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SU(1,1), SU(2), and SU(3) coherence-preserving Hamiltonians and time-ordering techniques

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The Wei-Norman ordering technique is utilized to present a unified approach to the time ordering for SU(2), SU(1,1), and SU(3) coherence-preserving Hamiltonians. It is shown that the characteristic equations of the ordering method can be cast in the form of a generalized Bloch equation in an SU(2), SU(1,1), and SU(3) space. The laws of conservation, linked to the Casimir invariants of the algebra, are also directly derived from the Wei-Norman characteristic equations.

I. INTRODUCTION

Algebraic methods have been recently exploited to deal with a very large number of problems in quantum electronics. They have been particularly useful for the analysis of quantum-mechanical systems, subjected to strong and time-dependent perturbations.[1]

Unitary symmetries, the well-known keynote to penetrate the ordering code of elementary particles[2] and nuclear spectra,[3] have indeed been shown to be a potentially unique tool to understand the dynamics of multilevel systems.

On analyzing the equations of motion of a three-level atom driven by an electromagnetic interaction, Elgin[4] and Hioe and Eberly[5] exploited the SU(3) symmetry and proved that the "torque" equation of the two-level case[6] can be generalized to an equation of motion describing a rotation in an eight-dimensional space. Furthermore, they also indicated how unitary groups may usefully be exploited to discover a number of previously unforeseen constants of motion. Later on the techniques suggested in the above quoted papers have been applied to a multilevel-type device like the free-electron laser[7] and it has been shown that its dynamics is governed by equations formally identical to those stated in Refs. 4 and 5.

The principle of unitary invariance applied to the study of N-level quantum systems was further extended by Hioe in a series of interesting and elegant papers.[8] In those notes the conditions have been discussed under which the dynamical space of a quantum system strongly interacting can be decomposed into independent subspaces, thus indicating subsets of independent constants of motion.[9] In developing this view to the N-level dynamics Hioe established a clear link with elementary particle physics, also suggesting the fascinating idea that a parallel can be drawn between the existence of the various types of quarks and the existence of the corresponding solitons in the decomposed dynamical subspace (see the second of Ref. 8).

More recently, renewed interest in group-theoretic methods has been motivated by the rediscovery of rigorous time-ordering techniques proposed more than two decades ago.[10,11] In fact, the ordering problems, arising in dealing with the evolution of quantum systems governed by Hamiltonian time-dependent linear combinations of Lie group generators, can be conveniently treated by means of the algebraic method developed by Wei-Norman.[11] This technique, resorted and suitably rehandled,[12] has proved a great help in treating a wide class of problems, ranging from the harmonic oscillator with time-dependent frequency to the propagation in optical fibers.[13] It has been stressed that the obtained results holds, *mutatis mutandis,* for any problem whose dynamical variables[1,12–14] can be embedded to form an SU(2) or SU(1,1) group. In this paper we also discuss the relevance of the Wei-Norman (WN) technique to the SU(3) case, solving for the first time the problem of finding the appropriate Weyl-type disentangling for this group. In so doing we indicate the possibility of treating, from a general point of view, the temporal behavior of quantum states ruled by Hamiltonian time-dependent linear combination of SU(3) group generators.

We develop in this note a unified view to the problem of the time ordering for SU(2), SU(1,1), and SU(3) coherence-preserving Hamiltonians.

We show that the characteristic equations of the ordering procedure can be cast in the form of generalized Bloch rotations in SU(2), SU(1,1), and SU(3) spaces, thus getting a direct connection with previous results (see, e.g., the already quoted Refs. 4–6 and 14).

The problems related to the laws of conservation are also discussed within a more direct framework and are derived from the Casimir invariants related to the generalized Bloch rotations.[15]

The paper is organized in three sections and one Appendix. In the forthcoming sections we discuss the ordering theorems and the relevant generalized Bloch equations. Finally, to stress the flexibility of the method, we show in the Appendix how the procedure can be extended to a more complicated Hamiltonian operator.

II. SU(2), SU(1,1), AND SU(3) ORDERING THEOREMS

To make the paper self-consistent we sketch briefly the main aspect of the WN method. We consider the following Hamiltonian operator:

$$\hat{H}(t)=\sum_{j=1}^{m\leq n} a_j(t)\hat{L}_j \ , \tag{1}$$

where $\hat{L}_j$ are the generators of the Lie algebra, $a_j(t)$ are linearly independent functions of t, and the index j runs from 1 to $m \leq n$, n being the dimensionality of the algebra.

The obliged step, once the time behavior of the quantum states driven by (1) is needed, is the search of the evolution operator $\hat{U}(t)$. Wei and Norman looked for a convenient form of this operator, which preserves its unitary nature without any *a priori* recourse to perturbative methods as in the more conventional Feynman-Dyson expansion (see Refs. 12 and 16 for further comments). The most natural expression for the $\hat{U}$ operator is the following:

$$\hat{U}(t)=\prod_{j=1}^{n} \exp[g_j(t)\hat{L}_j]\hat{1}, \ \ g_j(0)=0 \ . \tag{2}$$

The problem of finding the explicit expression of $\hat{U}(t)$ reduces therefore to the equivalent one of looking for the appropriate connection between $g_j(t)$ and $a_j(t)$ functions. Without entering the details of the derivation, otherwise quite straightforward, we recall that the two sets of functions are connected by[11,12]

$$\sum_{j=1}^{n} a_j(t)\hat{L}_j=\sum_{j=1}^{n}\sum_{i=1}^{n} \dot{g}_i^{(t)}\xi_{j,i}\hat{L}_j \ , \tag{3}$$

where the overdot means time derivative and $\xi_{i,j}$ depend on the algebra structure constants. From the already invoked linear independence, we finally get

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \xi_{1,2} & \cdots & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n,1} & \cdots & \xi_{n,n} \end{bmatrix} \begin{bmatrix} \dot{g}_1 \\ \vdots \\ \dot{g}_n \end{bmatrix} . \tag{4}$$

It is therefore clear that once the explicit forms of $a_j(t)$ and $\xi_{i,j}$ are known one can determine the functions $g_j(t)$ by solving a system of nonlinear differential equations.

