{n}_1(t)\rangle_{V}^{gr}\ ,$$

where $\langle \hat{n}_1(t)\rangle_{V}^{gr}$ and $\langle \hat{n}_2(t)\rangle_{V}^{gr}$ are given by (5.13a) and (5.13b) while we find

$$\langle \hat{n}_1^2(t)\rangle_{V}^{gr}=\overline{n_1^2}-[\langle \hat{n}_1(t)\rangle_{V}^{gr}-n_1] -2\sum_{n_1,n_2}\frac{\lambda_1^2n_1^2\sin^2(\mu' t)}{\mu'^2}P(n_1,n_2)\ , \qquad (5.16)$$

and the same expression for $\langle \hat{n}_2^2(t)\rangle_{V}^{gr}$ but with the subscripts 1 and 2 interchanged.

The summations appearing in the formulas (5.13) will be computed in Sec. VI for two squeezed modes of the field; and the phenomenon of collapses and revivals is discussed.

2. Atom initially in its intermediate state (of energy ω_2)

In this case we get

$$\langle \hat{n}_1(t)\rangle_{V}^{in}=\overline{n}_1-\sum_{n_1,n_2}\lambda_2^2(n_2+1)\,|A_1|^2P(n_1,n_2)\ , \qquad (5.17a)$$

and

$$\langle \hat{n}_2(t)\rangle_{V}^{in}=\overline{n}_2+\sum_{n_1,n_2}\lambda_2^2(n_2+1)\left[|A_1|^2+\frac{1}{\mu_1^2}\sin^2(\mu_1 t)\right]P(n_1,n_2)\ . \qquad (5.17b)$$

Compare these with (5.4) of the Λ configuration and atom in ground state.

The occupation numbers are given by

$$\langle \hat{S}_{33}(t)\rangle_V^{\text{in}} = \sum_{n_1,n_2} \lambda_2^2(n_2+1)\frac{\sin^2(\mu_1 t)}{\mu_1^2}P(n_1,n_2) \tag{5.18a}$$

while

$$\langle \hat{S}_{22}(t)\rangle_V^{\text{in}} = 1-\{\langle \hat{n}_2(t)\rangle_V^{\text{in}}-\bar{n}_2\} \tag{5.18b}$$

and

$$\langle \hat{S}_{11}(t)\rangle_V^{\text{in}} = -\{\langle \hat{n}_1(t)\rangle_V^{\text{in}}-\bar{n}_1\} \ . \tag{5.18c}$$

For the second-order operators, we find

$$\langle \hat{n}_1^2(t)\rangle_V^{\text{in}} = \overline{n_1^2}-[\langle \hat{n}_1(t)\rangle_V^{\text{in}}-\bar{n}_1]-2\sum_{n_1,n_2}\lambda_2^2 n_1(n_2+1)\,|A_1|^2P(n_1,n_2) \tag{5.19a}$$

and

$$\langle \hat{n}_2^2(t)\rangle_V^{\text{in}} = \overline{n_2^2}+[\langle \hat{n}_2(t)\rangle_V^{\text{in}}-\bar{n}_2]+2\sum_{n_1,n_2}\lambda_2^2 n_2(n_2+1)\left[|A_1|^2+\frac{\sin^2(\mu_1 t)}{\mu_1^2}\right]P(n_1,n_2) \ . \tag{5.19b}$$

3. *Atom initially in its upper state of energy ω_1*

When the atom starts in the upper state, we only observe that the same formulas of (5.17)—(5.19) go through with the interchange 1↔2. For example,

$$\langle \hat{n}_1(t)\rangle_V^{u} = \bar{n}_1+\sum_{n_1,n_2}\lambda_1^2(n_1+1)\left[|A_2|^2+\frac{\sin^2(\mu_2 t)}{\mu_2^2}\right]P(n_1,n_2) \tag{5.20}$$

which is (5.17b) with 1 and 2 exchanged.

The phenomenon of collapses and revivals is investigated when the summation appearing in this formula is calculated for two squeezed modes in Sec. VI.

