Nonlinear modulation of ion acoustic waves in a magnetized plasma

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The quasistatic plasma slow response to coherent ion acoustic waves in a magnetized plasma is considered. A multidimensional cubic nonlinear Schrödinger equation is derived. It is found that the ion acoustic waves remain modulationally stable against oblique perturbations.

Ion acoustic waves are one of the important low-frequency eigenmodes in plasmas. Such modes have been frequently observed in laboratory as well as in space plasmas. Since the existence of normal modes critically depends on their stability, it is of great interest to investigate the linear as well as the nonlinear instabilities that may hinder stable wave propagation.

A decade ago, Hasegawa[1] pointed out that the ion acoustic wave at a frequency smaller than the ion gyrofrequency Ω_i $(=eB_0/m_ic)$ can decay due to finite perpendicular wavelength effects. The dominant coupling comes from the interaction of the $\mathbf{E}\times\mathbf{B}_0$ flow with the ion-density fluctuations along the external magnetic field $B_0\hat{\mathbf{z}}$. On the other hand, Murtaza and Salahuddin[2] studied the modulation of ion-acoustic waves in a magnetized plasma by using the Krylov-Bogoliubov-Mitsopolsky (KBM) method.[3] They found that the inclusion of the harmonic-generation[4] nonlinearities produces wave instability in certain regions of parameter space.

In this paper we investigate the quasistatic[5] plasma slow response to coherent ion-acoustic waves in a magnetized plasma. Our present investigation extends a recent work[6] by including the effects of an external magnetic field. When the latter is incorporated, the problem becomes a multidimensional one in which the ion wave-group dispersions are anisotropic. Accounting for this fact, a two-dimensional nonlinear Schrödinger equation governing the dynamics of modulated wave packets is derived. It is found that the ion acoustic waves are modulationally stable against quasistatic oblique perturbations.

We consider the nonlinear propagation of electrostatic $(\mathbf{E}=-\nabla\phi)$, finite amplitude ion-acoustic waves in a homogeneous plasma embedded in an external magnetic field $B_0\hat{\mathbf{z}}$. All motion is restricted to the x-z plane. Thus $\nabla=\hat{\mathbf{x}}\partial_x+\hat{\mathbf{z}}\partial_z$ with $\partial_y\equiv 0$. For $\partial_t \ll \Omega_e$, the perpendicular (to $\mathbf{B}_0$) component of the electron fluid velocity is given by

$$\mathbf{v}_{e\perp}\simeq\hat{\mathbf{x}}(c/B_0\Omega_e)\partial^2{}_{tx}\phi \; , \tag{1}$$

where c is the speed of light, $\Omega_e=eB_0/m_ec$ is the electron gyrofrequency, and ϕ is the electrostatic potential. The above drift speed arises from the electron polarization drift, as in the chosen geometry the contribution of the $\mathbf{E}\times\mathbf{B}$ drift is zero. Since the parallel (to $\mathbf{B}_0$) phase velocity of the ion-acoustic waves is generally much smaller than the electron thermal velocity $v_{te}=(T_e/m_e)^{1/2}$, the inertialess electrons rapidly thermalize along the external magnetic field. The electron number-density perturbation associated with the ion-acoustic waves in the presence of the plasma slow motion is given by

$$\delta n_e/n_0=(1+\delta n_e^l/n_0)(e\phi/T_e) \; , \tag{2}$$

where n_0 is the average plasma density, and the superscript l denotes the corresponding quantities associated with the plasma slow motion. The leading order parallel (to $\mathbf{B}_0$) electron flow velocity obtained from the electron-continuity equation is given by

$$v_{ez}\approx-\partial_z^{-1}\partial_t(\delta n_e/n_0) \; . \tag{3}$$

Note that for the static modulations, $v_{ez}^l\approx 0$.

In the low-frequency limit ($\partial_t \ll \Omega_i$, where Ω_i is the ion gyrofrequency) the perpendicular component of the ion-fluid velocity is determined by the ion polarization drift, viz.,

$$\mathbf{v}_{i\perp}\simeq-\hat{\mathbf{x}}(c/B_0\Omega_i)\partial_{tx}^2\phi \; . \tag{4}$$