We can now utilize the above results to discuss the case of the real simple split three-dimensional Lie algebra. We therefore specialize the Hamiltonian operator (1) to

$$\hat{H}(t)=\frac{\omega(t)}{2}\hat{F}_0+\Omega^*(t)\hat{F}_+-\Omega(t)\hat{F}_- \ , \tag{5}$$

where $\omega(t)$ and $\Omega(t)$ are time-dependent nonsingular functions real and complex, respectively, and the $\hat{F}$ operators obey the commutation relations

$$[\hat{F}_0,\hat{F}_\pm]=\pm 2\lambda\hat{F}_\pm, \ \ [\hat{F}_+,\hat{F}_-]=-\delta\hat{F}_0 \ . \tag{6}$$

It is straightforwardly recognized that the SU(2) and SU(1,1) algebraic structures can be recovered by means of the correspondence

$$\begin{aligned} &\mathrm{SU}(2)\leftrightarrow \begin{cases}\hat{F}_0=2\hat{J}_3, \ \ \hat{F}_+=\hat{J}_+, \ \ \hat{F}_-=-\hat{J}_- \\ \lambda=\delta=1 \ ,\end{cases} \\ &\mathrm{SU}(1,1)\leftrightarrow \begin{cases}\hat{F}_0=2\hat{K}_3, \ \ \hat{F}_+=\hat{K}_+, \ \ \hat{F}_-=-\hat{K}_- \\ \lambda=-\delta=1 \ .\end{cases} \end{aligned} \tag{7}$$

According to (2) the evolution operator can be written as

$$\hat{U}(t)=\exp\left[\left[h(t)-i/2\int_0^t \omega(t')dt'\right]\hat{F}_0\right] \times\exp[g(t)\hat{F}_+]\exp[f(t)\hat{F}_-]\hat{1} \ . \tag{8}$$

Furthermore, from the relation (4) one gets the following system of differential equations for h, g, and f functions:

$$\begin{aligned} &\dot{h}(t)=\delta g(t)\dot{f}(t) \ , \\ &\dot{g}(t)=-i\Omega^*(t)\exp\left[-2h(t)+i\int_0^t\omega(t')dt'\right] \\ &\qquad -\dot{h}(t)g(t) \ , \\ &\dot{f}(t)=i\Omega(t)\exp\left[2h(t)-i\int_0^t\omega(t')dt'\right] \ . \end{aligned} \tag{9}$$

It is also well known that the solution of (9) depends on a single Riccati equation, namely

$$\dot{u}-u^2+p(t)u-\delta\,|\,\Omega(t)\,|^2=0, \ \ \dot{h}(t)=u(t) \ ,$$

$$u(0)=0, \ \ p(t)=-\frac{\dot{\Omega}(t)}{\Omega(t)}+i\omega(t) \ .$$

It is, however, more convenient to introduce the functions[1]

$$\begin{aligned} &\mathscr{H}(t)=e^{-h(t)}, \ \ \mathscr{H}(0)=1, \ \ \dot{\mathscr{H}}(0)=0 \ , \\ &\mathscr{F}(t)=f(t)e^{-h(t)}, \ \ \mathscr{F}(0)=0, \ \ \dot{\mathscr{F}}(0)=i\Omega(0) \ , \end{aligned} \tag{10}$$

which both obey the same second-order differential equation,

$$\ddot{y}(t)+p(t)\dot{y}+\delta\,|\,\Omega(t)\,|^2 y=0 \ . \tag{11}$$

The introduction of the functions (10) has a threefold motivation.

(1) They enter directly in the elements of the transition matrix, as shown below.

(2) They can be exploited to establish a connection between the ordering theorems and the SU(2) and SU(1,1) Bloch vector evolution as discussed in Sec. II A (see also Ref. 17).

(3) Finally, also in view of the second point, their generalization to the SU(3) case results in a simpler form of the differential equations involved in.

In the following we give an example of the structure of the matrix elements involved in the evolution of SU(2) and SU(1,1) states.

A. SU(2) transition matrix

The matrix elements relevant to this case take the form

$$S_{m,n}(t)=\langle J,n\mid \hat{U}(t)\mid J,m\rangle \tag{12}$$

(the ket$|J,m\rangle$ labels a generic angular momentum state) and can be calculated exploiting the properties of the angular momentum operators; after some algebra one gets

$$S_{m,n}(t)=\left[\begin{bmatrix}J+n_>\\J+n_<\end{bmatrix}\begin{bmatrix}J-n_<\\J-n_>\end{bmatrix}\right]^{1/2}\exp\left[-in\int_0^t\omega(\tau)d\tau\right]\mathcal{H}^{-(n+m)}[\mathrm{sgn}(n-m)|\mathcal{F}|]^{n_>-n_<}$$

$$\times\exp[-i\chi(n-m)]\,{}_2F_1(-J-n_<,J-n_<+1;n_>-n_<+1,|\mathcal{F}|^2)$$

$$[\chi=\arg(\mathcal{F}),\quad n_>=\max(m,n),\quad n_<=\min(m,n)]\ ,\qquad(13)$$

where ${}_2F_1(\ )$ is the hypergeometric function.