C. Ladder (cascade) configuration

The scheme in this case is shown in Fig. 2. The Hamiltonian takes the form

$$\hat{H} = \sum_{i=1}^{3}\omega_1\hat{S}_{ii}+\sum_{i=1}^{2}\Omega_i\hat{a}_i^{\dagger}\hat{a}_i+\lambda_1(\hat{S}_{1,2}\hat{a}_1+\hat{a}_1^{\dagger}\hat{S}_{21})+\lambda_2(\hat{S}_{23}\hat{a}_2+\hat{a}_2^{\dagger}\hat{S}_{32}) \ . \tag{5.21a}$$

The detuning parameter for this model is

$$\Delta = -(\omega_1-\omega_2-\Omega_1) = \omega_2-\omega_3-\Omega_2 \tag{5.21b}$$

while the constants of motion are

$$\hat{N}_1 = \hat{n}_1+\hat{S}_{11}, \quad \hat{N}_2 = \hat{n}_2-\hat{S}_{33} \ . \tag{5.21c}$$

Compare these with their counterparts in the other configurations (5.1) and (5.2) and (5.12b) and (5.12c). Here, it is the sum of the photon number in the first mode and the occupation number in the upper level which is a constant of motion, while the difference between the photon number in the second mode and the occupation number in the lower level is a constant of motion. We now look at the time evolution for the different operators when the atom is initially in one of its states, in this configuration.

1. *Atom initially in its ground state*

The expectation value for the photon number operators is found to be

$$\langle \hat{n}_1(t)\rangle_L^{\text{gr}} = \bar{n}_1-\sum_{n_1,n_2}P(n_1,n_2)\lambda_2^2 n_2\,|A_0|^2 \ , \tag{5.22a}$$

$$\langle \hat{n}_2(t)\rangle_L^{\text{gr}} = \bar{n}_2-\sum_{n_1,n_2}P(n_1,n_2)\lambda_2^2 n_2\left[|A_0|^2+\frac{\sin^2(\mu' t)}{\mu'^2}\right], \tag{5.22b}$$

where

$$A_0 = \frac{\lambda_1\sqrt{n_1}}{\nu'}\left[\cos(\mu' t)-e^{i(\Delta/2)t}+i\frac{\Delta}{2\mu'}\sin(\mu' t)\right], \tag{5.22c}$$

with μ' and ν' given by (5.14).

From these and the constants of motion in (5.12c), we can calculate the occupation numbers for the ground, intermediate and upper states. They are, respectively,

$$\langle \hat{S}_{33}(t)\rangle_L^{\text{gr}} = 1+[\langle \hat{n}_2(t)\rangle_L^{\text{gr}}-\bar{n}_2] \ , \tag{5.23a}$$

$$\langle \hat{S}_{22}(t)\rangle_L^{\text{gr}} = \sum_{n_1,n_2}P(n_1,n_2)\lambda_2^2 n_2\frac{\sin^2(\mu' t)}{\mu'^2} \ , \tag{5.23b}$$

and

$$\langle \hat{S}_{11}(t)\rangle_L^{\text{gr}} = \bar{n}_1-\langle \hat{n}_1(t)\rangle_L^{\text{gr}} \ . \tag{5.23c}$$

The second-order operators develop in time according to the following:

$$\langle \hat{n}_1^2(t)\rangle_L^{\rm gr}=\overline{n_1^2}-[\langle \hat{n}_1(t)\rangle_L^{\rm gr}-\bar{n}_1] -2\sum_{n_1,n_2} P(n_1,n_2)\lambda_2^2 n_2 \,|A_0|^2 , \quad (5.24a)$$

$$\langle \hat{n}_2^2(t)\rangle_L^{\rm gr}=\overline{n_2^2}-[\langle \hat{n}_2(t)\rangle_L^{\rm gr}-\bar{n}_2] -2\sum_{n_1,n_2} P(n_1,n_2)\lambda_2^2 n_2 \left[|A_0|^2+\frac{\sin^2(\mu' t)}{\mu'^2}\right] , \quad (5.24b)$$

while the correlation function

$$\langle \hat{n}_1(t)\hat{n}_2(t)\rangle_L^{\rm gr}=\overline{n_1 n_2}-\sum_{n_1,n_2} P(n_1,n_2)\lambda_2^2 n_2 \left[n_1 \frac{\sin^2(\mu' t)}{\mu'^2}+(n_1+n_2-1)|A_0|^2\right] . \quad (5.24c)$$