Noting that the total plasma density consists of three parts, namely, n_0, δn_e, and δn_e^l, we obtain from the ion-continuity equation and (4) an equation for the ion-acoustic waves including the plasma slow response,

$$[(1-\rho_s^2\partial_x^2-\lambda_D^2\nabla^2)\partial_t^2-c_s^2\partial_z^2]\Phi + [(1-\rho_s^2\partial_x^2)\partial_t^2-c_s^2\partial_z^2](\delta n_e^l/n_0)\Phi=0 \; , \tag{5}$$

where $\Phi=e\phi/T_e$. In deriving (5), we have made use of (2) and (4), the parallel component of the ion-momentum equation

$$\partial_t v_{iz}=-(e/m_i)\partial_z\phi \; , \tag{6}$$

and Poisson's equation. The ions are assumed to be much colder than the electrons. In the absence of the nonlinear interaction, linearization of (5) yields the dispersion relation[7]

$$\omega=k_zc_s/(1+k_x^2\rho_s^2+k^2\lambda_D^2)^{1/2}\equiv k_zc_s/b^{1/2} \; , \tag{7}$$

where $b=1+k_x^2\rho_s^2+k^2\lambda_D^2$, $c_s=(T_e/m_i)^{1/2}$ is the ion sound speed, $\lambda_D=c_s/\omega_{pi}$ is the electron Debye length, ω_{pi} is the ion plasma frequency, $\rho_s=c_s/\Omega_i$ is the ion Larmor radius at the electron temperature, $k^2=k_x^2+k_z^2$, and k_z (k_x) is the wave number parallel (perpendicular) to $B_0\hat{\mathbf{z}}$.

The dispersive terms $k_x^2\rho_s^2$ and $k^2\lambda_D^2$ come from the perpendicular ion inertia and charge-separation effects, respectively.

Assuming that the nonlinear interaction causes a slowly varying wave envelope, we let the ion-acoustic wave amplitude vary on slow time and space scales. Thus, within the WKB approximation,[8] we can write

$$\begin{aligned} &\Phi=\Phi(x,z,\tau)\exp(-i\omega t+ik\cdot\mathbf{x})+\text{c.c.}\ ,\\ &\partial_t-\partial_\tau-i\omega-\mathbf{v}_g\cdot\partial_\mathbf{x}\ ,\\ &\partial_x\to\partial_x+ik_x,\partial_z\to\partial_z+ik_z\ , \end{aligned} \tag{8}$$

where $\mathbf{v}_g=\hat{\mathbf{x}}v_{gx}+\hat{\mathbf{z}}v_{gz}$, with $v_{gx}=-\omega k_x\rho^2/b$, $v_{gz}=c_s(1+k_x^2\rho^2)/b^{3/2}$ and $\rho^2=\rho_S^2+\lambda_D^2$.

Inserting (8) into (5), using (7), one obtains a multidimensional nonlinear Schrödinger equation

$$i\partial_\tau\Phi+\frac{1}{2}(v_{gx}'\partial_x^2+v_{gz}'\partial_z^2+2T\partial_{xz}^2)\Phi-\frac{\omega}{2b}k^2\lambda_D^2(\delta n_e^l/n_0)\Phi=0\ , \tag{9}$$

where

$$v_{gx}'=\partial v_{gx}/\partial k_x=\omega\rho^2(2k_x^2\rho^2-1-k_z^2\lambda_D^2)/b^2\ ,$$
$$v_{gz}'=\partial v_{gz}/\partial k_z=-3\omega\lambda_D^2(1+k_x^2\rho^2)/b^2\ ,$$

and

$$T=k_x\omega\rho^2(2k_z^2\lambda_D^2-1-k_x^2\rho^2)/k_zb^2\ .$$

In deriving (9), the following ordering schemes have been introduced:

$$\begin{aligned} &\omega^{-1}\partial_\tau\sim O(\epsilon^3),\ \ \lambda_D\nabla\sim\rho_s\partial_x\sim O(\epsilon^{3/2})\ ,\\ &(k\lambda_D)^2\sim O(\epsilon),\ \ \Phi\sim O(\epsilon),\ \ \delta n_e^l/n_0\sim O(\epsilon^2)\ . \end{aligned} \tag{10}$$

Next, we present the plasma slow response to the ion-acoustic waves. For quasistatic modulations, the linear inertia of the low-frequency motion can be neglected. Thus, from the z component of the momentum balance, one obtains

$$m_e\langle\mathbf{v}_e\cdot\nabla v_{ez}\rangle=e\partial_z\phi^l-T_e\partial_z(\delta n_e^l/n_0)\ , \tag{11}$$

$$m_i\langle\mathbf{v}_i\cdot\nabla v_{iz}\rangle=-e\partial_z\phi^l-T_i\partial_z(\delta n_i^l/n_0)\ , \tag{12}$$

where the angular brackets denote averaging over the ion-acoustic wave period, and the left-hand sides of (11) and (12) represent the electron and the ion ponderomotive force associated with the ion-acoustic waves.

Adding (11) and (12), using (1), (3), (4), (6), and the quasineutrality $\delta n_e^l=\delta n_i^l$, one gets after some algebra

$$\delta n_e^l/n_0\approx-(1+k^2\lambda_D^2)\,|\Phi|^2/2(1+\sigma)\ , \tag{13}$$

where $\sigma=T_i/T_e$ is the ratio of the ion to electron temperatures, and we have noted that just as in the unmagnetized case,[6] the ponderomotive force acting on the ions is found to be much larger (by a factor of m_i/m_e) than that on the electrons. Clearly, the ion ponderomotive force gives rise to the field-aligned slow density variations.