B. SU(1,1) transition matrix

The matrix elements to be evaluated in SU(1,1) problems are of the type

$$S_{m,n}(t)=\langle n,k\,|\,\hat{U}(t)\,|\,m,k\rangle\ ,\qquad(14)$$

where m and n are integers and k is the Bergman index.[18] Using the properties of the $\hat{K}$ operators

$$\hat{K}_3\,|\,n,k\rangle=(n+k)\,|\,n,k\rangle\ ,$$

$$\hat{K}_+\,|\,n,k\rangle=\sqrt{(n+1)(n+2k)}\,|\,n+1,k\rangle\ ,$$

$$K_-\,|\,n,k\rangle=\sqrt{n(n+2k-1)}\,|\,n-1,k\rangle$$

(the notation $|\,n,k\rangle$ means $|\,2n\rangle$ or $|\,2n+1\rangle$ according to whether $k=\frac{1}{4}$ or $\frac{3}{4}$), we find the following expression:

$$S_{m,n}(t)=\left[\begin{bmatrix}n_>\\n_<\end{bmatrix}\begin{bmatrix}n_>+2k-1\\n_<+2k-1\end{bmatrix}\right]^{1/2}\exp\left[-i(n+k)\int_0^t\omega(t')dt'\right]\exp[-i\chi(n-m)]$$

$$\times\mathcal{H}^{-(n+m+2k)}[\mathrm{sgn}(n-m)|\mathcal{F}|]^{n_>-n_<}\,{}_2F_1(-n_<;-n_<-2k+1;n_>-n_<+1;-|\mathcal{F}|^2)\ .\qquad(15)$$

The above results complete our short analysis of the SU(2) and SU(1,1) time-ordering theorems. Their connection to the generalized Bloch vector dynamics and the possibility of getting exact solutions will be discussed in Sec. II C.

C. The SU(3) case

In this section we discuss the WN ordering technique applied to linear combination of SU(3) generators. We consider therefore the following Hamiltonian operator:

$$\hat{H}(t)=\omega_T(t)\hat{T}_3+\Omega_T^*(t)\hat{T}_++\Omega_T(t)\hat{T}_-$$
$$+\omega_u(t)\hat{U}_3+\Omega_u^*(t)\hat{U}_++\Omega_u(t)\hat{U}_-$$
$$+\omega_v(t)\hat{V}_3+\Omega_v^*(t)\hat{V}_++\Omega_v(T)\hat{V}_-\ .\qquad(16)$$

Just to share a common language with the Gell-Mann and Ne'eman eightfold way notation,[2] we have exploited the usual (T,U,V) spin notation, whose commutation relations are given in the Appendix. To be more precise the Hamiltonian (16) underlies an SU(2)⊗SU(2)⊗SU(2) group structure rather than SU(3), which can be recovered by embedding U_3 and V_3 to get the standard hypercharge operator. We remark that the ω's and Ω's are complex time-dependent nonsingular functions, real and complex, respectively. The Hamiltonian (16) is a rather direct generalization of the operator (5) and this is, in our opinion, one of the advantages of choosing its actual structure rather than starting with SU(3) from the very beginning.

It is more convenient to use the interaction representation, which yields an interaction Hamiltonian of the type

$$\hat{H}_{\mathrm{int}}(t)=[\bar{\Omega}_T^*(t)\hat{T}_++\mathrm{H.c.}]+[\bar{\Omega}_u^*(t)\hat{U}_++\mathrm{H.c.}]$$
$$+[\bar{\Omega}_v^*(t)\hat{V}_++\mathrm{H.c.}]\ ,\qquad(17)$$

where

$$\bar{\Omega}_T^*(t)=\Omega_T^*(t)\exp\left[i\int_0^t[2\omega_T(t')+\omega_u(t')-\omega_v(t')]dt'\right],$$

$$\bar{\Omega}_u^*(t)=\Omega_u^*(t)\exp\left[i\int_0^t[\omega_T(t')+2\omega_u(t')+\omega_v(t')]dt'\right],\qquad(18)$$

$$\bar{\Omega}_v^*(t)=\Omega_v^*(t)\exp\left[i\int_0^t[-\omega_T(t')+\omega_u(t')+2\omega_v(t')]dt'\right].$$

Within this framework the time evolution operator writes

$$\hat{U}(t)=\exp\left[-i\int_0^t\hat{H}_0(t')dt'\right]\hat{U}_{\mathrm{int}}(t)\ ,$$
$$\hat{H}_0(t')=\omega_T(t')\hat{T}_3+\omega_u(t')\hat{U}_3+\omega_v(t')\hat{V}_3\ .\qquad(19)$$

As for $\hat{U}_{\mathrm{int}}$ it is more convenient to deal with the following ordered expression:

$$\begin{aligned}\hat{U}_{\text{int}}(t)=&\exp[2h_T(t)\hat{T}_3]\exp[g_T(t)\hat{T}_+]\exp[f_T(t)\hat{T}_-]\\&\times\exp[2h_u(t)\hat{U}_3]\exp[g_u(t)\hat{U}_+]\exp[f_u(t)\hat{U}_-]\\&\times\exp[2h_v(t)\hat{V}_3]\exp[g_v(t)\hat{V}_+]\\&\times\exp[f_v(t)\hat{V}_-]\ ,\end{aligned}\tag{20}$$

which is a direct generalization, apart from an inessential minus sign in front of the f functions, of the ordered product (8).

According to what has been discussed in Sec. II we introduce the following functions:

$$\begin{aligned}&\mathscr{H}_\alpha(t)=\exp[-h_\alpha(t)],\ \ \mathscr{H}_\alpha(0)=1,\ \dot{\mathscr{H}}_\alpha(0)=0\ ,\\&\mathscr{F}_\alpha(t)=f_\alpha\exp[-h_\alpha(t)],\ \ \mathscr{F}_\alpha(0)=0,\ \dot{\mathscr{F}}_\alpha(0)=-i\overline{\Omega}_\alpha(0)\ ,\\&\mathscr{G}_\alpha(t)=g_\alpha\exp[+h_\alpha(t)],\ \ \mathscr{G}_\alpha(0)=0,\ \dot{\mathscr{G}}_\alpha(0)=-i\overline{\Omega}{}^*_\alpha(0)\ ,\\&\qquad\qquad(\alpha=T,U,V)\ .\end{aligned}\tag{21}$$