2. *Atom initially in its intermediate state*

The photon numbers in the two modes are given by

$$\langle \hat{n}_1(t)\rangle_L^{\rm in}=\overline{n_1}-\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2 n_1 \frac{\sin^2(\mu_1 t)}{\mu_1^2} \quad (5.25a)$$

and

$$\langle \hat{n}_2(t)\rangle_L^{\rm in}=\bar{n}_2+\sum_{n_1,n_2} P(n_1,n_2)\lambda_2^2(n_2+1)\frac{\sin^2(\mu_1 t)}{\mu_1^2} , \quad (5.25b)$$

which means stimulated absorption for mode one and emission for mode two. When we use the constants of motion of (5.21c), the occupation numbers in the atomic levels are obtained as follows:

$$\langle \hat{S}_{33}(t)\rangle_L^{\rm in}=\langle \hat{n}_2(t)\rangle_L^{\rm in}-\bar{n}_2 , \quad (5.26a)$$

$$\langle \hat{S}_{22}(t)\rangle_L^{\rm in}=\sum_{n_1,n_2} P(n_1,n_2)\left[\cos^2(\mu_1 t)+\frac{\Delta^2}{4\mu^2}\sin^2(\mu_1 t)\right] , \quad (5.26b)$$

and

$$\langle \hat{S}_{11}(t)\rangle_L^{\rm in}=\bar{n}_1-\langle \hat{n}_1(t)\rangle_L^{\rm in} . \quad (5.26c)$$

In this case, the expectation values for $\hat{n}_1^2$, $\hat{n}_2^2$, and $\hat{n}_1\hat{n}_2$ are

$$\langle \hat{n}_1^2(t)\rangle_L^{\rm in}=\overline{n_1^2}-[\langle \hat{n}_1(t)\rangle_L^{\rm in}-\bar{n}_1] -2\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2 n_1 \frac{\sin^2(\mu_1 t)}{\mu_1^2} , \quad (5.27a)$$

$$\langle \hat{n}_2^2(t)\rangle_L^{\rm in}=\overline{n_2^2}+[\langle \hat{n}_2(t)\rangle_L^{\rm in}-\bar{n}_2] +2\sum_{n_1,n_2} P(n_1,n_2)\lambda_2^2 (n_2+1) \frac{\sin^2(\mu_1 t)}{\mu_1^2} , \quad (5.27b)$$

and

$$\langle \hat{n}_1(t)\hat{n}_2(t)\rangle_L^{\rm in} =\overline{n_1 n_2}+[\langle \hat{n}_1(t)-\bar{n}_1] -\sum_{n_1,n_2} P(n_1,n_2) n_1 n_2 (\lambda_1^2+\lambda_2^2)\frac{\sin^2(\mu_1 t)}{\mu_1^2} . \quad (5.27c)$$

3. *Atom initially in its upper state*

The expectation values for the photon number operators in this case is given by

$$\langle \hat{n}_1(t)\rangle_L^{u}=\bar{n}_1+\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2(n_1+1) \times\left[\frac{\sin^2(\mu t)}{\mu^2}+|A|^2\right] , \quad (5.28a)$$

and

$$\langle \hat{n}_2(t)\rangle_L^{u}=\bar{n}_2+\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2(n_1+1)|A|^2 , \quad (5.28b)$$

where

$$A=\lambda_2\frac{\sqrt{n_2+1}}{\nu}\left[\cos(\mu t)-e^{i(\Delta/2)t}+\frac{i\Delta}{2}\frac{\sin(\mu t)}{\mu}\right] , \quad (5.29)$$

where μ and ν are given by (5.2c).