Substituting (13) into (9), we have a multidimensional cubic Schrödinger equation

$$i\partial_\tau\Phi+\mathscr{L}\Phi+Q\,|\Phi|^2\Phi=0\ , \tag{14}$$

where

$$2\mathscr{L}=v_{gx}'\partial_x^2+v_{gz}'\partial_z^2+2T\partial_{xz}^2\ ,$$

and

$$Q=\omega k^2\lambda_D^2(1+k^2\lambda_D^2)/4b(1+\sigma)\ .$$

Let us now enquire whether a constant amplitude pump wave would remain stable (or become unstable) against quasistatic self-modulation. For this purpose, we let

$$\Phi=(\Phi_0+\delta\Phi)\exp(-i\Delta\tau)\ , \tag{15}$$

where Φ_0 is the pump amplitude, $\delta\Phi(\ll\Phi_0)$ is the perturbation, and Δ is a nonlinear frequency shift. Inserting (15) into (14), one finds $\Delta=-Q\,|\Phi_0|^2$, and an evolution equation for the perturbation

$$i(\partial_\tau+\mathbf{v}_g\cdot\partial_\mathbf{x})\delta\Phi+\mathscr{L}\delta\Phi+Q\,|\Phi_0|^2(\delta\Phi+\delta\Phi^*)=0\ , \tag{16}$$

where we have included the group velocity of the wave packet and the asterisk denotes the complex conjugate. Assuming that the complex amplitude $\delta\Phi$ depends on $K_xx+K_zz-\Omega\tau(|\mathbf{K}|\ll|\mathbf{k}|,\Omega\ll\omega)$, we can analyze (16) following the standard technique[9] to obtain the dispersion relation for the electrostatic modulations. The result is

$$(\Omega-\mathbf{K}\cdot\mathbf{v}_g)^2=P_0^2/4-QP_0\,|\Phi_0|^2\ , \tag{17}$$

where

$$P_0=v_{gx}'K_x^2+v_{gz}'K_z^2+2TK_xK_z\ .$$

Letting $\Omega=\mathbf{K}\cdot\mathbf{v}_g+i\gamma$, we obtain from (18) the growth rate

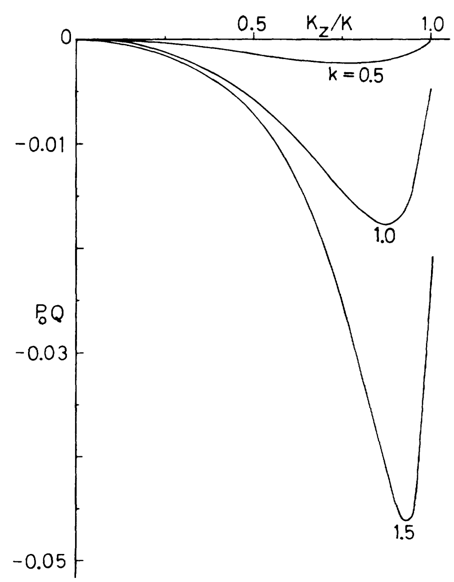

FIG. 1. Plot of the product P_0Q against the propagation angle. The parameter labeling the curves is $|\mathbf{k}|$.

$$\gamma = (QP_0 |\Phi_0|^2 - P_0^2/4)^{1/2} . \tag{18}$$

The term $P_0^2/4$ is small for long wavelengths ($|K|^{-1}$). Hence an instability may arise provided $P_0 Q > 0$. In Fig. 1 we have plotted the product $P_0 Q$ against K_z/K. We see that $P_0 Q$ is always negative. Thus we conclude that the ion-acoustic waves in a magnetized plasma are modulationally stable with respect to oblique (to $\mathbf{B}_0$) modulations.

We should like to contrast our work with that of Zakharov and Kuznetsov[7] who derived a multidimensional Korteweg—de Vries (KdV) equation and pointed out the possibility of a spherically symmetric stable nonenvelope ion-acoustic soliton. We note that the derivation of the KdV equation[7,10] includes the harmonic generation nonlinearities and does not account for the plasma slow response.[5] The inclusion of the latter in the analysis leads to the consideration of the two time and space scales.[5,8]

The present analysis, which accounts for the self-consistent density modulation of the ion-acoustic wave packet, differs from that of Murtaza and Salahuddin[2] who used the KBM method to consider the generation of second-order harmonics due to the nonlinear interaction.

In summary, we have investigated the quasistatic plasma slow response to the ion-acoustic waves in a magnetized plasma. It is found that a finite amplitude wave is not subjected to the modulational instability involving oblique perturbations. If this is the case, stable waves can propagate in a magnetized plasma. Such a phenomenon seems to be experimentally observed in the upstream region of the earth's bow shock.[11,12]

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