As already stated, the problem is now to find the proper link between the above functions and those entering the Hamiltonian (16). After very tedious algebra we get

$$\begin{aligned}&\frac{d}{dt}(\mathscr{H}_T\mathscr{H}_u)=-i(\overline{\Omega}_T\mathscr{G}_T\mathscr{H}_u+\overline{\Omega}_u\mathscr{G}_u)\ ,\\&\frac{d}{dt}(\mathscr{G}_T\mathscr{H}_u)=-i(\overline{\Omega}_v\mathscr{G}_u+\overline{\Omega}{}^*_T\mathscr{H}_T\mathscr{H}_u)\ ,\\&\dot{\mathscr{H}}_u\mathscr{H}_T=i\mathscr{G}_u[\overline{\Omega}_u(1+\mathscr{G}_T\mathscr{F}_T)-\overline{\Omega}_v\mathscr{F}_T\mathscr{H}_T]\ ,\\&\dot{\mathscr{G}}_u=-i(\overline{\Omega}{}^*_u\mathscr{H}_T\mathscr{H}_u+\overline{\Omega}{}^*_v\mathscr{G}_T\mathscr{H}_u)\ ,\\&\mathscr{H}_v\mathscr{H}_T\dot{\mathscr{F}}_T+\mathscr{H}_T^2\mathscr{F}_u\dot{\mathscr{G}}_v=-i\overline{\Omega}_T\mathscr{H}_v(1+\mathscr{G}_T\mathscr{F}_T)\ ,\\&\mathscr{H}_T(\mathscr{H}_u\dot{\mathscr{F}}_u-\dot{\mathscr{H}}_u\mathscr{F}_u)=-i\overline{\Omega}_u(1+\mathscr{G}_T\mathscr{F}_T)+i\overline{\Omega}_v\mathscr{F}_T\mathscr{H}_T\ ,\\&\mathscr{H}_u(\mathscr{H}_v\dot{\mathscr{F}}_v-\mathscr{F}_v\dot{\mathscr{H}}_v)=-i(\overline{\Omega}_v\mathscr{H}_T-\overline{\Omega}_u\mathscr{G}_T)\ ,\\&\mathscr{H}_u\dot{\mathscr{H}}_v=-i(\overline{\Omega}_v\mathscr{H}_T-\overline{\Omega}_u\mathscr{G}_T)\mathscr{G}_v\\&\dot{\mathscr{G}}_u=-i\frac{\mathscr{H}_v\mathscr{H}_u}{\mathscr{H}_T(1+\mathscr{G}_u\mathscr{F}_u)}[\overline{\Omega}{}^*_v(1+\mathscr{F}_T\mathscr{G}_T)+\overline{\Omega}{}^*_u\mathscr{F}_T\mathscr{H}_T]\ .\end{aligned}\tag{22}$$

The above system of differential equations looks quite complicated, it is, however, the most general answer to the problem of SU(3) time ordering. In any case, apart from simplifications inherent to a well specified problem under study, Eqs. (22) can be reduced to a simpler form, according to the following comments.

(1) The solution of the system (22) depends on the solutions of only three equations. Indeed the first four equations are solved once the following coupled two can be solved:

$$\begin{aligned}&\ddot{\mathscr{G}}_u-\frac{\dot{\overline{\Omega}}{}^*_v-i\overline{\Omega}{}^*_u\overline{\Omega}_T}{\overline{\Omega}{}^*_v}\dot{\mathscr{G}}_u+(|\overline{\Omega}_u|^2+|\overline{\Omega}_v|^2)\mathscr{G}_u=\left[i\overline{\Omega}{}^*_u\frac{d}{dt}\ln\left[\frac{\overline{\Omega}{}^*_v}{\overline{\Omega}{}^*_u}\right]-\overline{\Omega}{}^*_v\overline{\Omega}{}^*_T+\left[\frac{(\overline{\Omega}{}^*_u)^2\overline{\Omega}_T}{\overline{\Omega}{}^*_v}\right]\right](\mathscr{H}_T\mathscr{H}_u)\\&\frac{d^2}{dt^2}(\mathscr{H}_T\mathscr{H}_u)-\frac{\dot{\overline{\Omega}}_T-i\overline{\Omega}_u\overline{\Omega}{}^*_v}{\Omega_T}\frac{d}{dt}(\mathscr{H}_T\mathscr{H}_u)+(|\overline{\Omega}_T|^2+|\Omega_u|^2)(\mathscr{H}_T\mathscr{H}_u)\\&\qquad=\left[i\overline{\Omega}_u\frac{d}{dt}\ln(\overline{\Omega}_T\overline{\Omega}{}_u^{-1})-\overline{\Omega}_T\overline{\Omega}_u+\frac{\overline{\Omega}{}^2_u\overline{\Omega}{}^*_v}{\Omega_T}\right]\mathscr{G}_u\ .\end{aligned}\tag{23}$$

(2) Once the solution of the first set is found the remaining three can be solved if the following second-order differential equation admits a solution:

$$\ddot{\mathscr{H}}_v+\frac{\dot{\mathscr{A}}}{\mathscr{A}}\dot{\mathscr{H}}_v+\mathscr{B}\mathscr{H}_v=0,\quad \mathscr{A}=\frac{\mathscr{H}_u}{\overline{\Omega}_v\mathscr{H}_T-\overline{\Omega}_u\mathscr{G}_T},\quad \mathscr{B}=\frac{[\overline{\Omega}{}^*_v(1+\mathscr{F}_T\mathscr{G}_T)+\overline{\Omega}{}^*_u\mathscr{F}_T\mathscr{H}_T](\overline{\Omega}_v\mathscr{H}_T-\overline{\Omega}_u\mathscr{G}_T)}{\mathscr{H}_T(1+\mathscr{G}_u\mathscr{F}_u)}\ .\tag{24}$$

The main achievement of this section is the explicit derivation of the motion equations of the ordering procedure. In Sec. III we show how to get from Eqs. (11) and (22) a set of Bloch-type equations and how to derive from the ordering procedure the "intrinsic" laws of conservation.