The occupation numbers for the energy levels are

$$\langle \hat{S}_{33}(t)\rangle_L^{u}=\langle \hat{n}_2(t)\rangle_L^{u}-\bar{n}_2 , \quad (5.30a)$$

$$\langle \hat{S}_{22}(t)\rangle_L^{u}=\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2(n_1+1)\frac{\sin^2(\mu t)}{\mu^2} , \quad (5.30b)$$

$$\langle \hat{S}_{11}(t)\rangle_L^{u}=1-[\langle \hat{n}_1(t)\rangle_L^{u}-\bar{n}_1] . \quad (5.30c)$$

Compare these formulas where the stimulated emission is apparent with the formulas (5.22) and (5.23) where the stimulated absorption process takes place.

The time evolution for the second-order operators takes the following:

$$\langle \hat{n}_1^2(t)\rangle_L^u = \overline{n_1^2} - [\langle \hat{n}_1(t)\rangle_L^u - \bar{n}_1] + 2\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2(n_1+1)^2 \times \left[\frac{\sin^2(\mu t)}{\mu^2} + |A|^2\right], \quad (5.31a)$$

$$\langle \hat{n}_2^2(t)\rangle_L^u = \overline{n_2^2} - [\langle \hat{n}_2(t)\rangle_L^u - \bar{n}_2] + 2\sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2(n_1+1)(n_2+1)|A|^2, \quad (5.31b)$$

and the correlation function

$$\langle \hat{n}_1\hat{n}_2\rangle_L^u = \overline{n_1 n_2} + \sum_{n_1,n_2} P(n_1,n_2)\lambda_1^2(n_1+1) \times \left[\lambda_2^2 n_2 \frac{\sin^2(\mu t)}{\mu^2} + (n_1+n_2+1)|A|^2\right]. \quad (5.31c)$$

With this, we conclude the discussion of the different configurations for the three-level atom and two-mode system.

VI. INTERACTION WITH SQUEEZED LIGHT

We investigate in this section the phenomenon of collapses and revivals in the system of a three-level atom and two squeezed modes. The distribution function $P(n_1,n_2)$ which appears in Sec. V, is assumed to be in the form

$$P(n_1,n_2) = P(n_1)P(n_2), \quad (6.1)$$

which means that the two modes are initially decorrelated. For the squeezed mode $|\alpha,r\rangle$ the distribution function $P(n)$ is given by[42,43]

$$P(n) = [(n!)\cosh r]^{-1}(\tfrac{1}{2}\tanh r)^n |H_n[\beta/\sqrt{\sinh(2r)}]|^2 \times \exp[-|\beta|^2 + \tfrac{1}{2}\tanh r(\beta^2+\beta^{2*})], \quad (6.2)$$

where $\beta = \alpha\cosh r + \alpha^*\sinh r$, when $r\to 0$ we get the coherent state $|\alpha\rangle$.

We find the following values for the mean photon number $\bar{n}$ and variance:[42,43]

$$\bar{n} = |\alpha|^2 + \sinh^2 r \quad (6.3a)$$

and

$$\mathrm{var}(n) = \tfrac{1}{2}\sinh^2(2r) + |\alpha|^2[\cosh(2r) - \sinh(2r)\cos(2\theta)], \quad (6.3b)$$

where $r\geq 0$, and θ is the angle between the direction of the coherence excitation and the direction of squeezing.[44,17] When $|\alpha|$ is dominant in the mean photon number (i.e., $|\alpha| \gg r$) the distribution is sub-Poissonian for $\theta=0$ and super-Poissonian when $\theta=\pi/2$. When $\alpha\to 0$ the distribution is always broader than Poissonian.

$P(n_1)$ and $P(n_2)$ are given by the expression (6.2) with (α_1,r_1) and (α_2,r_2) for the two modes, respectively. We consider the effect of collapses and revivals in the system. We shall neglect the effect of detuning (i.e., write $\Delta=0$), and asymmetries in interaction (i.e., take $\lambda_1=\lambda_2=\lambda$). We concentrate on the sub- and super-Poissonian distribution effects on the phenomenon of collapses and revivals in this system.

To show these we compute the following quantities:

$$\Delta n_1^{(g)} = \langle \hat{n}_1(t)\rangle_V^{gr} - \bar{n}_1, \quad (6.4a)$$

where we use Eq. (5.13a), and

$$\Delta n_1^{(u)} = \langle n_1(t)\rangle_V^u - \bar{n}_1, \quad (6.4b)$$

where Eq. (5.20) is used. The behavior of these quantities is shown in the following set of figures.

We first consider sub-Poissonian distributions, i.e., we take $\theta_1=\theta_2=0$. In Figs. 3 and 4 we present the time development for $\Delta n_1^{(g)}$ and $\Delta n_1^{(u)}$, respectively, for $|\alpha_1| = |\alpha_2| = 4$ and $r_1=r_2=\sinh^{-1}(1)$. The collapses and revivals are very clear in these figures. They oscillate in a very fast way for short times. In Fig. 3, we note that the amplitudes after the first and second collapses are almost the same. While Fig. 4 shows a bigger amplitude after the second collapse in contrast to Fig. 3, and the case of the system of a two-level atom and a single mode.[17] This behavior is a characteristic feature of the three-level atom system.[36,33]

The same behavior appears in the interaction with two coherent modes.[36] We would expect this similarity because the coherent excitation is rather dominant in $\bar{n}$ of (6.3a) for the set of parameters taken ($|\alpha_1|^2 = 16 \gg \sinh^2 r_1 = 1$). The increase in the amplitude after the second collapse could be explained by writing (5.20) in the following:

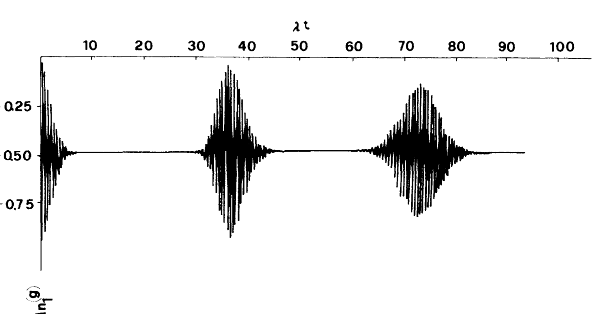

FIG. 3. Evolution of $\Delta n_1^{(g)}$ of Eq. (6.4a) against λt for $\Delta=0$, $\lambda_1=\lambda_2=\lambda$, $\theta_1=\theta_2=0$, $\alpha_1=\alpha_2=4$, and $r_1=r_2=\sinh^{-1}1$ (i.e., $\bar{n}_1=\bar{n}_2=17$).

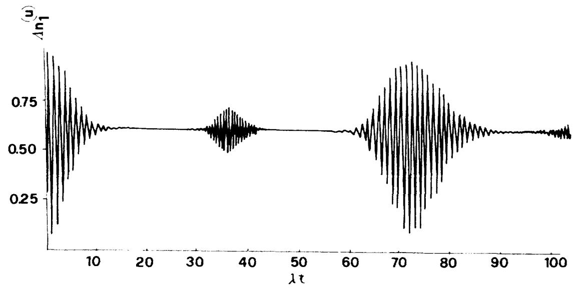

FIG. 4. Evolution of $\Delta n_1^{(u)}$ of Eq. (6.4b) against λt for the same parameters as in Fig. (3).

$$\Delta n_1^{(u)} = \sum_{n_1,n_2} P(n_1,n_2)\frac{n_1+1}{(n_1+n_2+1)^2} \times[(n_1+n_2+1)\sin^2(\sqrt{n_1+n_2+1}\lambda t) + 4n_2\sin^2(\tfrac{1}{2}\sqrt{n_1+n_2+1}\lambda t)] . \quad (6.5)$$

It is apparent that there are oscillations with frequencies $(\lambda/2)\sqrt{n_1+n_2+1}$, and with double its value. Since the distributions are dominated by the coherence excitations; thus they are peaked around $(\bar{n}_1+\bar{n}_2)$.[13] Therefore, the first part of the sum contributes mainly to the first main block, then the two parts contribute to the second block, and hence the increase in the amplitude.

Apart from this, the figures also differ from the two-level atom and a single-squeezed-mode system[17] in that the oscillations persist for a longer time before they collapse down, then the period of collapse is longer than in the mentioned system.[17] This could be understood when we use the saddle-point approximation,[13] but with the mean photon number $\bar{n}_1+\bar{n}_2=2\bar{n}_1$.