III. CONCLUSIONS

The results we have presented so far show that the WN ordering method is in principle a powerful tool to solve the Schrödinger equation when a time-dependent Lie-algebraic Hamiltonian is involved.

The actual forms of the Eqs. (11) and, in particular, of (22), may result complicated and, as they stand, not really helpful for the analysis of practical problems. It is worthwhile, therefore, to add some comments aimed to recover the connections with previous works. To this aim we will consider separately the SU(2), SU(1,1), and SU(3) cases.

A. SU(2) and SU(1,1)

We have shown that, in this case, the solution of the evolution problem depends on a single Riccati equation or, equivalently, on a second-order differential equation with time-dependent coefficients.

It is, however, well known that the SU(2) dynamics can be treated using a set of equations equivalent to those of nuclear magnetic resonance.[6] Indeed, the equations governing the evolution of the amplitude probabilities of a two-level problem can be cast in a vector form describing a rotation in an O(3) space.

More recently (see the last of Ref. 14) it has been shown that the SU(1,1) dynamics can be described by means of a rotation in a Lobatchevsky space [O(2,1)]. The SU(2) and SU(1,1) torque equations will be referred to as Euclidean and non-Euclidean Bloch equations, respectively. A relevant question is therefore whether a link exists between the Bloch-type dynamics and the characteristic equations of the ordering procedure.

Such a correspondence can be shown almost straightforwardly. Since Eq. (11) has the same structure of the amplitude probability equations, one can embed the two independent solutions $\mathcal{F}$ and $\mathcal{H}$ to form the following three variables:[17]

$$\begin{aligned} W&=|\mathcal{F}|^2-\delta|\mathcal{H}|^2 ,\\ U&=\sqrt{\delta}(\mathcal{F}\mathcal{H}+\mathcal{F}^*\mathcal{H}^*) ,\\ V&=-i\sqrt{\delta}(\mathcal{F}\mathcal{H}-\mathcal{F}^*\mathcal{H}^*) . \end{aligned} \tag{25}$$

Identifying (U,V,W) as the components of a vector $\mathcal{M}_\delta$, taking the time derivatives, and exploiting the equations (9), we find the following Bloch-type equations:

$$\dot{\mathcal{M}}_\delta=\mathbf{\Omega}_\delta\times\mathcal{M}_\delta , \tag{26}$$

where

$$\mathbf{\Omega}_\delta\equiv(2\delta^{3/2}\mathrm{Re}\Omega,2\delta^{3/2}\mathrm{Im}\Omega,+\omega) . \tag{27}$$

Equations (26) are Euclidean or non-Euclidean according to whether $\delta=\pm1$, respectively. A particularly important consequence one can derive from the above Bloch equation is the conservation law linked to the norm of $\mathcal{M}_\delta$, namely,

$$|\mathcal{M}_\delta|^2=|\mathcal{F}|^2+\delta|\mathcal{H}|^2=\delta . \tag{28}$$

Equation (28) is the Casimir invariant of the ordering procedure, and it can be shown to be linked to the average value of SU(2) or SU(1,1) Casimir invariant (for further comments see Ref. 17).

To give an example of the utility of the method just described, we will discuss its application to a particularly interesting problem. We will indeed consider the evolution of quantum states ruled by a harmonic oscillator Hamiltonian with a time-dependent mass and frequency. This kind of problem has been already discussed, in particular two of the present authors (Dattoli and Torre) and Solimeno have analyzed the case of the time-dependent frequency[13] while Abdalla[19] studied the time-dependent mass.

We dwell on this specific problem for its intrinsic importance and because it is mathematically equivalent to the analysis of other phenomena such as the wave propagation in self-focusing optical fibers.[20]

The Hamiltonian we will consider is of the type

$$\hat{H}(\hat{P},\hat{q},t)=\frac{\hat{P}^2}{2M(t)}+\tfrac{1}{2}\Omega^2(t)\hat{q}^2,\ \ \Omega(t)=\omega(t)M(t) , \tag{29}$$

where both $M(t)$ and $\Omega(t)$ are time-dependent nonsingular and real functions.

The relevance of the algebraic method to the Hamiltonian (29) is easily understood by noticing that the harmonic-oscillator dynamical variables can be cast to form a SU(1,1) noninvariance group as follows:[13]

$$\hat{k}_+=(i/2)\hat{q}^2,\ \ \hat{k}_-=(i/2)\hat{p}^2,\ \ \hat{k}_3=\tfrac{1}{2}(i\hat{q}\hat{p}+\tfrac{1}{2}) . \tag{30}$$

These operators can be identified as SU(1,1) generators with relations of commutation

$$[\hat{k}_+,\hat{k}_-]=-2\hat{k}_3,\ \ [\hat{k}_3,\hat{k}_\pm]=\pm\hat{k}_\pm . \tag{31}$$

The evolution operator can be therefore immediately written as

$$\hat{U}(t,t_0)=\exp(2h\hat{k}_3)\exp(g\hat{k}_+)\exp(-f\hat{k}_-) , \tag{32}$$

where, according to (9), the functions (h,g,f) obey the equations

$$\begin{aligned} \dot{h}&=-\frac{1}{M(t)}ge^{2h} ,\\ \dot{g}&=-\Omega^2(t)e^{-2h}-\dot{h}g ,\\ \dot{f}&=\frac{1}{M(t)}e^{2h},\ \ h(t_0)=f(t_0)=g(t_0)=0 , \end{aligned} \tag{33}$$

which can be solved once the following second-order equation for $\mathcal{H}$ and $\mathcal{F}$ [see Eq. (10)] can be solved:

$$\ddot{\xi}+\left[\frac{d}{dt}\ln[M(t)]\right]\dot{\xi}+\left[\frac{M(t)}{\Omega^2(t)}\right]^{-1}\xi=0 . \tag{34}$$