For distributions of initial super-Poissonian statistics, the behavior of $\Delta n_1^{(g)}$ and $\Delta n_1^{(u)}$ have some differences in the details from the foregoing mentioned discussions. Figures 5 and 6 show their behavior for the set of parameters $\theta_1=\theta_2=\pi/2$ and the same prescribed values for the other parameters (i.e., $|\alpha_1|=|\alpha_2|=4$ and $r_1=r_2=\sinh^{-1}1$). The first collapse occurs in the two figures,

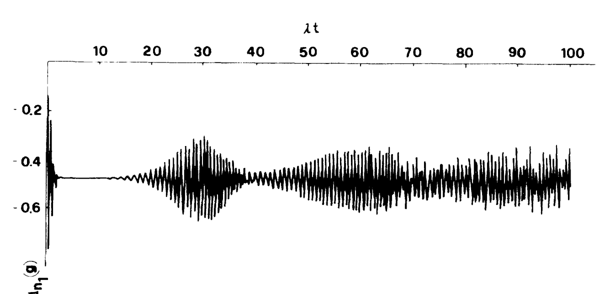

FIG. 5. Evolution of $\Delta n_1^{(g)}$ against λt for $\Delta=0$, $\lambda_1=\lambda_2=\lambda$, $\theta_1=\theta_2=\pi/2$, $|\alpha_1|=|\alpha_2|=4$, and $r_1=r_2=\sinh^{-1}1$.

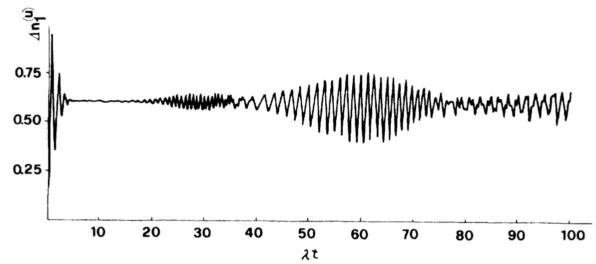

FIG. 6. Evolution of $\Delta n_1^{(u)}$ against λt for the same parameters as in Fig. (5).

while the second collapse disappears and chaotic behavior starts to build up after that. It is observed that the oscillations are somewhat slower while the times during which the oscillation take place are longer than in the sub-Poissonian case. Their centers are shifted towards the origin and the amplitudes are smaller than their corresponding values in the foregoing figures.

The chaotic behavior is the only feature that is present when $|\alpha|\to 0$. This is apparent in Figs. 7 and 8 for $\Delta n_1^{(g)}$ and $\Delta n_1^{(u)}$, respectively, where we have considered the following values for the different parameters $\theta_1=\theta_2=0$, $|\alpha_1|=|\alpha_2|=0$, and $r_1=r_2=\sinh^{-1}4$. Each of the two modes is initially in the squeezed vacuum state. The two distributions $P(n_1)$ and $P(n_2)$ are very broad and peaked around $\bar{n}=0$, Ref. 17. Thus, the chaotic behavior is apparent from the beginning. It does not start somewhat later as in the thermal distributions considered in Ref. 36, where the distributions are peaked around $\bar{n}\neq 0$.

VII. MULTIPHOTON PROCESSES

In this section, we look at processes where more than one photon are involved in the interaction. We assume that the transition between the two levels ω_1 and ω_{j+1} is effected through m_j photons of energy Ω_j of the jth mode. Thus, instead of the form (1.2) for the Hamiltonian we now have

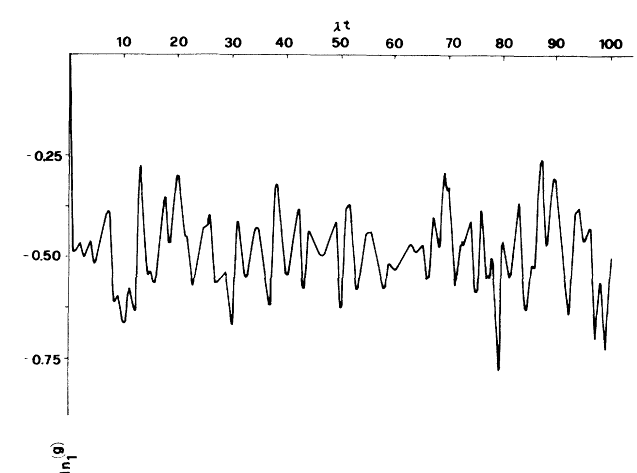

FIG. 7. Evolution of $\Delta n_1^{(g)}$ against λt for $\Delta=0$, $\lambda_1=\lambda_2=\lambda$, $\theta_1=\theta_2=0$, $|\alpha_1|=|\alpha_2|=0$, and $r_1=r_2=\sinh^{-1}4$.

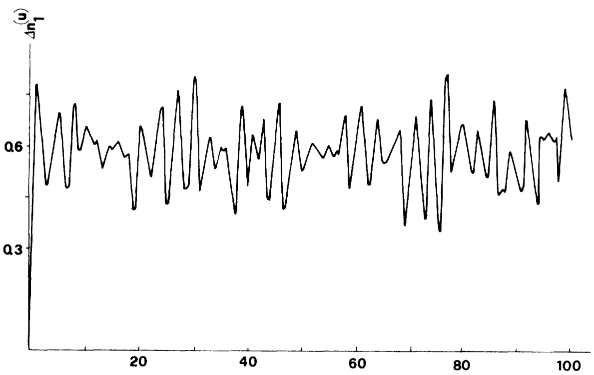

FIG. 8. Evolution of $\Delta n_1^{(u)}$ against λt for the same parameters as in Fig. (7).

$$\hat{H}=\sum_{j=1}^{N}\omega_j\hat{S}_{j,j}+\sum_{j=1}^{N-1}\Omega_j\hat{a}_j^\dagger\hat{a}_j+\sum_{j=1}^{N}\lambda_j[\hat{S}_{1,j+1}\hat{a}_j^{m_j}+(\hat{a}_j^\dagger)^{m_j}\hat{S}_{j+1,1}] . \quad (7.1)$$

This form can be cast in a form close to (1.2), by using the generalized boson operators $\hat{b}_i^\dagger$ defined as follows:[45]

$$\hat{a}_i^{m_i}=\hat{b}_i\left[\frac{(n_i)!}{(n_i/m_i)(n_i-m_i)!}\right]^{1/2}=\hat{b}_if_i(\hat{n}_i) . \quad (7.2a)$$

The relation between the operators $\hat{n}_{b_i}=\hat{b}_i^\dagger\hat{b}_i$ and $\hat{n}_i=\hat{a}_i^\dagger\hat{a}_i$ is given by[45]

$$\hat{n}_i=m_i\hat{n}_{b_i} . \quad (7.2b)$$

The form (7.1) now takes the form

$$\hat{H}=\sum_{j=1}^{N}\omega_j\hat{S}_{j,j}+\sum_{j=1}^{N-1}m_j\Omega_j\hat{b}_j^\dagger\hat{b}_j+\sum_{j=1}^{N}\lambda_j[\hat{S}_{1,j+1}\hat{b}_jf(\hat{n}_j)+f(\hat{n}_j)\hat{b}_j^\dagger\hat{S}_{j+1,1}] . \quad (7.3)$$

For this model to be exactly solvable, detuning parameter Δ of (1.1) now takes the form

$$\Delta=\omega_1-\omega_{j+1}-m_j\Omega_j \quad (j=1,\dots,N-1) . \quad (7.4)$$

The whole argument of Secs. II and III go through, but with each operator in (2.4) $\hat{a}_i$ replaced by $\hat{b}_if(\hat{n}_i)$. For example, we find that the corresponding formula to (4.7a) in this case is given by

$$\langle\hat{n}_j(t)\rangle_{\text{multi}}^{k+1}=\bar{n}_j+m_j\sum_{\{j_j\}}P(\{\hat{n}_j\})\lambda_j^2\lambda_k^2\left[\frac{(n_j+m_j)!}{n_j!}\right]\left[\frac{n_k!}{(n_k-m_k)!}\right]|M_k|^2,\ j\neq k , \quad (7.5a)$$