We notice that the procedure so far developed is significantly simpler than the canonical treatment proposed by Abdalla[19] which leads to characteristic equations more complicated than (34). Furthermore, the solution of the Schrödinger equation can be found almost straightforwardly. Indeed, in the hypothesis that initially the system is described by harmonic-oscillator functions, the wave function at a generic time is

$$\begin{aligned} \psi(t)&=\exp[i\theta_n(t)]\frac{(\mathcal{H}^2+\mathcal{F}^2)^{-1/4}}{(n!2^n\sqrt{\pi})^{1/2}}\\ &\quad\times H_n\left[\frac{\eta}{(\mathcal{H}^2+\mathcal{F}^2)^{1/2}}\right]\\ &\quad\times\exp\left[-\frac{\eta^2}{2(\mathcal{F}^2+\mathcal{H}^2)}\right],\ \ \eta=\sqrt{M\omega}q ,\\ \theta_n(t)&=\frac{\eta^2}{2\mathcal{H}}\left[\dot{\mathcal{H}}+\frac{\mathcal{F}}{\mathcal{H}^2+\mathcal{F}^2}\right]-(n+\tfrac{1}{2})\arctan[f(t)] \end{aligned} \tag{35}$$

[$H_n(\ )$ are the Hermite polynomials].

B. SU(3)

We have already stressed that Eqs. (22) are the most general answer to the problem of the SU(3) time ordering.

The characteristic equations, as they stand, do not appear easily solvable even in the case in which all the couplings are constants. Furthermore, it is not evident what the link between the result (22) and the previous works is.

In this section we will discuss the problem in full analogy to the SU(2) and SU(1,1) case by showing how Bloch-type motion equations can be derived and what kind of physical information can be drawn from the above analysis.

Before going into more technical details, let us stress that the characteristic equations of the SU(2) case can be immediately recovered from (22) by setting two of the three coupling constants equal to zero.

One of the most remarkable successes of the introduction of the SU(3) symmetry in dealing with a three-level system has been the possibility of characterizing its evolution by means of a generalized Bloch equation in an eight-dimensional space.

The natural question is whether we can derive, as in the previous case, a generalized torque equation directly from (22).

To this aim, following Refs. 15 and 21, we introduce the new variables

$$m'=\mathcal{H}_T\mathcal{H}_U,\quad n'=\mathcal{G}_T\mathcal{H}_U,\quad p'=\mathcal{G}_U\ , \tag{36}$$

which, as immediately verified from the first three equations of (22), obey the following equations of motion:

$$\begin{bmatrix}\dot m'\\ \dot n'\\ \dot p'\end{bmatrix}=\begin{bmatrix}0 & -i\bar\Omega_T & -i\bar\Omega_U\\ -i\bar\Omega_T^* & 0 & -i\bar\Omega_v\\ -i\bar\Omega_U^* & -i\bar\Omega_v^* & 0\end{bmatrix}\begin{bmatrix}m'\\ n'\\ p'\end{bmatrix}. \tag{37}$$

The structure of Eq. (37) is immediately recognized as that of three coupled harmonic oscillators.

By simply applying the Schwinger-Wigner realization of SU(3) (Ref. 22), we construct an eight-dimensional vector w' with components

$$\begin{aligned}&w_1'=2\,\mathrm{Re}(m'p'^*),\quad w_2'=2\,\mathrm{Im}(m'p'^*)\ ,\\ &w_3'=|p|^2-|m|^2,\quad w_4'=\mathrm{Re}(p'n'^*)\ ,\\ &w_5'=2\,\mathrm{Im}(p'n'^*),\quad w_6'=2\,\mathrm{Re}(m'n'^*)\ ,\\ &w_7'=2\,\mathrm{Im}(m'n'^*),\quad w_8'=1/\sqrt{3}(2|n'|^2-|p'|^2-|m'|^2)\ .\end{aligned} \tag{38}$$

After some algebra it can be checked that the time derivative of the vector w' takes the form

$$\dot w'_\lambda=f_{\lambda,\beta,\gamma}w'_\beta\Omega_\gamma\quad(\lambda\neq\beta\neq\gamma=1,\ldots,8)\ , \tag{39}$$

where $f_{\lambda,\beta,\gamma}$ are SU(3) structure constants and Ω_γ is a component of the vector

$$\Omega\equiv(-2\,\mathrm{Re}\bar\Omega_u,-2\,\mathrm{Im}\bar\Omega_u,0,-2\,\mathrm{Re}\bar\Omega_v,-2\,\mathrm{Im}\bar\Omega_v,-2\,\mathrm{Re}\bar\Omega_T,-2\,\mathrm{Im}\bar\Omega_T,0)\ . \tag{40}$$

Equation (39) is the SU(3) Bloch equation, and has the same structure of that proposed by Elgin[4] and Hioe and Eberly.[5]

The combination (36) is not unique. Indeed from (22) one can pick out further combinations satisfying the same equation (37). In particular we have

$$\begin{aligned}&m''=\mathcal{F}_T\mathcal{H}_v+\mathcal{F}_u\mathcal{H}_T\mathcal{G}_v\ ,\\ &n''=\mathcal{H}_v\mathcal{H}_T^{-1}(1+\mathcal{G}_T\mathcal{F}_T)+\mathcal{F}_u\mathcal{G}_T\mathcal{G}_v\ ,\\ &p''=\mathcal{H}_u^{-1}\mathcal{G}_v(1+\mathcal{G}_u\mathcal{F}_u)\ ,\end{aligned} \tag{41}$$

and

$$\begin{aligned}&m'''=\mathcal{F}_u\mathcal{H}_T\mathcal{H}_v^{-1}(1+\mathcal{G}_v\mathcal{F}_v)+\mathcal{F}_T\mathcal{F}_v\ ,\\ &n'''=\mathcal{F}_v/\mathcal{H}_T(1+\mathcal{G}_T\mathcal{F}_T)\\ &\qquad+\mathcal{G}_T\mathcal{F}_u\mathcal{H}_v^{-1}(1+\mathcal{F}_v\mathcal{G}_v)\ ,\\ &p'''=(\mathcal{H}_v\mathcal{H}_u)^{-1}(1+\mathcal{G}_u\mathcal{F}_u)(1+\mathcal{G}_v\mathcal{F}_v)\ .\end{aligned} \tag{42}$$

The above relations can be implemented with the following ones, also derived from (22):

$$\begin{aligned}&(m')^*=\frac{(1+\mathcal{G}_T\mathcal{F}_T)(1+\mathcal{G}_u\mathcal{F}_u)}{\mathcal{H}_T\mathcal{H}_u}\ ,\\ &(n')^*=-\mathcal{F}_T\mathcal{H}_u^{-1}(1+\mathcal{F}_u\mathcal{G}_u)\ ,\\ &(p')^*=-\mathcal{F}_u\ ,\end{aligned} \tag{43a}$$

and

$$\begin{aligned}&(m'')^*=\frac{\mathcal{G}_u\mathcal{F}_v}{\mathcal{H}_T}(1+\mathcal{G}_T\mathcal{F}_T)-\frac{\mathcal{G}_T}{\mathcal{H}_v}(1+\mathcal{G}_v\mathcal{F}_v)\ ,\\ &(n'')^*=\frac{\mathcal{H}_T(1+\mathcal{F}_v\mathcal{G}_v)}{\mathcal{H}_v}-\mathcal{F}_T\mathcal{G}_u\mathcal{F}_v\ ,\\ &(p'')=-\mathcal{F}_u\mathcal{H}_u\ ,\end{aligned} \tag{43b}$$

and

$$\begin{aligned}&(m''')^*=\mathcal{G}_v\mathcal{G}_T-\frac{\mathcal{H}_v\mathcal{G}_u}{\mathcal{H}_T}(1+\mathcal{G}_T\mathcal{F}_T)\ ,\\ &(n''')^*=-\mathcal{H}_T\mathcal{G}_v+\mathcal{F}_T\mathcal{G}_u\mathcal{H}_v\ ,\\ &(p''')^*=\mathcal{H}_u\mathcal{H}_v\ .\end{aligned} \tag{43c}$$

It goes without saying that from (41)–(43) one can get a Bloch-like equation identical to (39). It is also clear that, once one has the solutions for (m,n,p) one can get the explicit time dependence of the $(\mathcal{F}_\alpha,\mathcal{G}_\alpha,\mathcal{H}_\alpha)$ functions.

Let us therefore comment about the possibility of solving Eq. (37). When all the $\bar\Omega$ are time independent the solution is almost straightforward. When $\bar\Omega$ are time-dependent functions the solution cannot be found for any time dependence. However, if $\Omega_\alpha=i\,|\Omega_\alpha(t)|$, Eq. (37) can be rewritten as

$$\dot{\mathscr{P}}=\mathscr{P}\times\mathbf{\Omega}\ ,$$
$$\mathscr{P}\equiv(m,n,p)\ , \tag{44}$$
$$\mathbf{\Omega}\equiv(\,|\bar{\Omega}_v\,|,-\,|\bar{\Omega}_u\,|,|\bar{\Omega}_T\,|\,)\ ,$$

i.e., the three-level problem can be "mapped" on a two-level problem, and thus it can be solved using well-established techniques.[24]

Before concluding this paper let us add a few comments about the possibility of deriving from the ordering equations the invariants of motion of the three-level dynamics.

As to the SU(2) and SU(1,1) case we have seen that the intrinsic law of conservation is that related to the Casimir invariant of the characteristic equations. To derive the laws of conservation relevant to the three-level dynamics we must therefore construct the Casimir invariants associated to the SU(3) characteristic equations. This can be done almost straightforwardly. Following indeed a standard procedure we get[25]

$$\begin{aligned}|w|^2&=\tfrac{1}{2}\sum_{\beta,\lambda=1}^{3}A^{\lambda}_{\beta}A^{\beta}_{\lambda}=\sum_{i=1}^{8}w_i^2\ ,\\ \chi^3&=\sum_{\beta,\lambda,\mu=1}^{8}A^{\lambda}_{\beta}A^{\mu}_{\lambda}A^{\beta}_{\lambda}\ ,\end{aligned} \tag{45}$$

where

$$\begin{aligned}&A^1_1=w_3+1/\sqrt{3}w_8,\quad A^2_1=w_1+iw_2\ ,\\ &A^1_2=(A^2_1)^*,\quad A^2_2=-w_3+1/\sqrt{3}w_8\ ,\\ &A^3_1=w_1+iw_3,\quad A^1_3=(A^3_1)^*\ ,\\ &A^3_3=-\sqrt{2/3}w_8,\quad A^3_2=w_2+iw_3,\quad A^2_3=(A^3_2)^*\ .\end{aligned} \tag{46}$$

The invariants (45) have been constructed by adapting to the present case the procedure of De Swart to derive the SU(3) algebraic invariants. We must stress that the invariants (28) and (45) are intrinsic in the sense that they are implicitly contained in the ordering procedure. However, their physical interpretation needs specification on the nature of the initial wave function on which the evolution operator is acting (for further comments see Ref. 15).

As a final comment we must underline that the problem of SU(3) ordering [as well as of SU(2)] depends on the solution of a set of equations which are fully equivalent to those governing the amplitude probabilities evolution [see Eqs. (11) and (37)]. Therefore, from the mathematical point of view, the degree of complexity of directly tackling the solution of the Schrödinger equation or the Heisenberg equations of motion is the same.

An exact global solution of the SU(1,1), SU(2), and SU(3) problem is hampered by the impossibility of finding an analytical solution to Eqs. (11) and (37) for any time dependence of $\Omega(t)$. However, for a large class of $\Omega(t)$ functions solutions in terms of ${}_pF_q(\ )$ hypergeometric functions can be found (see also Ref. 24). This last problem and specific applications of the technique developed in this paper will be discussed elsewhere.