and the corresponding formula to (4.7b) is given by

$$\langle\hat{n}_k(t)\rangle_{\text{multi}}^{k+1}=\bar{n}_k-m_k\sum_{\{n_j\}}P(\{n_j\})\lambda_k^2\left[\frac{n_k!}{(n_k-m_k)!}\right]\left[\frac{\sin^2(\mu_kt)}{\mu_k^2}+(\nu_k-\lambda_k^2)\left[\frac{n_k!}{(n_k-m_k)!}\right]|M_k|^2\right] , \quad (7.5b)$$

where the ν_k and μ_k in this case are given by

$$\nu_k=\lambda_k^2\left[\frac{n_k!}{(n_k-m_k)!}\right]+\sum_{\substack{i\\(i\neq k)}}\lambda_i^2\frac{(n_i+m_i)!}{n_i!} , \quad (7.6a)$$

$$\mu_k^2=\nu_k+\frac{\Delta^2}{4} ,$$

while

$$M_k=\nu_k^{-1}\left[\cos(\mu_kt)-e^{i(\Delta/2)t}+i\frac{\Delta}{2\mu_k}\sin(\mu_kt)\right] . \quad (7.6b)$$

When the atom is initially in its upper state, we find in this case that

$$\langle\hat{n}_j(t)\rangle_{\text{multi}}^{u}=\bar{n}_j+m_j\sum_{\{n_i\}}P(\{n_i\})\lambda_j^2\left[\frac{(n_j+m_j)!}{n_j!}\right]\frac{\sin^2(\mu t)}{\mu^2}$$

$$j=1,2,\dots,N-1 \quad (7.7)$$

with

$$\mu^2=\nu^2+\frac{\Delta^2}{4},\quad \nu^2=\sum_i\lambda_i^2\frac{(n_i+m_i)!}{n_i!} , \quad (7.8)$$

which corresponds to (4.10a) for this case.

When the number N of the energy levels is restricted to three, we get the case of the three-level atom in interaction with multiphotons of the two modes.

VIII. DISCUSSION AND CONCLUSIONS

The model for the interaction between an N-level atom and $N-1$ modes presented in this investigation contains a detuning parameter Δ, which is a generalization to earlier models.[33–36,41] In general, the following features are manifest in this model.

Independent of the specific form of the initial probability distribution function of the photons in the modes; the system develops in time when the atom starts from its uppermost state of energy ω_1, whatever the number of photons in the field modes, even if all modes are in vacuum state. On the other hand, the system never develops in time when the atom starts from any state of energy $\omega_{j+1}\neq\omega_1$, and the mode that connects this energy level with ω_1 is in vacuum. When this mode is not in the vacuum state the system evolves. For $t>0$ the mean photon number in the mode that connects the energy level that was occupied at $t=0$ with the uppermost level, is always less than its initial value, while in the rest of modes, it exceeds its initial value. It is noted that the mean photon numbers in all the modes and the occupation number in the uppermost state, i.e., $(\hat{S}_{11}+\sum_{i=1}^{N-1}\hat{n}_i)$ is a constant of motion.

The case of the three-level atom and two-modes is discussed in some detail with the different configurations considered. The relations between the statistical quantities in the different configurations are shown. Then, the interaction with two modes in the squeezed state is considered numerically and the phenomenon of collapses and revivals is shown. However, some differences from the case of the interaction between a two-level atom and a squeezed mode[17] are exhibited. These differences are characteristic to the three-level-atom system, and they appear in the interaction with coherent modes.[33,36] Squeezed light gives an opportunity to discuss the effect of sub- and super-Poissonian distributions for the two modes on the collapses and revival phenomena in the system. This is apparent when we compare Figs. 3 and 4 and Figs. 5 and 6. It is obvious that for highly squeezed light and for longer periods the system develops to a chaotic behavior. On the other hand, when the coherence excitation is predominant, the system develops almost like in interaction with coherent light, and it takes a very long time before chaotic behavior starts to appear especially for sub-Poissonian distribution ($\theta_1=\theta_2=0$). However, chaotic behavior is more pronounced when we take $\theta_1=\theta_2=\pi/2$, i.e., super-Poissonian distribution.

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*Present address: Department of Mathematics, Faculty of Education, Umm-Al-Qurra University, Taif, Saudi Arabia.

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