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APPENDIX

Recall that introducing the notation

$$\hat{F}\equiv[\hat{T}_1,\hat{T}_2,\hat{T}_3,\hat{V}_1,\hat{V}_2,\hat{U}_1,\hat{U}_2,\hat{Y}]\ , \tag{A1}$$

where the operator "hypercharge" $\hat{Y}$ is given by

$$\hat{Y}=\frac{1}{\sqrt{3}}(\hat{U}_3+\hat{V}_3)\ , \tag{A2}$$

the SU(3) group is characterized by the following commutation relations:

$$[\hat{F}_\alpha,\hat{F}_\beta]=if_{\alpha,\beta,\gamma}\hat{F}_\gamma\quad(\alpha\neq\beta\neq\gamma=1,\ldots,8)\ . \tag{A3}$$

The structure constants $f_{\alpha,\beta,\gamma}$ are explicitly given by

$$\begin{aligned}&f_{1,2,3,}=1,\quad f_{1,4,7}=-\tfrac{1}{2},\quad f_{1,5,6}=\tfrac{1}{2}\ ,\\ &f_{2,4,6}=\tfrac{1}{2},\quad f_{2,5,7}=\tfrac{1}{2},\quad f_{3,4,5}=-\tfrac{1}{2}\ ,\\ &f_{3,6,7}=\tfrac{1}{2},\quad f_{4,5,8}=\frac{\sqrt{3}}{2},\quad f_{6,7,8}=\frac{\sqrt{3}}{2}\ ,\end{aligned} \tag{A4}$$

and zero otherwise.

In the paper we have mentioned the SU(1,1) group, which has recently gained significant attention within the framework of parameter amplifiers.[23] It is well known that given two harmonic oscillators a straightforward realization of this group is the following:[23]

$$K_+=\hat{a}^\dagger_1\hat{a}^\dagger_2,\quad \hat{K}_-=\hat{a}_1\hat{a}_2,\quad \hat{K}_0=\tfrac{1}{2}(\hat{a}^\dagger_1\hat{a}_1+\hat{a}_2\hat{a}^\dagger_2)\ . \tag{A5}$$

Furthermore, introducing the "vector" with components

$$\hat{K}_1=\frac{\hat{K}_++\hat{K}_-}{2},\quad \hat{K}_2=\frac{\hat{K}_+-\hat{K}_-}{2i},\quad \hat{K}_3=\hat{K}_0\ , \tag{A6}$$

the following angular-momentum-like commutation relations can be stated:[14]

$$[\hat{K}_\alpha,\hat{K}_\beta]=i\tilde{\epsilon}_{\alpha,\beta,\gamma}\hat{K}_\gamma\ , \tag{A7}$$

where $\tilde{\epsilon}_{\alpha,\beta,\gamma}$ is linked to the Ricci tensor by

$$\tilde{\epsilon}_{\alpha,\beta,\gamma}=(-1)^{\delta_{\gamma 3}}\epsilon_{\alpha,\beta,\gamma} \tag{A8}$$

(δ is the Krönecker symbol and ϵ is the Ricci tensor), the evident analogy with the SU(2) case, allows one to extend the considerations developed in the paper and relevant to SU(3) to group structures of the type SU(1,1)⊗SU(1,1)⊗SU(2), whose generators can be recognized as

$$\begin{aligned}&\tilde{T}_3=\tfrac{1}{2}(\hat{a}^\dagger_1\hat{a}_1+\hat{a}_2\hat{a}^\dagger_2),\quad \tilde{V}_3=\tfrac{1}{2}(\hat{a}^\dagger_3\hat{a}_3-\hat{a}^\dagger_2\hat{a}_2)\ ,\\ &\tilde{T}_+=\hat{a}^\dagger_1\hat{a}^\dagger_2,\quad \tilde{V}_+=\hat{a}_2\hat{a}^+_3\ ,\\ &\tilde{T}_-=\hat{a}_1\hat{a}_2,\quad \tilde{V}_-=\hat{a}^\dagger_2\hat{a}_3\ ,\\ &\tilde{U}_3=\tfrac{1}{2}(\hat{a}^\dagger_1\hat{a}_1+\hat{a}_3\hat{a}^\dagger_3),\quad \tilde{U}_+=\hat{a}^\dagger_1\hat{a}^\dagger_3,\quad \tilde{U}_-=\hat{a}_1\hat{a}_3\ .\end{aligned} \tag{A9}$$

We can therefore define an eight-dimensional vector

$$\widetilde{F} \equiv (\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3, \widetilde{V}_1, \widetilde{V}_2, \widetilde{U}_1, \widetilde{U}_2, \widetilde{Y}) , \tag{A10}$$

thus finding

$$[\widetilde{F}_\alpha, \widetilde{F}_\beta] = i\widetilde{f}_{\alpha,\beta,\gamma}\widetilde{F}_\gamma , \tag{A11}$$

where the new structure constants are linked to the old ones by

$$\begin{aligned} \widetilde{f}_{\alpha,\beta,\gamma} &= f_{\alpha,\beta,\gamma}(-1)^{\delta_{\gamma 3}}, \quad \alpha,\beta,\gamma = 1,2,3 \text{ or } 6,7,8 , \\ \widetilde{f}_{\alpha,\beta,\gamma} &= f_{\alpha,\beta,\gamma}(-1)^{\delta_{\gamma 5}}, \quad \alpha,\beta,\gamma = 1,5,6 , \\ \widetilde{f}_{\alpha,\beta,\gamma} &= f_{\alpha,\beta,\gamma}(-1)^{\delta_{\gamma 4}}, \quad \alpha,\beta,\gamma = 1,7,4 , \\ \widetilde{f}_{\alpha,\beta,\gamma} &= f_{\alpha,\beta,\gamma}(-1)^{\delta_{\gamma 8}}, \quad \alpha,\beta,\gamma = 6,7,8 . \end{aligned} \tag{A12}$$

It is clear that the whole previous formalism for the time ordering applies with only few changes even for this new structure (the details will be given elsewhere[15]).

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